Epis and Permutative Posemigroups

[1]Shabir Ahmad Mir and [2]Rama Kumari

[1,2]Department of Mathematics DeshBhagat University Punjab India
Email: 1theshabir2@gmail.com and 2vinayrama87@gmail.com

Abstract: We show that the permutation identity $x_1 x_2 x_3 = x_{i_1} x_{i_2} x_{i_3}$ is preserved by every posemigroup epimorphism. Next, we determine an identity I other than permutation idenity such every epimorphisim from a permutative posemigroup satisfying the identity I is surjective. Finally, we show that any identity other than permutation identity satisfying by a permutative posemigroup is preserved under epimorphisms.

1. Introduction and Preliminaries

A partially ordered semigroup, briefly posemigroup is a pair $(S, \leq)$ comprising a semigroup S and a partial order $\leq$ on S that is compatible with its binary operation, i.e. for all $s_1, s_2, t_1, t_2 \epsilon S$, $s_1 \leq t_1$ and $s_2 \leq t_2$ implies that $s_1 s_2 \leq t_1 t_2$. If S is a monoid, we call $(S, \leq)$ partially ordered monoid, shortly pomonoid. We call $(U, \leq_U)$ a subposemigroup of a posemigroup $(S, \leq s)$, if U is subsemigroup of the semigroup S and $\leq_U = \leq s \cap (U \times U)$. The corresponding notion of subpomonoids is defined analogously.

A posemigroup morphism $f : (S, \leq_S) \to (T, \leq_T)$ is a monotone map i.e. $(x \leq_S y = \Rightarrow f(x) \leq_T f(y))$ which is also a semigroup morphism of the underlying semigroups.

We shall also denote posemigroups (pomonoids) by S, T etc. whenever no explicit mention of the order relation is required.

Let S and T be posemigroups and f : $S \to$ T be a posemigroup morphism. Then f is said be an epimorphism (epi for short) if for any posemigroup W and any posemigroup morphisms $\alpha, \beta : T \to W, \alpha \circ f = \beta \circ f$ implies $\alpha = \beta$. We observe that $f : S \to T$ is necessarily a posemigroup epimorphism if $f : S \to T$ is a semigroup epimorphism, where in the latter case we disregard the orders (and hence the monotonocity) and treat S and T as semigroups.

Let U be a subposemigroup of a posemigroup S and $d \in S$. We say that U dominates d if for all posemigroup epimorphisms $\alpha, \beta : S \to T$, such that $\alpha(u) = \beta(u)$ for all $u \in U$, one has $\alpha(d) = \beta(d)$. The set of all elements of S that are dominated by U is called the posemigroup dominion of U in S and is denoted by $\mathrm{Dom}_{\leq}(U, S)$. One can easily verify that $\mathrm{Dom}_{\leq}(U, S)$ is a subposemigroup of S containing U. A posemigroup U is said to be saturated if $\mathrm{Dom}_{\leq}(U, S) \neq S$ for every posemigroup S containing U properly as a subposemigroup. It can be easily verified that a posemigroup morphism $f : S \to T$ is an epi if and only if the inclusion $i : f(S) \to T$ is epi and the inclusion $i : U \to S$ is epi if and only if $\mathrm{Dom}_{\leq}(U, S) = S$.

An identity $u = v$ is said to be preserved under posemigroup epis if for any posemigroup S satisfying the identity $u = v$ and epimorphism $\alpha : S \to T$ implies that T also satisfies the identity $u = v$.

13220

Equivalently, we say that if U is a subposemigroup of a posemigroup S such that $\mathrm{Dom}_{\leq}(U, S) = S$ and U satisfies the idenity $u = v$ implies that S will also satisfy the same idenity.

The semigroup theoretic notations and conventions of Howie [4] will be used throughout without explicit mention.

The following result is known as the Zigzag Theorem for posemigroups provided by Sohailand Tart [6] and will frequently be used in what follows

Theorem 1.1.[6], Theorem 5 Let U be a subposemigroup of a posemigroup S. Then we have $d \in \mathrm{Dom}_{\leq}(U, S)$ if and only if $d \in U$ or

$$d \leq x_1 u_0, \qquad u_0 \leq u_1 y_1$$
$$x_i u_{2i-1} \leq x_{i+1} u_{2i}, \qquad u_{2i} y_i \leq u_{2i+1} y_{i+1}, 1 \leq i \leq m - 1$$
$$x_m u_{2m-1} \leq u_{2m}, \qquad u_{2m} y_m \leq d$$
$$(1.1) \qquad v_0 \leq s_1 v_1, \qquad d \leq v_0 t_1$$
$$s_j v_{2j} \leq s_{j+1} v_{2j+1} \qquad v_{2j-1} t_{j-1} \leq v_{2j} t_j, 1 \leq j \leq m' - 1$$
$$s_{m'} v_{2m'} \leq d, \qquad v_{2m'-1} t_{m'} \leq v_{2m'},$$

$where, u_0, v_0 \ldots u_{2m}, v_{2m'} \in U, x_1, y_1, \ldots, x_m, y_m, s_1, t_1, \ldots, s_{m'}, t_{m'} \in S.$

Let us call the above inequalities posemigroup zigzag inequalities in S over U with value d and length (m, m') and we say that it is of minimal length (m, m') if m and m' are the least positive integers. The first scheme of the zigzag inequalities (1.1) gives:

$$d \leq x_1 u_0 \leq x_1 u_1 y_1 \leq x_2 u_2 y_1 \leq \cdots \leq x_m u_{2m-1} y_m \leq u_{2m} y_m \leq d.$$

This gives,

$(1.2) \qquad d = x_1 u_0 = x_1 u_1 y_1 = x_2 u_2 y_1 = \cdots = x_m u_{2m-1} y_m = u_{2m} y_m.$

Similarly the second scheme of the zigzag inequalities (1.1) gives:

$$d \leq v_0 t_1 \leq s_1 v_1 t_1 \leq s_1 v_2 t_2 \leq \cdots \leq s_{m'} v_{2m'-1} t_{m'} \leq s_{m'} v_{2m'} \leq d.$$

This gives,

$(1.3) \qquad d = v_0 t_1 = s_1 v_1 t_1 = s_1 v_2 t_2 = \cdots = s_{m'} v_{2m'-1} t_{m'} = s_{m'} v_{2m'}.$

The next theorems are from [1] and are very important for our investigations.

Theorem 1.2. [1], Lemma 3.2

Let $d \in \mathrm{Dom}_{\leq}(U, S) \setminus U$ and (1.1) be the zigzag inequalities for d of minimal length (m, m'), then $x_i, y_i \in S \setminus U$ for $i = 1, 2. \ldots m$ and $s_j, t_j \in S \setminus U$ for all $j = 1, 2, \ldots m'$.

Theorem 1.3.([1], Lemma 3.3)

For any $d \in S \setminus U$ and for any positive integers k and k' there exist $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_{k'} \in U$ and $d_k, d_{k'} \in S \setminus U$ such that $d = u_1 u_2 \ldots u_k d_k = d_{k'} v_{k'} v_{k'-1} \ldots v_2 v_1$

Bracketed statements whenever used shall mean the dual to the other statements.

