On the Relative Strength of Double Fourier Series and its Allied Series

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Abstract:

In this study, we have established four theorems concerning the generalizability of postponed Borel sums of double Fourier series. Our results are special examples of results achieved by others, such as the generalized Borel summability of double Fourie series and its linked series.

Keywords: double Fourier series, allied series, summability method.

Let us suppose that $\zeta(\alpha,\ \beta)$ be a periodic function with a period 2π, which is integrable in the Lebesgue sense over the square $R(-\pi,\ \pi;-\pi,\ \pi)$. The Fourier series of $\zeta(\alpha,\ \beta)$ is

$$\sum_{u=0}^{\infty}\ \sum_{v=0}^{\infty}\ \xi_{u,\,v}\Big[p_{u,\,v}\cos u\alpha\ \cos v\beta+q_{u,\,v}\sin u\alpha\ \cos v\beta$$

$$+r_{u,\,v}\cos u\alpha\ \sin v\beta+s_{u,\,v}\sin u\alpha\sin v\beta\Big] \qquad \ldots \qquad (1)$$

where,

$$\xi_{u,\,v}=\begin{cases}\dfrac{1}{4} & for\ u=0,\ v=0\\ \dfrac{1}{2} & for\ u=0,\ v>0;\ u>0,\ v=0\\ 1 & for\ \ u>0,\ v>0\end{cases}$$

also

$$p_{u.\,v}=\frac{1}{\pi^2}\iint_R \zeta(u,\ v)\cos u\alpha\ \cos\beta\ d\alpha\, d\beta \qquad \ldots \qquad (2)$$

$$q_{u,\,v}=\frac{1}{\pi^2}\iint_R \zeta(u,\ v)\ \sin u\alpha\ \cos v\beta\ d\alpha\, d\beta \qquad \ldots \qquad (3)$$

$$r_{u,v} = \frac{1}{\pi^2} \iint\limits_R \zeta(u, v) \cos u\alpha \, \sin v\beta \, d\alpha \, d\beta \qquad \ldots \qquad (4)$$

$$s_{u,v} = \frac{1}{\pi^2} \iint\limits_R \zeta(u, v) \sin u\alpha \, \sin v\beta \, d\alpha \, d\beta \qquad \ldots \qquad (5)$$

The series (1) can also be written as

$$\zeta(\propto, \beta) \sim \sum_{1}^{\infty} \sum_{1}^{\infty} (p, q, r, s; \alpha, \beta)_{u,v} \qquad \ldots \qquad (6)$$

The allied series of (6) can be obtained by differentiating first with respect to α, then with respect to β, and then with respect to α and β simultaneously and omitting the constant.

Thus, we get allied series

$$\sum_{1}^{\infty} \sum_{1}^{\infty} (s, -r, -q, p; \alpha, \beta)_{u,v} \qquad \ldots \qquad (7)$$

$$\sum_{1}^{\infty} \sum_{1}^{\infty} (r, s, -p, -q; \alpha, \beta)_{u,v} \qquad \ldots \qquad (8)$$

$$\sum_{1}^{\infty} \sum_{1}^{\infty} (q, -p, s, -r; \alpha, \beta)_{u,v} \qquad \ldots \qquad (9)$$

From here on out, the first, second, and third allied series will refer to these.

Let $\overline{\zeta_1}(\alpha, \beta)$, $\overline{\zeta_2}(\alpha, \beta)$ and $\overline{\zeta_3}(\alpha, \beta)$ be three conjugate functions of $\zeta(\alpha, \beta)$, associated with the first, second and third allied series respectively,

where,

$$\overline{\zeta_1}(\alpha, \beta) = \frac{1}{2\pi} \int_0^{\pi}\int_0^{\pi} \chi_1(\lambda, \varepsilon) \cot\frac{\lambda}{2} \cot\frac{\varepsilon}{2} \, d\lambda \, d\varepsilon \qquad \ldots \qquad (10)$$

$$\overline{\zeta_2}(\alpha, \beta) = \frac{1}{2\pi} \int_0^{\pi}\int_0^{\pi} \chi_2(\lambda, \varepsilon) \cot\frac{\lambda}{2} \, d\lambda \, d\varepsilon \qquad \ldots \qquad (11)$$

