More on IF-(α,β)-n-Binormal Operator on IFH-Space

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Abstract
In this paper we introduce IF-(α,β)-n-Binormal operator on Hilbert space $\mathcal{H}$.We give some basic properties of these operator .An operator δ is an intuitionistic fuzzy-(α,β)-n-binormal operator if $\alpha^2\delta^n\delta^*\delta^*\delta^n \le \delta^*\delta^n\delta^n\delta^* \le \beta^2\delta^n\delta^*\delta^*\delta^n$ i.e. δ commutes with its intuitionistic fuzzy adjoint with $0\le \alpha \le 1 \le \beta$

Keywords: Self adjoint , intuitionistic fuzzy(α,β)- n-binormal operator , intuitionistic fuzzy n-binormal operator , partial isometry

1. Introduction

Let IFB(H) be the set of all IF-Bounded Linear operators on IFH-space .On Intuitionistic Fuzzy Norm have been defined by saadati [6] . Then Goudarzi et al . [4] in 2009 ,introduced Intuitionistic Fuzzy Inner product space(IFIP-Space).

Majumdhar and Samanta [7] defined IFIP-Space in 2011 .In 2018

Radharamani et al .[1], [2] have given the definition and properties of Intuitionistic Fuzzy Hilbert space (IFH-Space) $\mathcal{H}$ as a triplet $(\mathcal{H},\mathcal{F}_{\mu,\nu},\mathcal{T})$ and also the cancept of intuitionistic fuzzy adjonint and self-adjont operators (IFA and IFSA , operators) in IFH-space .IFH-space .IF$\delta \in \mathrm{IFB}(\mathrm{H}), \ni <\delta x, y> = <x\delta^* y>, \forall x,y \in \mathcal{H}$ *Also* δ *is an* IFSA-operator if $S = S^*$. In this paper we introduce n-binormal operators acting on a Hilbert space H. An operator T ∈ L(H) is n-binormal ifT^*T^n commutes with T^nT^* or $[T^*T^n, T^nT^*] = 0$ and it is denoted by $[nBN]$. An operator $T \in L(H)$ is called normal if $T^*T = T^*T$, n-normal if $T^*T^n = T^nT^*$, binormal if T^*T commutes with T^*T, isometry if $T^*T = I$, 2-isometry if $T^{*2}T^2 - 2T^*T + I = 0$, 3-isometry if $T^{*3}T^3 - 3T^{*2}T^2 + 3T^*T - I = 0$, n-isometry if $\sum_{k=0}^{n}(-1)^k\binom{n}{k}T^{*n-k}T^{n-k} = 0$ or $T^{*n}T^n - \binom{n}{1}T^{*n-1}T^{n-1} + \binom{n}{2}T^{*n-2}T^{n-2}\dots\dots\dots(-1)^{n-1}\binom{n}{n-1}T^*T + (-1)^n I = 0$ and n-binormal if $T^*T^nT^nT^* = T^nT^*T^*T^n$.

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Now we introduced intuitionistic fuzzy(α,β)-n-binormal operator on $\mathcal{H}$,if for real number α,β with $0\leq \alpha \leq 1 \leq \beta$ $then\ \alpha^2T^nT^*T^*T^n \leq T^*T^nT^nT^* \leq B^2T^nT^*T^*T^n$

Have we estadlish some theorems and an example for intuitionistic fuzzy(α,β)- n-binormal operator like addition and multiplication of intuitionistic fuzzy(α,β)- n-normal operator.

2. PRELIMINARIES

Definition 2.1: [4]A continuous t – norm T is called continuous t- representable iff ∃ a continuous t- norm * and a continuous t- conform ◊ on the interval [0,1] such that for all $x=(x_1,x_2)$, $y=(y_1,y_2) \in L^*$, $\mathcal{T}(x,y) = (x_1 * y_1\ , x_2 \lozenge y_2)$.