2. Permutative posemigroups

A semigroup S is said to be permutative if S satisfies a permutation identity

(2.1) $\qquad x_1 x_2 \ldots x_n = x_{i_1} x_{i_2 \ldots} x_{i_n}$,

Where i is a non trivial permutation of the set $\{1, 2, \ldots, n\}$.

We call a posemigroup S a permutative posemigroup if it is such as a semigroup. In [1], Ahanger and Shah have shown that if U is a commutative posemigroup then for any containing posemigroup S, $\text{Dom}_{\leq}(U, S)$ is also a commutative posemigroup. In particular it shows that commutativity is preserved under epimorphisms in the category of posemigroups. In this paper we extend this result to the case when U is any permutative posemigroup satisfying the permutation identity,

(2.2) $\qquad x_1 x_2 x_3 = x_{i_1} x_{i_2} x_{i_3}$

where i is any non-trivial permutation on $\{1, 2, 3\}$.

The permutative identity (2.2) can have any of the following form:

$\qquad x_1 x_2 x_3 = x_1 x_3 x_2$ [left normal]

$\qquad x_1 x_2 x_3 = x_2 x_1 x_3$ [right normal]

$\qquad x_1 x_2 x_3 = x_3 x_2 x_1$ [semi-commutative]

$\qquad x_1 x_2 x_3 = x_2 x_3 x_1$ [CT1-Permutative]

$\qquad x_1 x_2 x_3 = x_3 x_1 x_2$ [CT2-Permutative]

In this paper by permutative posemigroup we shall mean a posemigroup S satisfying the permutation identity (2.2).

Let U be a subposemigroup of a posemigroup S such that $\text{Dom}_{\leq}(U, S) = S$ and let $d_k \in S \setminus U$, $k = 1, 2, 3$. By Theorem 1.1, each d_k has zigzag inequalities of type (1.1) in S over U of minimal length (m_k, m_k'),

$$d_k = x_1^{(k)} u_0^{(k)}, \qquad u_0^{(k)} \leq u_1^{(k)} y_1^{(k)}$$

$$x_i^{(k)} u_{2i-1}^{(k)} \leq x_{i+1}^{(k)} u_{2i}^{(k)}, u_{2i}^{(k)} y_i^{(k)} \leq u_{2i+1}^{(k)} y_{i+1}^{(k)}, 1 \leq i \leq m_k - 1$$

$$(2.3) \qquad x_{m_k}^{(k)} u_{2m_k-1}^{(k)} \leq u_{2m_k}^{(k)}, \quad u_{2m_k}^{(k)} y_{m_k}^{(k)} = d_k$$

$$s_j^{(k)} v_{2j}^{(k)} \leq s_{j+1}^{(k)} v_{2j+1}^{(k)} \qquad v_{2j-1}^{(k)} t_{j-1}^{(k)} \leq v_{2j}^{(k)} t_j^{(k)}, \qquad 1 \leq j \leq m_{k'} - 1$$

$$s_{m_k'}^{(k)} v_{2m_k'}^{(k)} = d_k, \qquad\qquad v_{2m'_k-1}^{(k)} t_{m_k'}^{(k)} \leq v_{2m_k'}^{(k)},$$

13222

Where, $u_0^{(k)}, v_0^{(k)}, \ldots., u_{2m_k}^{(k)}, v_{2m_k'}^{(k)}, \in U$ *and by Theorem* 1.2, $x_1^{(k)}, y_1^{(k)}, \ldots\ldots, x_{m_k}^{(k)}, y_{m_k}^{(k)},$

$s_1^{(k)}, t_1^{(k)}, \ldots., s_{m_k'}^{(k)}, t_{m_k'}^{(k)}, \in S \backslash U.$

In Lemmas 2.1 to 2.6, U is a permutative posemigroup satisfying the permutation identity (2.2) and S is any posemigroup containing U properly as a subposemigroup such that $\text{Dom}_{\leq}(U, S) = S$.

Lemma 2.1. If U is right [left] normal posemigroup, then so is S.

Proof: We prove the lemma for the case when U is right normal posemigroup, the other case follows on similar lines. We have,

(2.4) $\qquad u_1 u_2 u_3 = u_2 u_1 u_3$ for all $u_1, u_2, u_3 \in U.$

We show that S is also right normal posemigroup, that is:

(2.5) $\qquad d_1 d_2 d_3 = d_2 d_1 d_3$ for all $d_1, d_2, d_3 \in U.$

If all of $d_1, d_2, d_3 \in U$, then there is nothing to prove. So assume that not all of d_1, d_2, d_3 are in U. Now we have the following cases:

Case 1: $\qquad d_1 \in S \setminus U$ and $d_2 = u_2, d_3 = u_3 \in U$. Then we have,

$$d_1 u_2 u_3 \leq x_1^{(1)} u_0^{(1)} u_2 u_3 \text{ (by zigzag inequalities (2.3)}$$
$$= x_1^{(1)} u_2 u_0^{(1)} u_3 (\text{as } U \text{ satisfies the identity (2.4))}$$
$$\leq x_1^{(1)} u_2 u_1^{(1)} y_1^{(1)} u_3 \big(\text{by zigzag inequalities (2.3)}\big).$$

Next for $l = 1, 2, \ldots, m_1 - 1$, we have

$$x_l^{(1)} u_2 u_{2l-1}^{(1)} y_l^{(1)} u_3 = x_l^{(1)} u_2 u_{2l-1}^{(1)} u^{(1)} y^{(1)} u_3 \ (where \ u^{(1)} y^{(1)} = y_l^{(1)}, u^{(1)} \in U, y^{(1)}$$
$$\in S \backslash U \ by \ Theorem \ 1.3 \ as \ y_l^{(1)} \in S \backslash U)$$

$$= x_l^{(1)} u_{2l-1}^{(1)} u_2 u^{(1)} y^{(1)} u_3 \big(as \ U \ satisfies \ the \ identity \ (2.4)\big)$$

$$\leq x_{l+1}^{(1)} u_{2l}^{(1)} u_2 u^{(1)} y^{(1)} u_3 \ (by \ zigzag \ inequalities \ (2.3))$$

$$= x_{l+1}^{(1)} u_2 u_{2l}^{(1)} u^{(1)} y^{(1)} u_3 \ (as \ U \ satifies \ the \ identity \ (2.4)$$

$$= x_{l+1}^{(1)} u_2 u_{2l}^{(1)} y_l^{(1)} u_3 \ (as \ u^{(1)} y^{(1)} = y_l^{(1)})$$

$$= x_{l+1}^{(1)} u_2 u_{2l+1}^{(1)} y_{l+1}^{(1)} u_3 \big(by \ zigzag \ inequalities \ (2.3)\big).$$