$$\overline{\zeta_3}(\alpha, \beta) = \frac{1}{2\pi} \int_0^{\pi}\int_0^{\pi} \chi_3(\lambda, \varepsilon) \cot\frac{\varepsilon}{2} \, d\lambda \, d\varepsilon \qquad \ldots \qquad (12)$$

$$\chi_1(\lambda, \varepsilon) = \frac{1}{4}\left[\zeta(\alpha+\lambda, \beta+\varepsilon) - \zeta(\alpha+\lambda, \beta-\varepsilon)\right.$$

$$-\zeta(\alpha-\lambda,\ \beta+\varepsilon)+\zeta(\alpha-\lambda,\ \beta-\varepsilon)\Big] \quad \ldots$$
(13)

$$\chi_2(\lambda,\ \varepsilon)=\frac{1}{4}\Big[\zeta(\alpha+\lambda,\ \beta+\varepsilon)+\zeta(\alpha+\lambda,\ \beta-\varepsilon)$$

$$-\zeta(\alpha-\lambda,\ \beta+\varepsilon)-\zeta(\alpha-\lambda,\ \beta-\varepsilon)\Big] \quad \ldots$$
(14)

$$\chi_3(\lambda,\ \varepsilon)=\frac{1}{4}\Big[\zeta(\alpha+\lambda,\ \beta+\varepsilon)-\zeta(\alpha+\lambda,\ \beta-\varepsilon)$$

$$+\zeta(\alpha-\lambda,\ \beta+\varepsilon)-\zeta(\alpha-\lambda,\ \beta-\varepsilon)\Big] \quad \ldots$$
(15)

Also, we consider *a* function $\varphi(\lambda,\ \varepsilon)$,

$$\varphi(\lambda,\ \varepsilon)=\frac{1}{4}\Big[\zeta(\alpha+\lambda,\ \beta+\varepsilon)+\zeta(\alpha+\lambda,\ \beta-\varepsilon)$$

$$+\zeta(\alpha-\lambda,\ \beta+\varepsilon)+\zeta(\alpha-\lambda,\ \beta-\varepsilon)-4s\Big] \quad \ldots$$
(16)

1. Definition:

Let $t_{u,\,v}=\sum_{j=0}^{u}\ \sum_{k=0}^{v}\ g_{jk}$ …
(17)

An infinite series $\sum_{u=0}^{\infty}\ \sum_{v=0}^{\infty}\ g_{u,v}$ with the sequence of partial sums $\{t_{u,\,v}\}$

is said to be summable by Borel’s exponential method (or summable B)

if $\theta_{a,\,b}=\lim_{a\to\infty}\ \lim_{b\to\infty}\ e^{-(a+b)}\ \sum_{u=0}^{\infty}\ \sum_{v=0}^{\infty}\ \frac{a^u}{u\,!}\ \frac{b^v}{v\,!}\ t_{u,\,v}$ …
(18)

has *a* finite value *t*.

With this we may define Borel summability for double series, as well as single ones [1].

Takahashi and Wang [2] and Sahney [3] both talk about how Borel summability can be used with Fourier series. Both the resulting Fourier series and its conjugate series were shown to be Borel summable by the work of Sahney [4] and Sinvhal [5]. In the special case of conjugate series, Kathl [6] found a distinct set of requirements.

In this study, we set out to show the following theorems about double Fourier series and related series.

Theorem 1: If as $\lambda \to +0,\ \varepsilon \to +0$,

$$\psi(\lambda, \varepsilon) = \int_0^\lambda da \int_0^\varepsilon |\varphi(a, b)| db = o\left(\frac{\lambda}{\log \lambda^{-1}}\ \frac{\varepsilon}{\log \varepsilon^{-1}}\right) \quad \ldots (19)$$

$$\int_0^\pi db \left| \int_0^\lambda \varphi(a, b) da \right| = o\left(\frac{\lambda}{\log \lambda^{-1}}\right) \quad \ldots (20)$$

$$\int_0^\pi da \left| \int_0^\varepsilon \varphi(a, b) db \right| = o\left(\frac{\varepsilon}{\log \varepsilon^{-1}}\right) \quad \ldots (21)$$

then the double Fourier series of function $\zeta(\lambda, \varepsilon)$ is summable by Borel exponential mean to the sum *t* at $\lambda = \alpha,\ \varepsilon = \beta$.