Definition 2.2: [4] Let $\mu : V^2 \times (0,+\infty) \to [0,1]$ and $\vartheta: V^2 \times (0,\infty) \to [0,1]$ be Fuzzy sets , such that $\mu(x,y,t) + \vartheta(x,y,t) \leq 1, \forall\, x,y \in V\ \&\ t > 0$. An Intuitionistic Fuzzy Inner Product Space

(IFIP-Space) is a triplet $\left(v,\mathcal{F}_{\mu,v}\ ,\mathcal{T}\right)$, whrer V is real Vector Space ,$\mathcal{T}$ is a continuous t – representable and $\mathcal{F}_{\mu,v}$ is an Intuitionistic Fuzzy set on $V^2 \times \mathbb{R}$ satisfying the following conditions for all $x,y,z \in V$ and $s,r,t \in \mathbb{R}$: (IFI-1) $\mathcal{F}_{\mu,v}(x,y,0) = 0$ and $\mathcal{F}_{\mu,v}(x,y,t) > 0$, for every $t > 0$.

(IFI-2) $\mathcal{F}_{\mu,v}(x,y,t) = \mathcal{F}_{\mu,v}(y,x,t)$.(IFI-3) $\mathcal{F}_{\mu,v}(x,x,t) \neq H(t)$ for some $t \in \mathbb{R}$ iff $x \neq 0$

Where $H(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$

(IFI-4) For any $\propto \in \mathbb{R}$,

$$\text{(IFI}-5)\sup\left\{\mathcal{T}\left(\mathcal{F}_{\mu,v}(x,z,s),\mathcal{F}_{\mu,v}(y,z,r)\right)\right\} = \mathcal{F}_{\mu,v}(x+y,y,t).$$

$$\text{(IFI}-6)\mathcal{F}_{\mu,v}(x,y,.): \mathbb{R} \to [0,1] \text{ is Continuous on } \mathbb{R}\backslash \{0\}.$$

$$\text{(IFI}-7)\ \text{Iim}_{t\to 0}.\mathcal{F}_{\mu,v}(x,y,t) = 1$$

Note 2.3: [4]

(i) Here the standard negator$\mathcal{N}_s(x) = 1 - x, \quad \forall\, x \in [0,1]$

(ii)By putting $\langle x,y\rangle = \mathcal{F}_{\mu,v}(x,y,\ .)$, it is very simple to show that the Intuitionistic Fuzzy Inner Product acts quite similarly as theOrdinary Inner Product.

Definition 2.4: (IFSA – operator) [2] Let $(V,\mathcal{F}_{\mu,v},\mathcal{T})$ be an IFH-Space with IP: $\langle x,y\rangle$= sup $\{t \in \mathbb{R}: \mathcal{F}_{\mu,v}(x,y,t) < 1\}, \forall\, x,y \in v$ and let $\mathcal{S} \in$ IFB (V). Then $\mathcal{S}$ is Intuitionistic Fuzzy Self- Ad joint operator , if $\mathcal{S} = \mathcal{S}^*$,where of $\mathcal{S}^*$is Intuitionistic Fuzzy Self- ad joint of $\mathcal{S}$.

Theorem 2.5: [2]Let$\left(v,\mathcal{F}_{\mu,v}\ ,\mathcal{T}\right)$be an IFIP – Space , where $\mathcal{T}$is a

continuoust– representable for every x, y $\in$ V,sup { t $\in \mathbb{R}$
$\mathcal{F}_{\mu,v}$ x, y, t $< 1 < \infty$. $Define$.,.: V $\times$ V $\to \mathbb{R}$ by x, y =sup
t $\in \mathbb{R}$: $\mathcal{F}_{\mu,vx,y,t<1}$.Then(V, ., .)is an IFIP-space, so that$(V, \mathcal{P}_{\mu,v})$is normed
space , where $\mathcal{P}_{\mu,v}(x,t) = \langle x, x\rangle^{1/2} \forall$ x $\in$ V

3. MAIN RESULTS

In this section , we introduced the definition of intuitionistic fuzzy(α,β)- n-Biormal operator in IFH - space and also explain some elementary properties of IF-(α,β)- n-Binormal operator in IFH, space in detail.