Therefore we have the following

$$du_2 u_3 \leq x_m^{(1)} u_2 u_{2m-1}^{(1)} y_m^{(1)} u_3$$

$$= x_{m_1}^{(1)} u_2 u_{2m_1-1}^{(1)} u^{(1)} y^{(1)} u_3 \ (where \ u^{(1)} y^{(1)} = y_{m_1}^{(1)}, u^{(1)} \in U, y^{(1)}$$
$$\in S \backslash U \ by \ Theorem \ 1.3 \ as \ y_{m_1}^{(1)} \in S \backslash U)$$

$$= x_{m_1}^{(1)} u_{2m_1-1}^{(1)} u_2 u^{(1)} y^{(1)} u_3 \ (as \ U \ satifies \ the \ identity \ (2.4))$$

13223

$$\leq u_{2m_1}^{(1)} u_2 u^{(1)} y^{(1)} u_3 \ (by \ zigzag \ inequalities \ (2.3))$$

$$= u_2 u_{2m_1}^{(1)} u^{(1)} y^{(1)} u_3 \ (as \ U \ satifies \ the \ identity \ (2.4))$$

$$= u_2 u_{2m_1}^{(1)} y_{m_1}^{(1)} u_3 \ (as \ u^{(1)} y^{(1)} = y_{m_1}^{(1)})$$

$$= u_2 d_1 u_3 \big(by \ zigzag \ inequalities \ (2.3) \big).$$

On similar lines by using the second scheme of the zigzag inequalities (2.3), we obtain $u_2 d_1 u_3 \leq d_1 u_2 u_3$. Therefore $d_1 u_2 u_3 = u_2 d_1 u_3$.

Case 2: $\quad\quad d_1, d_2 \in S \backslash U \ and \ d_3 = u_3 \in U. \ Then \ we \ have,$

$$d_1 d_2 u_3 \leq d_1 x_1^{(2)} u_0^{(2)} u_3 \ (by \ zigzag \ inequlities \ (2.3))$$

$$\leq u_0^{(2)} d_1 x_1^{(2)} u_3 \ (by \ Case \ 1)$$

$$\leq u_1^{(2)} y_1^{(2)} d_1 x_1^{(2)} u_3 \ (by \ zigzag \ inequality \ 2.3)).$$

Next for $l = 1, 2, \ldots, m_2 - 1$, we have

$$u_{2l-1}^{(2)} y_l^{(2)} d_1 x_l^{(2)} u_3 = y_l^{(2)} d_1 x_l^{(2)} u_{2l-1}^{(2)} u_3 \ (by \ Case \ 1)$$

$$\leq y_l^{(2)} d_1 x_{l+1}^{(2)} u_{2l}^{(2)} u_3 \ (by \ zigzag \ inequalities \ (2.3))$$

$$= u_{2l}^{(2)} y_l^{(2)} d_1 x_{l+1}^{(2)} u_3 \ (by \ Case \ 1)$$

$$\leq u_{2l+1}^{(2)} y_{l+1}^{(2)} d_1 x_{l+1}^{(2)} u_3 \big(by \ zigzag \ inequalities \ (2.3) \big).$$

Therefore we have the following

$$d_1 d_2 u_3 \leq u_{2m_2-1}^{(2)} y_{m_2}^{(2)} d_1 x_{m_2}^{(2)} u_3$$

$$= y_{m_2}^{(2)} d_1 x_{m_2}^{(2)} u_{2m_2-1}^{(2)} u_3 \ (by \ Case \ 1)$$

$$\leq y_{m_2}^{(2)} d_1 u_{2m_2}^{(2)} u_3 \ (by \ zigzag \ inequalities \ (2.3))$$

$$= u_{2m_2}^{(2)} y_{m_2}^{(2)} d_1 u_3 \ (by \ Case \ 1)$$

$$= d_2 d_1 u_3 \big(by \ zigzag \ inequalities \ (2.3) \big).$$

On similar lines by using the second scheme of the zigzag inequalities (2.3), we obtain $d_2 d_1 u_3 \leq d_1 d_2 u_3. \ Therefore \ d_1 d_2 u_3 = d_2 d_1 u_3$

Case 3: $\quad\quad d_1, d_2, d_3 \in S \backslash U. \ Then \ we \ have$

$$d_1 d_2 d_3 = d_1 d_2 v_0^3 t_1^3 \ (by \ zigzag \ inequalities \ (2.3))$$

$$= d_2 d_1 v_0^3 t_1^3 \ (by \ Case \ 2)$$

$$= d_2 d_1 d_3 \big(by \ zigzag \ inequalities \ (2.3) \big),$$

as required.

Lemma 2.2. If U is semicommutative posemigroup, then so is S.

Proof: Since U is semicommutative, therefore, we have,

$(2.6) u_1 u_2 u_3 = u_3 u_2 u_1 \ \text{for all} \ u_1, u_2, u_3 \in U.$

We show that S is also semicommutative, that is,

$(2.7) \qquad\qquad d_1 d_2 d_3 = d_3 d_2 d_1 \ for \ all \ d_1, d_2, d_3 \in S.$

If all of $d_1, d_2, d_3 \in U$, there is nothing to prove. So assume that not all of $d_1, d_2, d_3 \ are \ in \ U.$ Now we have the following cases:

Case 1: $\qquad d_1 \in S \backslash U \ and \ d_2 = u_2, d_3 = u_3 \in U. \ Then \ we \ have,$

$$d_1 u_2 u_3 \leq x_1^{(1)} u_0^{(1)} u_2 u_3 \ (by \ zigzag \ inequalities \ (2.3))$$

$$= x_1^{(1)} u_3 u_2 u_0^{(1)} \ (as \ U \ satifies \ the \ identity \ (2.6))$$

$$\leq x_1^{(1)} u_3 u_2 u_1^{(1)} y_1^{(1)} \ (by \ zigzag \ inequalities \ (2.3))$$

$$= x_1^{(1)} u_1^{(1)} u_2 u_3 y_1^{(1)} \big(as \ U \ satifes \ the \ identity \ (2.6)\big).$$

Next for $l=1,2,\ldots,m_1$-1, we have

$$x_l^{(1)} u_{2l-1}^{(1)} u_2 u_3 y_l^{(1)} = x_l^{(1)} u_3 u_2 u_{2l-1}^{(1)} y_l^{(1)} \ (as \ U \ satifies \ the \ identity \ (2.6))$$

$$\leq x_l^{(1)} u_3 u_2 u_{2l}^{(1)} y_{l+1}^{(1)} \ (by \ zigzag \ inequalities \ (2.3))$$

$$\leq x_l^{(1)} u_{2l}^{(1)} u_2 u_3 y_{l+1}^{(1)} \ (as \ U \ satisfies \ the \ identity \ (2.6))$$

$$\leq x_{l+1}^{(1)} u_{2l+1}^{(1)} u_2 u_3 y_{l+1}^{(1)} \big(by \ zigzag \ inequalities \ (2.3)\big).$$