Theorem 2: If, as $\lambda \to +0,\ \varepsilon \to +0$,

$$F(\lambda, \varepsilon) = \int_0^\lambda da \int_0^\varepsilon |\chi_1(a, b)| db = o\left(\frac{\lambda}{\log \lambda^{-1}}\ \frac{\varepsilon}{\log \varepsilon^{-1}}\right) \quad \ldots (22)$$

$$\int_0^\pi db \left| \int_0^\lambda \chi_1(a, b) da \right| = o\left(\frac{\lambda}{\log \lambda^{-1}}\right) \quad \ldots (23)$$

$$\int_0^\pi da \left| \int_0^\varepsilon \chi_1(a, b) db \right| = o\left(\frac{\varepsilon}{\log \varepsilon^{-1}}\right) \quad \ldots (24)$$

If the conjugate integral has a value of B, then the linked series (7) can be summed. $\overline{\zeta_1}(\alpha, \beta)$, given by (10), provided the integral exists.

Theorem 3: If, as $\lambda \to +0,\ \varepsilon \to +0$,

$$F(\lambda, \varepsilon) = \int_0^\lambda da \int_0^\varepsilon |\chi_1(a, b)| db = o\left(\frac{\lambda}{\log \lambda^{-1}}\ \frac{\varepsilon}{\log \varepsilon^{-1}}\right) \quad \ldots (25)$$

$$\int_0^{\pi} db \left| \int_0^{\lambda} \chi_2(a, b)\, da \right| = o\left(\frac{\lambda}{\log \lambda^{-1}} \right) \qquad \ldots (26)$$

$$\int_0^{\pi} da \left| \int_0^{\varepsilon} \chi_2(a, b)\, db \right| = o\left(\frac{\varepsilon}{\log \varepsilon^{-1}} \right) \qquad \ldots (27)$$

If this is the case, then the associated series (8) can be summed up to the value of the conjugate integral. $\overline{\zeta_2}(\alpha, \beta)$, given by (11), provided the integral exists.

Theorem 4: If, as $\lambda \to +0, \ \varepsilon \to +0$,

$$F(\lambda, \varepsilon) = \int_0^{\lambda} da \int_0^{\varepsilon} |\chi_3(a, b)|\, db = o\left(\frac{\lambda}{\log \lambda^{-1}} \ \frac{\varepsilon}{\log \varepsilon^{-1}} \right) \qquad \ldots (28)$$

$$\int_0^{\pi} db \left| \int_0^{\lambda} \chi_2(a, b)\, da \right| = o\left(\frac{\lambda}{\log \lambda^{-1}} \right) \qquad \ldots (29)$$

$$\int_0^{\pi} da \left| \int_0^{\varepsilon} \chi_3(a, b)\, db \right| = o\left(\frac{\varepsilon}{\log \varepsilon^{-1}} \right) \qquad \ldots (30)$$

If the conjugate integral has a value of B, then the linked series (9) can be summed. $\overline{\zeta_3}(\alpha, \beta)$, given by (12), provided the integral exists.

These theorems extend the theorems of Sahney [4], for double Fourier series. Sahney [3] also proved a theorem which is similar to theorem 1, which is weaker than our result.

Proof of theorem 1: The $(u, v)^{th}$ partial sum of double Fourier series of the function $\zeta(\lambda, \varepsilon)$ is

$$t_{u,v}(\lambda, \varepsilon) = \frac{1}{\pi^2} \int_0^{\pi}\int_0^{\pi} \varphi(\lambda, \varepsilon) \frac{\sin u\lambda}{\lambda} \frac{\sin v\varepsilon}{\varepsilon} d\lambda\, d\varepsilon$$

using the result of Hardy [1],

we have

$$\pi^2 \theta_{a,b} = e^{-(a+b)} \int_0^{\pi}\int_0^{\pi} \frac{\varphi(\lambda, \varepsilon)}{\lambda \varepsilon} \sum_{u,v=0}^{\infty} \left(\sin u\lambda \frac{a^u}{u!} \sin v\varepsilon \frac{b^v}{v\,!} \right) d\lambda\, d\varepsilon$$