Definition 3.1: let $(\nu_1, F_{\mu,v}, T)$ be IFH-space and let $\tau \in IFB(\nu)$ Then T is called IF-(α,β)n-normal if for real number α,β with $0 \leq \alpha \leq 1 \leq \beta$,
$\alpha^2 T^n T^* T^* T^n \leq T^* T^n T^n T^* \leq B^2 T^n T^* T^* T^n$An immediate consequence of a bove definition
$\alpha^2 \langle T^n T^* T^* T^n x, x\rangle \leq \langle T^* T^n T^n T^* x, x\rangle \leq \beta^2 \langle T^n T^* T^* T^n x, x\rangle$For all $x \in H$

Theorem 3.2:if $S_1 S_2$ are commuting IF-(α,β) n-binormal operator,then $S_1 S_2$ is an IF-(α,β) n-binormal operator.

Proposition 3.3: let T,S is be a commuting IF-(α,β)- n-binormal operator such that $(S+T)^*$commutes with $\sum_{k=1}^{n-1}\binom{n}{k} S^{n-k}T^k$ Then $(S+T)$ is an IF-$(\alpha,\beta)-$ n-binormal operator.

Theorem 3.4: Let $T \in IFB(H)$Then T is $\mathrm{IF} - (\alpha,\beta) - \mathrm{n} - \mathrm{binormal}$ operator Then T^* is $\in [IF - (\alpha,\beta) - n - BN]$ operator .

Theorem 3.5: Let $T_1 \ldots \ldots . . T_m$ be commute IF-(α,β)-n -binormal operator in IFB(H) then $(T_1 \oplus \ldots \ldots . . \oplus T_m)$ are IF-(α,β)-n -binormal operator

Theorem 3.6: Let $T_1 \ldots \ldots . . T_m$ be commute IF-(α,β)-n -binormal operator in IFB(H)then$(T_1 \otimes \ldots \ldots . . \otimes T_m)$ are IF-(α,β)-n-binormal operator

Theorem 3.7 :Let $S_1 \in [IF - (\alpha,\beta) - n - BN]$ and $S_2 \in [IF - (\alpha,\beta) - n - BN]$ If S_1 and S_2 are doubly commuting . Then $S_1 S_2 \in [IF - (\alpha^2,\beta^2) - n - BN]$

Theorem 3.8: Let $T \in B(V)$be an IF-(α,β)-n-binormal operator, if $0 \leq p \leq 1$ or $p \geq 2$, and $(\mathbf{1}/\boldsymbol{p}) + (\mathbf{1}/\boldsymbol{q}) = \mathbf{1}$. Then we have

$$\left(\boldsymbol{P}_{\mu,v}(\boldsymbol{T}^*\boldsymbol{T}^n\boldsymbol{T}^n\boldsymbol{T}^*\boldsymbol{x} + \boldsymbol{T}^n\boldsymbol{T}^*\boldsymbol{T}^*\boldsymbol{T}^n\boldsymbol{x},\boldsymbol{t})\right)^{\boldsymbol{p}} + \left(\boldsymbol{P}_{\mu,v}(\boldsymbol{T}^*\boldsymbol{T}^n\boldsymbol{T}^n\boldsymbol{T}^*\boldsymbol{x} - \boldsymbol{T}^n\boldsymbol{T}^*\boldsymbol{T}^*\boldsymbol{T}^n\boldsymbol{x},\boldsymbol{t})\right)^{\boldsymbol{p}} \geq \mathbf{2}(\mathbf{1}+\boldsymbol{\alpha}^{\mathbf{2}\boldsymbol{q}})^{\boldsymbol{p}-\mathbf{1}}\left((\boldsymbol{P}_{\mu,v}(\boldsymbol{T}^n\boldsymbol{T}^*\boldsymbol{T}^*\boldsymbol{T}^n\boldsymbol{x}\,,\boldsymbol{t})\right)^{\boldsymbol{p}} \quad (3.1)$$

Proof: use the following known inequality :

$$\left(\boldsymbol{P}_{\mu,v}(\boldsymbol{a}+\boldsymbol{b},\boldsymbol{t})\right)^{\boldsymbol{p}} + \left(\boldsymbol{P}_{\mu,v}(\boldsymbol{a}-\boldsymbol{b},\boldsymbol{t})\right)^{\boldsymbol{p}} \geq \mathbf{2}\left[\left(\boldsymbol{P}_{\mu,v}(\boldsymbol{a},\boldsymbol{t})\right)^{\boldsymbol{q}} + \left(\boldsymbol{P}_{\mu,v}(\boldsymbol{b},\boldsymbol{t})\right)^{\boldsymbol{q}}\right]^{\boldsymbol{p}-\mathbf{1}} \quad (3.2)$$

Which is valid any $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{H}$ where $\boldsymbol{H}$ is Hilbert space .