Therefore we have the following

$$d_1 u_2 u_3 \leq x_{m_1}^{(1)} u_{2m_1-1}^{(1)} u_2 u_3 y_{m_1}^{(1)}$$

$$\leq u_{2m_1}^{(1)} u_2 u_3 y_{m_1}^{(1)} \ (by \ zigzag \ inequalities \ (2.3))$$

$$= u_3 u_2 u_{2m_1}^{(1)} y_{m_1}^{(1)} (as \ U \ satifies \ the \ identity \ (2.6))$$

$$= u_3 u_2 d_1 \big(by \ zigzag \ inequalities \ (2.3)\big).$$

On Similar lines by using the second scheme of the zigzag inequalities $(2.3),$

$we \ obtain, u_3 u_2 d_1 \leq d_1 u_2 u_3. \ Therefore \ d_1 u_2 u_3 = u_3 u_2 d_1.$

Case 2: $\qquad d_1, d_2 \in S \backslash U \ and \ d_3 = u_3 \in U. \ Then \ we \ have,$

$$d_1 d_2 u_3 \leq d_1 x_1^{(2)} u_0^{(2)} u_3 \ (by \ zigzag \ inequalities \ (2.3))$$

$$= u_3 u_0^{(2)} d_1 x_1^{(2)} \ (by \ Case \ 1)$$

13225

$$\leq u_3 u_1^{(2)} y_1^{(2)} d_1 x_1^{(2)} \left(by\ zigzag\ inequalities\ (2.3)\right).$$

For all $l = 1,2,\ldots,m_2 - 1$ and by applying zigzag inequalities (2.3) and Case 1 it can be easily verified that

$$u_3 u_{2l-1}^{(2)} y_l^{(2)} d_1 x_l^{(2)} \leq u_3 u_{2l+1}^{(2)} y_{l+1}^{(2)} d_1 x_{l+1}^{(2)}.$$

Therefore we have the following

$$d_1 d_2 u_3 \leq u_3 u_{2m_2-1}^{(2)} y_{m_2}^{(2)} d_1 x_{m_2}^{(2)}$$

$$= y_{m_2}^{(2)} d_1 x_{m_2}^{(2)} u_{2m_2-1}^{(2)} u_3\ (by\ Case\ 1)$$

$$\leq y_{m_2}^{(2)} d_1 u_{2m_2}^{(2)} u_3\ (by\ zigzag\ inequalities\ (2.3))$$

$$= u_3 u_{2m_2}^{(2)} y_{m_2}^{(2)} d_1\ (by\ Case\ 1)$$

$$= u_3 d_2 d_1 \left(by\ zigzag\ inequalities\ (2.3)\right).$$

By a similar token, we have, $u_3 d_2 d_1 \leq d_1 d_2 u_3.$ Therefore $u_3 d_2 d_1 \leq d_1 d_2 u_3.$

Case 3: $\qquad d_1, d_2, d_3 \in S \backslash U.$ Then we have,

$$d_1 d_2 d_3 = d_1 d_2 v_0^{(3)} t_1^{(3)}\ (by\ zigzag\ inequalities\ (2.3))$$

$$= v_0^{(3)} d_2 d_1 t_1^{(3)}\ (by\ Case\ 2)$$

$$\leq s_1^{(3)} v_1^{(3)} d_2 d_1 t_1^{(3)} \left(by\ zigzag\ inequalities\ (2.3)\right).$$

$Next, for\ all\ l = 1,2,\ldots,m_3 - 1, we\ have$

$$s_l^{(3)} v_{2l-1}^{(3)} d_2 d_1 t_l^{(3)} = s_l^{(3)} d_1 d_2 v_{2l-1}^{(3)} t_l^{(3)}\ (by\ Case\ 2)$$

$$\leq s_l^{(3)} d_1 d_2 v_{2l}^{(3)} t_{l+1}^{(3)}\ (by\ zigzag\ inequalities\ (2.3))$$

$$= s_l^{(3)} v_{2l}^{(3)} d_2 d_1 t_{l+1}^{(3)}\ (by\ Case\ 2)$$

$$\leq s_{l+1}^{(3)} v_{2l+1}^{(3)} d_2 d_1 t_{l+1}^{(3)} \left(by\ zigzag\ inequalities\ (2.3)\right).$$

Thus we have the following

$$d_1 d_2 d_3 \leq s_{m_3}^{(3)} v_{2m_3-1}^{(3)} d_2 d_1 t_{m_3}^{(3)}$$

$$= s_{m_3}^{(3)} d_1 d_2 v_{2m_3-1}^{(3)} t_{m_3}^{(3)}\ (by\ Case\ 2)$$

$$\leq s_{m_3}^{(3)} d_1 d_2 v_{2m_3}^{(3)}\ (by\ zigzag\ inequalities\ (2.3))$$

$$= s_{m_3}^{(3)} v_{2m_3}^{(3)} d_2 d_1\ (by\ Case\ 2)$$

$$\leq d_3 d_2 d_1 \left(by\ zigzag\ inequalities\ (2.3)\right).$$

13226

On similar lines, we have, $d_3 d_2 d_1 \leq d_1 d_2 d_3$.Therefore $d_1 d_2 d_3 = d_3 d_2 d_1$.Hence the lemma follows.□

Lemma 2.3. *If U is $CT1$ $[CT2]$ posemigroup then so is S.*

Proof: We prove the lemma for CT1 posemigroup, the other case follows on similar lines. We have,

$(2.8) u_1 u_2 u_3 = u_2 u_3 u_1$ *for all $u_1, u_2, u_3 \in U$.*

We show that S is also $CT1$, that is,

$$(2.9) d_1 d_2 d_3 = d_2 d_3 d_1 \text{ for all } d_1, d_2, d_3 \in S$$

If all of $d_1, d_2, d_3 \in U$, the result follows trivially. So assume that not all d_1, d_2, d_3 *are in U*. Therefore we have the following cases:

Case 1:*If $d_1 \in S \backslash U$ and $d_2 = u_2, d_3 = u_3 \in U$.Then we have,*

$$d_1 u_2 u_3 = x_1^{(1)} u_0^{(1)} u_2 u_3 \text{ (by zigzag inequalities (2.3))}$$

$$= x_1^{(1)} u_2 u_3 u_0^{(1)} \text{ (as U satifies (2.9))}$$

$$\leq x_1^{(1)} u_2 u_3 u_1^{(1)} y_1^{(1)} \text{ (by zigzag inequalities (2.3))}$$

$$= x_1^{(1)} u_1^{(1)} u_2 u_3 y_1^{(1)} \quad \text{(as U satisfies (2.9)).}$$

By zigzag inequalities and the identity (2.9) it can be easily verified that for all $l = 1, 2, \ldots m_1 - 1$

$$x_l^{(1)} u_{2l-1}^{(1)} u_2 u_3 y_l^{(1)} \leq x_{l+1}^{(1)} u_{2l+1}^{(1)} u_2 u_3 y_{l+1}^{(1)}.$$

Thus we have the following

$$d_1 u_2 u_3 \leq x_{m_1}^{(1)} u_{2m_1-1}^{(1)} u_2 u_3 y_{m_1}^{(1)}$$

$$\leq u_{2m_1}^{(1)} u_2 u_3 y_{m_1}^{(1)} \text{ (by zigzag inequalities (2.3))}$$

$$= u_2 u_3 u_{2m_1}^{(1)} y_{m_1}^{(1)} \text{ (as U satifies (2.9))}$$

$$= u_2 u_3 d_1 \big(\text{by Zigzag inequalities (2.3)}\big).$$

On similar lines, we can show that, $d_1 u_2 u_3 \geq u_2 u_3 d_1$. Therefore, $d_1 u_2 u_3 = u_2 u_3 d_1$. By a similar token we can show that $u_2 u_3 d_1 = u_3 d_1 u_2$. Therefore $d_1 u_2 u_3 = u_2 u_3 d_1 = u_3 d_1 u_2$.