$$= \left(\int_0^{\frac{1}{a^m}} \int_0^{\frac{1}{b^n}} + \int_0^{\frac{1}{a^m}} \int_{\frac{1}{b^n}}^{\pi} + \int_{\frac{1}{a^m}}^{\pi} \int_0^{\frac{1}{b^n}} + \int_{a^m}^{\pi} \int_{\frac{1}{b^n}}^{\pi} \right) \frac{\varphi(\lambda, \varepsilon)}{\lambda \varepsilon} \frac{\sin(a \sin \lambda)}{e^{a(1-\cos \lambda)}} \frac{\sin(b \sin \varepsilon)}{e^{b(1-\cos \varepsilon)}} \, d\lambda \, d\varepsilon,$$

where

$$0 < m < \frac{1}{2},$$
$$0 < n < \frac{1}{2}$$

$$= I_1 + I_2 + I_3 + I_4 \quad \text{say} \qquad \dots (31)$$

Now, applying second mean value theorem for double integral,

$$I_4 = \frac{a^m \ b^n}{e^{\left[a \, 2\sin^2 \frac{1}{2\, a^m} + b \, 2\sin^2 \frac{1}{2b^n} \right]}} \int_{\frac{1}{a^m}}^{\tau} \int_{\frac{1}{b^n}}^{\delta} \left\{ \varphi(\lambda, \varepsilon) \sin(a \sin \lambda) \sin(b \sin \varepsilon) \right\} d\lambda \, d\varepsilon$$

where $\frac{1}{a^m} < \tau < \pi$,

$$\frac{1}{b^n} < \delta < \pi .$$

Thus,

$$|I_4| = \frac{a^m \ b^n}{e^{a^{1-2m} + b^{1-2n}}} \int_{\frac{1}{a^m}}^{\tau} \int_{\frac{1}{b^n}}^{\delta} |\varphi(\lambda, \varepsilon)| \, d\lambda \, d\varepsilon$$

$= O(1)$, as $a, b \to \infty$,

Also,

$$I_3 = \left(\int_{\frac{1}{a^m}}^{\pi} \int_0^{\frac{1}{b}} + \int_{\frac{1}{a^m}}^{\pi} \int_{\frac{1}{b}}^{\frac{1}{b^n}} \right) \frac{\varphi(\lambda, \varepsilon)}{\lambda \varepsilon} \frac{\sin(a \sin \lambda)}{e^{a(1-\cos \lambda)}} \frac{\sin(b \sin \varepsilon)}{e^{b(1-\cos \varepsilon)}} \, d\lambda \, d\varepsilon$$

$$= I_{3,1} + I_{3,2}, \quad \text{say} \qquad \dots (32)$$

By second mean value theorem, we have

$$I_{3,1} = \frac{a^m}{e^{a.2\sin^2\frac{1}{2a^m}}} \int_{\frac{1}{a^m}}^{\tau} \int_{0}^{\frac{1}{a}} \frac{\varphi(\lambda,\, \varepsilon)}{\varepsilon} \sin(a\sin\lambda)\, \frac{\sin(b\sin\varepsilon)}{e^{b(1-\cos\varepsilon)}}\, d\lambda\, d\varepsilon$$

$$\left|I_{3,1}\right| = \frac{a^m}{e^{a^{1-2m}}} \int_{\frac{1}{a^m}}^{\tau} \int_{0}^{\frac{1}{a}} \frac{\left|\varphi(\lambda,\, \varepsilon)\right|}{\varepsilon}\; o(b\varepsilon)\; d\lambda\, d\varepsilon$$

$$= \frac{ba^m}{e^{a^{1-2m}}}\, o\left(\frac{\frac{1}{b}}{\log b}\right), \qquad \text{(using (20) and (21))}$$

$$= O(1) \qquad \ldots \qquad (33)$$

and

$$\left|I_{3,2}\right| = \frac{a^m}{e^{a\,2\sin^2\frac{1}{2a^m}}}\; \frac{1}{e^{b\,2\sin^2\frac{1}{2b}}} \int_{\frac{1}{a^m}}^{\tau} \int_{\frac{1}{b}}^{\frac{1}{b^{n^1}}} \frac{\left|\varphi(\lambda,\, \varepsilon)\right|}{\varepsilon}\; d\lambda\, d\varepsilon\,,$$