Now , if we take $\boldsymbol{a} = \boldsymbol{T}^n\boldsymbol{T}^*\boldsymbol{T}^*\boldsymbol{T}^n\boldsymbol{x}$ and $\boldsymbol{b} = \boldsymbol{T}^*\boldsymbol{T}^n\boldsymbol{T}^n\boldsymbol{T}^*\boldsymbol{x}$in *(3.2)* , Then for any $\boldsymbol{x} \in \boldsymbol{V}$ we get

$$\left(\boldsymbol{P}_{\mu,v}(\boldsymbol{T}^n\boldsymbol{T}^*\boldsymbol{T}^*\boldsymbol{T}^n\boldsymbol{x} + \boldsymbol{T}^*\boldsymbol{T}^n\boldsymbol{T}^n\boldsymbol{T}^*\boldsymbol{x},\boldsymbol{t})\right)^{\boldsymbol{p}} + \left(\boldsymbol{P}_{\mu,v}(\boldsymbol{T}^n\boldsymbol{T}^*\boldsymbol{T}^*\boldsymbol{T}^n\boldsymbol{x} - \boldsymbol{T}^*\boldsymbol{T}^n\boldsymbol{T}^n\boldsymbol{T}^*\boldsymbol{x},\boldsymbol{t})\right)^{\boldsymbol{p}} \geq$$

$$\mathbf{2}\ \left[\left(P_{\mu,v}(T^nT^*T^*T^nx\ ,t)\right)^q + \left(P_{\mu,v}(T^*T^nT^nT^*x,t)\right)^q\right]^{p-1}$$

$$\geq \mathbf{2}\ \left[\left(P_{\mu,v}(T^nT^*T^*T^nx\ ,t)\right)^q + \alpha^{2q}\left(P_{\mu,v}(T^nT^*T^*T^nx,t)\right)^q\right]^{p-1}$$

$$= \mathbf{2}\ \ (\mathbf{1}+\alpha^{2q})^{p-1}\left[\left(P_{\mu,v}(T^nT^*T^*T^n\ x,t)\right)^q\right]^{p-1} \quad (3.3)$$

$$= \mathbf{2}\ \ (\mathbf{1}+\alpha^{2q})\ \ \left[\left(P_{\mu,v}(T^nT^*T^*T^nx,t)\right)^q\right]$$

Now , Taking the supremum over $P_{\mu,v}(x,t)=\mathbf{1}$ in (3.3) , We get

$$\left(P_{\mu,v}(T^nT^*T^*T^nx+T^nT^*T^*T^nx,t)\right)^p +$$
$$\left(P_{\mu,v}(T^nT^*T^*T^nx-\ T^*T^nT^nT^*x,t)\right)^p \geq$$
$$\mathbf{2}(\mathbf{1}+\alpha^{2q})^{p-1}\left((P_{\mu,v}(T^nT^*T^*T^nx\ ,t)\right)^p. \qquad \square$$

Theorem (3.9): Assume that T is $[IF-(\alpha,\beta)-n-BN]$ operator .Then for any real s with $\mathbf{0}\leq s\leq \mathbf{1}$,we have

$$\left[\left(\left(\frac{\mathbf{1}-s}{\beta^4}\right)+s\right)\left((\mathbf{1}-s)+\frac{s}{\beta^4}\right)\left(P_{\mu,v}(T^nT^*T^*T^n,t)\right)^4\right]$$
$$\leq [(\mathbf{1}-s)+s\beta^4]\left(P_{\mu,v}(T^nT^*T^*T^nx,t)\right)^2$$
$$\left(P_{\mu,v}(T^*T^nT^nT^*-T^nT^*T^*T^n\ ,t)\right)^2+\omega((T^*T^nT^nT^*)^*T^nT^*T^*T^n))^2.$$