Case 2: *If $d_1, d_2 \in S \backslash U$ and $d_3 = u_3 \in U$.* We first verify that for all $s \in S \backslash U$ and $v \in U, sv = vs$. By Theorem 1.3, $s = s'v'$, where $s' \in S \backslash U$ and $v' \in U$, as $s \in S \backslash U$.Now we have

$$sv = s'v'v$$

$$= vs'v' (By \ Case \ 1)$$

(2.10) $\qquad\qquad\qquad\qquad\qquad\qquad = vs.$

By using the equation (2.10) it can be easily verified that,

$(2.11) d_1 d_2 = d_2 d_1$

Now,

$$d_1 d_2 u_3 = d_2 d_1 u_3 \; (by \; equation \; (2.11))$$

$$= d_2 u_3 d_1 \big(by \; equation \; (2.10)\big).$$

Again, we have $d_1 d_2 u_3 = u_3 d_1 d_2$. Therefore $d_1 d_2 u_3 = d_2 u_3 d_1 = u_3 d_1 d_2$

Case 3: If $d_1, d_2, d_3 \in S \backslash U.$ *Then we have,*

$$d_1 d_2 d_3 = d_2 d_3 d_1 \big(by \; equation \; (2.11)\big),$$

as required. $\square$

Theorem 2.4. Let U be a permutataive posemigroup satisfying the permutation identity (2.2) and S be any posemigroup containing U properly as a subposemigroup such that $\mathrm{Dom}_{\leq}(U,S) = S$. Then S is also permutative posemigroup satisfying the permutation identity (2.2).

Corollary 2.5. Any permutative identity of the type (2.2) is preserved under epimorphisms of posemigroups.

3. Saturated Variety of Permutative Posemigroups

Ahanger and Shah [1], have shown that the commutative posemigroups is saturated if and only if it satisfies a non-trivial identity I such that I is not a permutation identity and at least one side of I has no repeated variable. In particular, it shows that if U is a commutative posemigroup satisfying an identity I other than a permutation identity and at least one side of I has no repeated variable, then every epi from U is surjective. In this paper we extend this result to permutative posemigroups satisfying the permutation identity (2.2).

Lemma 3.1. Let S be any permutative posemigroup satisfying the permutative identity (2.2). Then the following hold:

(i)   If neither $x_1 x_2 \; nor \; x_2 x_3$ are subwords of $x_{i_1} x_{i_2} x_{i_3}$, then $x_1 x y x_2 x_3 = x_1 y x x_2 x_3 \; and \; x_1 x_2 x y x_3 = x_1 x_2 y x x_3$ respectively.

(ii)   If $i_1 \neq 1 \; [i_3 \neq 3],$ then S satisfying the following permutation identity

$$x y x_1 x_2 x_3 = y x x_1 x_2 x_3 \; [x_1 x_2 x_3 x y = x_1 x_2 x_3 y x]$$

Proof: The proof follows from [5], Proposition 4.1. $\square$

From Lemma 3.2 to Lemma 3.6, U is a permutative posemigroup satisfying the permutation identity (2.2) and S is any posemigroup containing U properly as a subposemigroup such that $\mathrm{Dom}_{\leq}(U,S) = S$.

Lemma 3.2. *For any $x, y \in S$ and $s, t \in S \backslash U, \; sxyt = syxt.$*

Proof: Since by Theorem 1.3, $s = s_1 s_2 s_3$ and $t = t_1 t_2 t_3$, where $s_i, t_i \in S$, for $i = 1,2,3$ and either $i_1 \neq 1 \ or \ i_3 \neq 3$, therefore the result follows from Lemma 3.1 (ii). $\square$

Lemma 3.3. Let j be any permutation of the set $\{1,2,\ldots,k\}$, then

$$sx_1 x_2 \ldots x_k t = sx_{j1} x_{j2} \ldots x_{jk} t \ for \ all \ s, t \ \in S \backslash U, x_1, x_2, \ldots, x_k \in S.$$

Proof: It follows from Lemma 3.2. $\square$

Lemma 3.4. If $i_1 \neq 1 \ [i_3 \neq 3]$ then $xyt = yxt \ [sxy = syx]$ for all $x, y \ \in S$ and

$$t \in S \backslash U \ [s \in S \backslash U].$$

Proof: It follows from Lemma 3.1 (ii). $\square$

Lemma 3.5.

If $i_1 \neq 1 \ [i_3 \neq 3]$ then $x_1 x_2 \ldots x_k t = x_{j1} x_{j2} \ldots x_{jk} t \ [sx_1 x_2 \ldots x_k = sx_{j1} x_{j2} \ldots x_{jk}]$ for   all $x_1 x_2 \ldots x_k \in S \ and \ t \in S \backslash U \ [s \in s \backslash U]$ for any permutation j of the set $\{1,2,\ldots,k\}$.

Proof: It follows from Lemma 3.4. $\square$

Lemma 3.6. Let U satisfies the non-trivial identity I such that one side of I has no repeated variable. Then for any $u \in U$ and $x, t \in S \backslash U, sut = su^q t$ for some $q \geq 2$.

Proof: As in([5], Theorem 5.1) without loss of generality we can assume that I has the following form:

$$(3.1) \qquad\qquad z_1 z_2 \ldots z_n = w(z_1, z_2, \ldots, z_n),$$

Where $|z_i|_w \geq 1 \ for \ all \ i = 1,2, \ldots, n \ and \ |z_j|_w \geq 2 \ for \ some \ j \in \{1,2, \ldots, n\}$

($|z|_w \ for \ any \ variable \ z \ is \ the \ number \ of \ occurences \ of \ the \ variable \ z \ in \ the \ word \ w$).

By theorem 1.3, we have,

$$(3.2) \qquad t = u_1 u_2 \ldots u_n t_n \ (for \ some \ t_n \in S \backslash U \ and \ u_1, u_2, .., u_n \in U.$$

Therefore we have,

$sut = suu_1 u_2 \ldots u_n t_n \ (by \ equation \ (3.2))$

$\qquad = su_1 u_2 \ldots (uu_j) \ldots u_n t_n \ (by \ Lemma \ 3.3 \ as \ s, t_n \in S \backslash U)$

$\qquad = sw(u_1, u_2, .., uu_j, .., u_n) t_n \ (as \ U \ satifies \ the \ identity \ (3.1))$

$\qquad = su^q w(u_1, u_2, \ldots, u_n) t_n \ (by \ Lemma \ 3.3, where \ q = |z_j|_w \geq 2)$

$\qquad = su^q u_1 u_2 \ldots u_n t_n \ (as \ U \ satifies \ the \ identity \ (3.1))$

$\qquad = su^q t \ (by \ equation \ (3.2)),$

as required. $\square$

Lemma 3.7. For all $u \in U, s, t \in S^1 \ and \ x, y \ \in S \backslash U, xsuty = xsu^q ty$ for some $q \geq 2$.