where $0 < n < n' < \frac{1}{2}$

By partial integration by parts, we get

$$\left|I_{3,2}\right| = O(1)\left[\int_{\frac{1}{a^m}}^{\tau} \left\{\psi\left(\lambda,\, b^{n'}\right)b^{n'} + \int_{\frac{1}{b}}^{\frac{1}{b^{n'}}} \psi(\lambda,\, \varepsilon)\, \frac{1}{\varepsilon^2}\, d\varepsilon\right\} d\lambda\right]$$

where $\varphi(\lambda,\, \varepsilon) = \int_{0}^{\varepsilon} \left|\varphi(\lambda,\, b)\right| db\,,$

provided the integral exists and is ∞ otherwise.

Therefore, $\left|I_{3,2}\right| = O\left(\frac{1}{\log b}\right) + O(\log n')$, (by (20) and (21))

$$= O(1) \qquad \ldots \qquad (34)$$

Thus, $\left|I_3\right| = O(1) \qquad \ldots \qquad (35)$

Similarly, $\left|I_2\right| = O(1)$ … (36)

Also,

$$I_1 = \left[\int_0^{\frac{1}{a}}\int_0^{\frac{1}{b}} + \int_{\frac{1}{a}}^{\frac{1}{a^m}}\int_0^{\frac{1}{b}} + \int_0^{\frac{1}{a}}\int_{\frac{1}{b}}^{\frac{1}{b^n}} + \int_{\frac{1}{a}}^{\frac{1}{a^m}}\int_{\frac{1}{b}}^{\frac{1}{b^n}}\right] \frac{\varphi(\lambda, \varepsilon)}{\lambda\varepsilon}\, \frac{\sin(a\sin\lambda)}{e^{a(1-\cos\lambda)}}\, \frac{\sin(b\sin\varepsilon)}{e^{b(1-\cos\varepsilon)}}\, d\lambda\, d\varepsilon$$

$= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}$, say … (37)

For $I_{1,1}$,

$$\left|I_{1,1}\right| = \int_0^{\frac{1}{a}}\int_0^{\frac{1}{b}} \frac{\left|\varphi(\lambda, \varepsilon)\right|}{\lambda\varepsilon}\, O(a\lambda)\, O(b\varepsilon)\, d\lambda\, d\varepsilon$$

$= O\left(\frac{1}{\log a \, \log b}\right)$, (using (19))

$= O(1)$ … (38)

For $I_{1,2}$,

$$\left|I_{1,2}\right| = \frac{1}{e^{a^{1-2m}}} \int_{\frac{1}{a}}^{\frac{1}{a^m}}\int_0^{\frac{1}{b}} \frac{\left|\varphi(\lambda, \varepsilon)\right|}{\lambda\varepsilon}\, O(b\varepsilon)\, d\lambda\, d\varepsilon$$

$= O(1)$, as in $I_{3,2}$ … (39)

Similarly, $\left|I_{1,3}\right| = O(1)$ … (40)

For $I_{1,4}$,

using second mean value theorem,

we have

$$I_{1,4}=\frac{1}{e^{a\,2\sin^2\frac{1}{2a}+b\,2\sin^2\frac{1}{2b}}}\int_{\frac{1}{a}}^{\frac{1}{a^{m'}}}\int_{\frac{1}{b}}^{\frac{1}{b^{n'}}}\left\{\frac{\varphi(\lambda,\varepsilon)}{\lambda\varepsilon}\sin(a\sin\lambda)\sin(b\sin\varepsilon)\right\}d\lambda\,d\varepsilon$$

Where $0<m<m'<\frac{1}{2},\ 0<n<n'<\frac{1}{2}.$

$$|I_{1,4}|=O(1)\int_{\frac{1}{a}}^{\frac{1}{a^{m'}}}\int_{\frac{1}{b}}^{\frac{1}{b^{n'}}}\frac{|\varphi(\lambda,\varepsilon)|}{\lambda\varepsilon}d\lambda\,d\varepsilon$$