Proof . By [9, Theorem 2.6](see also [10, Theorem 2.4]) we have

$$\left[(\mathbf{1}-s)\left(P_{\mu,v}(a,t)\right)^2+s\left(P_{\mu,v}(b,t)\right)^2\right]\left[(\mathbf{1}-s)\left(P_{\mu,v}(b,t)\right)^2\right.$$
$$\left.+s\left(P_{\mu,v}(a,t)\right)^2\right]-|\langle a,b\rangle|^2\leq\left[(\mathbf{1}-s)\left(P_{\mu,v}(a,t)\right)^2+s\left(P_{\mu,v}(b,t)\right)^2\right]$$
$$\left[(\mathbf{1}-s)\left(P_{\mu,v}(b-ma,t)\right)^2+s\left(P_{\mu,v}(mb-a,t)\right)^2\right] \qquad (3.4)$$

Where $\mathbf{0}\leq s\leq\mathbf{1}, m\in\mathbb{R}$ and $a,b\in\mathcal{V}$. By taking $m=\mathbf{1}, a=T^nT^*T^*T^nx$ and $b=T^*T^nT^nT^*x$in (3.4) ,we have

$$\left[(\mathbf{1}-s)\left(P_{\mu,v}(T^nT^*T^*T^nx,t)\right)^2\right.$$
$$\left.+s\left(P_{\mu,v}(T^*T^nT^nT^*x,t)\right)^2\right]\left[(\mathbf{1}-s)\left(P_{\mu,v}(T^*T^nT^nT^*x,t)\right)^2\right.$$
$$\left.+s\left(P_{\mu,v}(T^nT^*T^*T^nx,t)\right)^2\right]$$
$$-|\langle T^nT^*T^*T^nx,T^*T^nT^nT^*x\rangle|^2$$
$$\leq\left[(\mathbf{1}-s)\left(P_{\mu,v}(T^nT^*T^*T^nx,t)\right)^2\right.$$
$$\left.+s\left(P_{\mu,v}(T^*T^nT^nT^*x,t)\right)^2\right]\left[(\mathbf{1}-s)\left(P_{\mu,v}(T^*T^nT^nT^*x-T^nT^*T^*T^nx,t)\right)^2\right.$$
$$\left.+s\left(P_{\mu,v}(T^*T^nT^nT^*x-T^nT^*T^*T^nx,t)\right)^2\right] \quad (3.5)$$

Thus ,we have

$$\left[\left(\frac{1-s}{\beta^4}\right)\left(P_{\mu,v}(T^*T^nT^nT^*x,t)\right)^2 + s\left(P_{\mu,v}(T^*T^nT^nT^*x,t)\right)^2\right]\Big[(1 - s)\left(P_{\mu,v}(T^*T^nT^nT^*x,t)\right)^2 + \frac{s}{\beta^4}\left(P_{\mu,v}(T^*T^nT^nT^*x,t)\right)^2\Big]$$

$$-|\langle (T^*T^nT^nT^*x)^*T^nT^*T^*T^nx,\rangle|^2$$

$$\leq \Big[(1-s)\left(P_{\mu,v}(T^nT^*T^*T^nx,t)\right)^2 + s\beta^4\left(P_{\mu,v}(T^nT^*T^*T^nx,t)\right)^2\Big]\left(P_{\mu,v}(T^*T^nT^nT^*x - T^nT^*T^*T^nx,t)\right)^2$$

Finally , we take supremum over $\mathbf{P}_{\mu,\mathbf{v}}(x,t) = 1$

$$\left[\left(\left(\frac{1-s}{\beta^4}\right)+s\right)\left((1-s)+\frac{s}{\beta^4}\right)\left(P_{\mu,v}(T^nT^*T^*T^n,t)\right)^4\right]$$

$$\leq [(1-s)+s\beta^4]\left(P_{\mu,v}(T^nT^*T^*T^nx,t)\right)^2$$

$$\left(P_{\mu,v}(T^*T^nT^nT^* - T^nT^*T^*T^n\ ,t)\right)^2 + \omega((T^*T^nT^nT^*)^*T^nT^*T^*T^n))^2. \square$$

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