Proof: It follows from the proof of the Lemma 3.6. $\square$

Lemma 3.8. If $i_1 \neq 1$ $[i_3 \neq 3]$, then $xyz = yxz$ $[xyz = xzy]$ for all $x, z \in S$ and $y \in S \backslash U$

Proof: We prove the lemma for the case $i_1 \neq 1$, the other case follows on similar lines. Let (1.1) be the zigzag inequality for y and minimal length (m, m'). If $i_3 = 3$. Then $i_2 \neq 2$ and therefore the result follows trivially. If $i_3 \neq 3$, then we have,

$$xyz \leq xv_0 t_1 z \ (by \ zigzag \ inequalities \ (1.1))$$

$$= v_0 x t_1 z \ (by \ Lemma \ 3.4 \ as \ t_1 \in S \backslash U \ and \ i_1 \neq 1)$$

$$\leq s_1 v_1 x t_1 z \ (by \ zigzag \ inequalities \ (1.1))$$

$$= s_1 x v_1 t_1 z \ (by \ Lemma \ 3.4 \ as \ t_1 \in S \backslash U).$$

Next by applying the Lemma 3.4 and zigzag inequalities (1.1) it can be easily verified that for all $l = 1, 2, \ldots, m' - 1$, we have

$$s_l x v_{2l-1} t_l z \leq s_{l+1} x v_{2l+1} t_{l+1} z.$$

Thus we have following

$$xyz \leq s_{m'} x v_{2m'-1} t_{m'} z$$

$$\leq s_{m'} x v_{2m'} z \ (by \ zigzag \ inequalities \ (1.1))$$

$$= s_{m'} v_{2m'} x z \ (by \ Lemma \ 3.4 \ as \ s_{m'} \in S \backslash U \ and \ i_3 \neq 3)$$

$$= yxz \ \big(by \ zigzag \ inequalites \ (1.1)\big).$$

On similar lines, we can show that $xyz \geq yxz$ and therefore $xyz = yxz$ as required.

Theorem 3.9. A permutative posemigroup satisfying the permutation identity (2.2) is saturated if and only if it admits the not-trivial identity I such that I is not a permutation identity and one side of I has no repeated variable.

Proof. Assume on contrary, then there exist a permutative posemigroup U satisfying the permutation identity (2.2) and the identity I and a posemigroup S containing U properly as a subposemigroup such that $\text{Do}m_{\leq}(U, S) = S$. By Theorem 2.4, S is also permutative posemigroup. Now to complete the proof of the theorem take any $d \in S \backslash U$ and let (1.1) be the zigzag inequalities for d of minimal length (m, m'). Note that for the non-trivial permutation i of the set $\{1, 2, 3\}$ either $i_1 \neq 1$ or $i_3 \neq 3$. We only consider the case $i_1 \neq 1$, the other case follows dually. Now,

$$d = s_1 v_1 t_1 \ (by \ zigzag \ equations \ (1.3))$$

$$= s_1 v_1^q t_1 \ (by \ Lemma \ 3.6)$$

$$= s_1 v_1^{q-1} v_1 t_1$$

$$= v_1^{q-1} s_1 v_1 t_1 (by \ Lemma \ 3.5).$$

Next for $j = 1, 2, \ldots, m' - 1$, we have

$$\prod_{k=1}^{l} v_{2k-1}^{q-1} s_l v_{2l-1} t_l = \prod_{k=1}^{l} v_{2k-1}^{q-1} s_{l+1} v_{2l+1} t_{l+1} \ (by \ zigzag \ equations \ (1.3))$$

$$= \prod_{k=1}^{l} v_{2k-1}^{q-1} s_{l+1} v_{2l+1}^{q} t_{l+1} \ (by \ Lemma \ 3.6)$$

$$= \prod_{k=1}^{l} v_{2k-1}^{q-1} v_{2l+1}^{q-1} s_{l+1} v_{2l+1} t_{l+1} \ (by \ Lemma \ 3.5)$$

$$= \prod_{k=1}^{l+1} v_{2k-1}^{q-1} s_{l+1} v_{2l+1} t_{l+1}$$

Therefore we have the following

$$d = \prod_{k=1}^{m'} v_{2k-1}^{q-1} s_{m'} v_{2m'-1} t_{m'}$$

$$= \prod_{k=1}^{m'} v_{2k-1}^{q-1} s_{m'} v_{2m'} \ (by \ zigzag \ equation \ (1.3))$$

$$= \prod_{k=1}^{m'} v_{2k-1}^{q-2} \prod_{k=1}^{m'} v_{2k-1} s_{m'} v_{2m'} \ (by \ Lemma \ 3.3 \ as \ i_1 \neq 1 \ and \ s_{m'} \in S \backslash U)$$

$$= \prod_{k=1}^{m'} v_{2k-1}^{q-2} \prod_{k=1}^{m'-1} v_{2k-1} \ v_{2m'-1} s_{m'} v_{2m'}$$

$$= \prod_{k=1}^{m'} v_{2k-1}^{q-2} \prod_{k=1}^{m'-1} v_{2k-1} s_{m'} v_{2m'-1} v_{2m'} (by \ Lemma \ 3.8 \ as \ i_1 \neq 1 \ and \ s_{m'} \in S \backslash U)$$

$$\geq \prod_{k=1}^{m'} v_{2k-1}^{q-2} \prod_{k=1}^{m'-1} v_{2k-1} s_{m'-1} v_{2m'-2} v_{2m'} (by \ zigzag \ inequalities \ (1.1)).$$

Continuing in this way we get,

$$d \geq \prod_{k=1}^{m'} v_{2k-1}^{q-2} v_1 s_1 \prod_{k=1}^{m'} v_{2k}$$

$$= \prod_{k=1}^{m'} v_{2k-1}^{q-2} s_1 v_1 \prod_{k=1}^{m'} v_{2k} \ (by \ Lemma \ 3.8 \ as \ i_1 \neq 1 \ and \ s_1 \in S \backslash U)$$

$$\geq \prod_{k=1}^{m'} v_{2k-1}^{q-2} \prod_{k=0}^{m'} v_{2k} \ (by \ zigzag \ inequalities \ (1.1)).$$

By a similar token, we obtain , $d \leq \prod_{k=1}^{m'} v_{2k-1}^{q-2} \prod_{k=0}^{m'} v_{2k}$ and therefore $d = \prod_{k=1}^{m'} v_{2k-1}^{q-2} \prod_{k=0}^{m'} v_{2k} \in U$, a contradiction as required.

13231

The posemigroup V (as every semigroup is a posemigroup with trivial order) or [3] is commutative, so is permutative satisfying the permutation identity (2.2) and also satisfies the identity of the type whose both sides have repeated variables. But Higgins in [3], Lemma 3 proved that V is not saturated. Therefore if U is a permutative posemigroup satisfying the permutation identity (2.2) is saturated then U must satisfies the identity I. This proves the theorem completely. □

Theorem 3.10. Let U be a permutative posemigroup satisfying the permutation identity (2.2) and admits the not-trivial identity I such that I is not a permutation identity and one side of I has no repeated variable then every epimorphism from U is surjective.

Proof: Sine every identity is preserved under every morphisms. Therefore the result follows from the Theorem 3.9. □

4. Epimorphically Closed Varieties Of Permutative Posemigroup

In [1], Theorem 2.5 Ahanger and Shah have shown that any identity of commutative posemigroups is preserved under epis. We extend this result to a permutative posemigroup satisfying the permutation identity (2.2).

In Lemma 4.1 and Lemma 4.2, U is a permutative posemigroup satisfying the permutation identity (2.2) and S is any posemigroup containing U properly as a subposemigroup such that $\text{Dom}_\leq(U,S) = S$. Moreover by Theorem 2.4, S is a permutative posemigroup satisfying the permutation identity (2.2).

Lemma 4.1. *If $i_1 = 1$ [$i_3 = 3$] and w is any word in variables $y_1, y_2, \ldots, y_p$*

Then

$$w(t_1 s_1, t_2 s_2, \ldots, t_p s_p) = w(t_1, t_2, \ldots, t_p) w(s_1, s_2, \ldots, s_p)$$

for all $t_j, s_j \in S^1$, $j = 1,2, \ldots, p$.

Proof: Since $i_1 = 1$, therefore $s_1 s_2 s_3 = s_1 s_3 s_2$ for all $s_1, s_2, s_3 \in S$. Therefore $w(t_1 u_1, t_2 u_2, \ldots, t_p u_p) = w(t_1, t_2, \ldots, t_p) w(u_1, u_2, \ldots, u_p)$ follows trivially. The case $i_3 = 3$ follows on similar lines. □

Now let $S^{(p)}$ denotes the Cartesian product of p copies of S. Then one can easily verify that $S^{(p)}$ is a posemigroup under componentwise binary operation and order. Also $U^{(p)}$ is a subposemigroup of $S^{(p)}$. We first show that $\text{Dom}_\leq(U^{(p)}, (S^1)^{(p)}) = S^{(p)}$. Take any $(d_1, d_2, \ldots, d_p) \in S^{(p)}$.

As in ([1], Lemma 2.4) each $d_k, 1 \leq k \leq p$, have the following zigzag inequalities in S^1 over U of some common length (m, m'):

$$d_k = x_1^{(k)} u_0^k , \qquad u_0^{(k)} \leq u_1^{(k)} y_1^{(k)}$$

$$x_i^{(k)} u_{2i-1}^{(k)} \leq x_{i+1}^{(k)} u_{2i}^{(k)} , \qquad u_{2i}^{(k)} y_i^{(k)} \leq u_{2i+1}^{(k)} y_{i+1}^{(k)} \ (i = 1,2, \ldots, m-1)$$

$$x_m^{(k)} u_{2m-1}^{(k)} \leq u_{2m}^{(k)} , \qquad u_{2m}^{(k)} y_m^{(k)} = d_k , and$$

(4.1) $\qquad v_0^{(k)} \leq s_1^{(k)} v_1^{(k)} , \qquad d_k = v_0^{(k)} t_1^{(k)}$

13232

$$s_j^{(k)}v_{2j}^{(k)} \le s_{j+1}^{(k)}v_{2j+1}^{(k)}, \qquad v_{2j-1}^{(k)}t_j^{(k)} \le v_{2j}^{(k)}t_{j+1}^{(k)} \; (j = 1,2,\dots,m'-1)$$

$$s_{m'}^{(k)}v_{2m'}^{(k)} = d_k, \qquad v_{2m'-1}^{(k)}t_{m'}^{(k)} \le v_{2m'}^{(k)}.$$

where $u_i^{(k)}, v_j^{(k)} \in U$ ($i = 1,2,\dots 2m, \; j = 1,2,\dots,2m'$) *and* $x_i^{(k)}, y_i^{(k)}, s_j^{(k)}, t_j^{(k)} \in$
S ($i = 1,2,\dots,m, \; j = 1,2,\dots,m'$) and by Theorem 1.2, for any $d_j \in S\backslash$
$U, x_i^{(j)}, y_i^{(j)}, s_k^{(j)}, t_k^{(j)} \in S\backslash U$.

Now we use the following notation:

$$\bar{x} = (x_1, x_2, \dots, x_k)$$

In this notation, we have,

$$\bar{d} = (d_1, d_2, \dots, d_k)$$

$$\bar{u}_p = \left(u_p^{(1)}, u_p^{(2)}, \dots, u_p^{(k)}\right) (p = 0,1,\dots,2m),$$

$$\bar{x}_i = \left(x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(k)}\right) (i = 1,2,\dots,m),$$

$$\bar{y}_i = \left(y_i^{(1)}, y_i^{(2)}, \dots, y_i^{(k)}\right) (i = 1,2,\dots,m),$$

$$\bar{v}_q = \left(v_q^{(1)}, v_q^{(2)}, \dots, v_q^{(k)}\right) (q = 0,1,\dots,2m'),$$

$$\bar{s}_j = \left(s_j^{(1)}, s_j^{(2)}, \dots, s_j^{(k)}\right) (j = 1,2,\dots,m'),$$

$$\bar{t}_j = \left(t_j^{(1)}, t_j^{(2)}, \dots, t_j^{(k)}\right) (j = 1,2,\dots,m').$$

Lemma 4.2. Every element of $S^{(k)}$ is dominated by $U^{(k)}$.

Proof. Let $(d_1, d_2, \dots, d_k) \in S^{(k)}$ and (4.1) be the zigzag inequalities of $d_r, 1 \le r \le k$.
Now one can easily verify the following.

$$\bar{d} = \bar{x}_1\bar{u}_0, \qquad \bar{u}_0 \le \bar{u}_1\bar{y}_1$$

$$\bar{x}_i\bar{u}_{2i-1} \le \bar{x}_{i+1}\bar{u}_{2i}, \qquad u_{2i}y_i \le \bar{u}_{2i+1}\bar{y}_{i+1}, 1 \le i \le m-1$$

$$\bar{x}_m\bar{u}_{2m-1} \le \bar{u}_{2m}, \qquad \bar{u}_{2m}\bar{y}_m \le \bar{d}$$

$$(4.2) \qquad \bar{v}_0 \le \bar{s}_1\bar{v}_1, \qquad \bar{d} = \bar{v}_0\bar{t}_1$$

$$\bar{s}_j\bar{v}_{2j} \le \bar{s}_{j+1}\bar{v}_{2j+1}, \qquad \bar{v}_{2j-1}\bar{t}_{j-1} \le \bar{v}_{2j}\bar{t}_j, 1 \le j \le m'-1$$

$$\bar{s}_{m'}\bar{v}_{2m'} \le \bar{d}, \qquad \bar{v}_{2m'-1}\bar{t}_{m'} \le \bar{v}_{2m'}.$$

Therefore by Theorem 1.1, $d \in Dom_{\le}(U^{(k)}, S^{(k)})$, which proves the Lemma. $\square$

Theorem 4.3. Let U be a permutative posemigroup satisfying the permutation identity (2.2).
Then all the identities of U are preserved under posemigroupepimorphisms.

Proof. Let S be a posemigroup with $Dom_{\le}(U, S) = S$ and U satisfying an identity

$$(4.3) \qquad\qquad u(\bar{x}) = v(\bar{x})$$

13233

By Theorem 2.4, S is also a permutative posemigroup satisfying the permutation identity (2.2). We show that S satisfies the identity (4.3), that is $,u(\bar{d}) = v(\bar{d})$, where $\bar{d} = (d_1, d_2, \ldots, d_k)$ and $d_r \in S, 1 \leq r \leq k$. If each $d_r \in U$, then result follows trivially. Assume that there exists $d_l, 1 \leq l \leq k$ such that $d_l \in S\backslash U$. Clearly either $i_1 \neq 1$ or $i_3 \neq 3$. If $i_3 \neq 3$ and $i_1 \neq 1$. By Lemma 3.6, both the words $u(\bar{d})$ and $v(\bar{d})$ begin with d_l. To prove the theorem completely we take the following two cases:

Case 1: $i_3 \neq 3$ and $i_1 \neq 1$. Then we have,

$$u(\bar{d}) = u(\bar{x}_1\bar{u}_0) \quad (by\ zigzag\ inequalities\ 4.2)$$

$$= u(\bar{x}_1)u(\bar{u}_0) \quad (as\ i_3 \neq 3\ and\ i_1 \neq 1)$$

$$= u(\bar{x}_1)v(\bar{u}_0) \quad (as\ U\ satisfies\ (4.3))$$

$$\leq u(\bar{x}_1)v(\bar{u}_1\bar{y}_1) \quad (by\ zigzag\ inequlities\ 4.2)$$

$$= u(\bar{x}_1)v(\bar{u}_1)v(\bar{y}_1) \quad (as\ i_3 \neq 3\ and\ i_1 \neq 1)$$

$$= u(\bar{x}_1)u(\bar{u}_1)v(\bar{y}_1) \quad (as\ U\ satisfies\ 4.3)$$

$$= u(\bar{x}_1\bar{u}_1)v(\bar{y}_1) \quad (as\ i_3 \neq 3\ and\ i_1 \neq 1).$$

Next for $i = 1, 2, \ldots, m-1$, it can be easily verified that

$$u(\bar{x}_i\bar{u}_{2i-1})v(\bar{y}_i) = u(\bar{x}_{i+1}\bar{u}_{2i+1})v(\bar{y}_{i+1}).$$

Now,

$$u(\bar{d}) = u(\bar{x}_m\bar{u}_{2m-1})v(\bar{y}_m)$$

$$\leq u(\bar{u}_{2m})v(\bar{y}_m) \quad (by\ zigzag\ inequalities\ 4.2)$$

$$= v(\bar{u}_{2m})v(\bar{y}_m) \quad (as\ U\ satisfies\ (4.3))$$

$$= v(\bar{u}_{2m}\bar{y}_m) \quad (as\ i_3 \neq 3\ and\ i_1 \neq 1)$$

$$= v(\bar{d}) (by\ zigzag\ inequalities\ 4.2).$$

By similar argument, we obtain, $v(\bar{d}) \leq u(\bar{d})$. Therefore $v(\bar{d}) = u(\bar{d})$ and hence the theorem follows in this case.

Case 2: If either $i_1 = 1$ or $i_3 = 3$. Then we have,

$$u(\bar{d}) = u(\bar{x}_1\bar{u}_0) \quad (by\ zigzag\ inequalities\ 4.2)$$

$$= u(\bar{x}_1)u(\bar{u}_0) \quad (By\ Lemma\ 4.1\ as\ i_1 = 1\ or\ i_3 = 3)$$

$$= u(\bar{x}_1)v(\bar{u}_0) \quad (as\ U\ satifies\ (4.3))$$

$$\leq u(\bar{x}_1)v(\bar{u}_1\bar{y}_1) \quad (by\ zigzag\ inequlities\ 4.2)$$

$$= u(\bar{x}_1)v(\bar{u}_1)v(\bar{y}_1) \quad (By\ Lemma\ 4.1\ as\ i_1 = 1\ or\ i_3 = 3)$$

$$= u(\bar{x}_1)u(\bar{u}_1)v(\bar{y}_1) \quad (as\ U\ satifies\ 4.3)$$

13234

$$= u(\bar{x}_1\bar{u}_1)v(\bar{y}_1)(By\ Lemma\ 4.1\ as\ i_1 = 1\ or\ i_3 = 3\ ).$$

Next for $i=1,2,\ldots,m$ -1,we have,

$$u(\bar{x}_i\bar{u}_{2i-1})v(\bar{y}_i) = u(\bar{x}_{i+1}\bar{u}_{2i+1})v(\bar{y}_{i+1}).$$

Therefore we have

$$u(\bar{d}) = u(\bar{x}_m\bar{u}_{2m-1})v(\bar{y}_m)$$

$$\leq u(\bar{u}_{2m})v(\bar{y}_m)\ \ (by\ zigzag\ inequalities\ 4.2)$$

$$= v(\bar{u}_{2m})v(\bar{y}_m)\ \ (as\ U\ satisfies\ (4.3))$$

$$= v(\bar{u}_{2m}\bar{y}_m)\ \ (By\ Lemma\ 4.1\ as\ i_1 = 1\ or\ i_3 = 3\ )$$

$$\leq\ v(\bar{d})(by\ zigzag\ inequalities\ 4.2).$$

On similar lines, we have $v(\bar{d}) \leq u(\bar{d}).\ Therefore\ u(\bar{d}) = v(\bar{d})$. Hence the theorem is proved completely.

References

[1] S. A. Ahanger, A. H. Shah, Epimorphisms, Dominions and Varieties of Commutative Posemigroups, Asian-European Journal of Mathematics 14 (4) (D.O.I: 10.1142/S1793557121500480) (14 pages) (2021).

[2] L. S. Bloom, Variety of ordered algebras, J. Comput. System Sci. 13, 200-212 (1976).

[3] P. M. Higgin, Saturated and Epimorphically Closed Varieties of Semigroups, J. Austral. Math. soc.(Series A)36,153-175 (1984).

[4] J. M. Howie, Fundamentals of Semigroup Theory, Clarendon Press, Oxford (1995).

[5] N. M. Khan, On Saturated Varieties and Consequences of Permutative Identities, J. Austral. Math. soc. (Series A)38, 186-197 (1985).

[6] S. Nasir and L. Tart, Dominions, Zizags and epimorphisms for partially ordered semigroup, ActaEtCommentationesUniversitatisTartuensis De Mathematica 18(1) 81-91 (2014).