$$=\psi\left(\frac{1}{a^{m'}},\frac{1}{b^{n'}}\right)+\int_{\frac{1}{a}}^{\frac{1}{a^{m'}}}\psi\left(\lambda,\frac{1}{b^{n'}}\right)\frac{1}{\lambda^2}b^{n'}\,d\lambda+\int_{\frac{1}{b}}^{\frac{1}{b^{n'}}}\psi\left(\frac{1}{a^{m'}},\varepsilon\right)\frac{1}{\in^2}a^{m'}\,d\varepsilon$$

$$+\int_{\frac{1}{a}}^{\frac{1}{a^{m'}}}\int_{\frac{1}{b}}^{\frac{1}{b^{n'}}}\frac{\psi(\lambda,\varepsilon)}{\lambda^2\,\varepsilon^2}d\lambda\,d\varepsilon$$

$$=O(1)+O\left(\int_{\frac{1}{a}}^{\frac{1}{a^{m'}}}\frac{1}{\lambda\log\lambda^{-1}}d\lambda\right)+O\left(\int_{\frac{1}{b}}^{\frac{1}{b^{n'}}}\frac{1}{\in\log\in^{-1}}d\varepsilon\right)+O\left(\int_{\frac{1}{a}}^{\frac{1}{a^{m'}}}\int_{\frac{1}{b}}^{\frac{1}{b^{n'}}}\frac{1}{\lambda\log\lambda^{-1}}\frac{1}{\in\log\varepsilon^{-1}}d\lambda\,d\varepsilon\right)$$

$$=O(1)+O(\log m')+O(\log n')+O(\log m'\log n')$$

$$=O(1).$$

The theorem has now been proven.Proof of theorem 2:

The $(u,v)^{th}$ partial sum of the allied series (7) is

$$\tilde{t}_{u,v}(\lambda,\varepsilon)=\frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}\chi_1(\lambda,\varepsilon)\frac{\cos\frac{\lambda}{2}-\cos\left(u+\frac{1}{2}\right)\lambda}{\sin\frac{\lambda}{2}}\frac{\cos\frac{\varepsilon}{2}-\cos\left(v+\frac{1}{2}\right)\varepsilon}{\sin\frac{1}{2}}d\lambda\,d\varepsilon$$

taking $\theta_{a,b}$ as Borel transform for allied series (7), and avoiding the detail by omitting some similar integrals, in our calculate, we have

$$4\pi^2\theta_{a,b}-\int_{\frac{\pi}{a^m}}^{\pi}\int_{\frac{\pi}{b^n}}^{\pi}\chi_1(\lambda,\varepsilon)\cot\frac{\lambda}{2}\cot\frac{\varepsilon}{2}d\lambda\,d\varepsilon$$

$$= \int_0^{\frac{\pi}{a^m}} \int_0^{\frac{\pi}{b^n}} \chi_1(\lambda, \varepsilon)\, e^{-(a+b)} \left(\sum_{u=0}^{\infty} \frac{\cos\frac{\lambda}{2} - \cos\left(u+\frac{1}{2}\right)\lambda}{\sin\frac{\lambda}{2}} \frac{a^u}{u!} \right)$$

$$\left(\sum_{v=0}^{\infty} \frac{\cos\frac{\varepsilon}{2} - \cos\left(v+\frac{1}{2}\right)\varepsilon}{\sin\frac{\varepsilon}{2}} \frac{b^v}{v!} \right) d\lambda\, d\varepsilon -$$

$$\int_{\frac{\pi}{a^m}}^{\pi} \int_0^{\frac{\pi}{b^n}} \chi_1(\lambda, \varepsilon)\, e^{-(a+b)} \left(\sum_{u=0}^{\infty} \frac{\cos\left(u+\frac{1}{2}\right)\lambda}{\sin\frac{\lambda}{2}} \frac{a^u}{u!} \right) \left(\sum_{v=0}^{\infty} \frac{\cos\frac{\varepsilon}{2} - \cos\left(v+\frac{1}{2}\right)\varepsilon}{\sin\frac{\varepsilon}{2}} \frac{b^v}{v!} \right) d\lambda\, d\varepsilon$$

$$-\int_0^{\frac{\pi}{a^m}} \int_{\frac{\pi}{b^n}}^{\pi} \chi_1(\lambda, \varepsilon)\, e^{-(a+b)} \left(\sum_{v=0}^{\infty} \frac{\cos\left(v+\frac{1}{2}\right)\varepsilon}{\sin\frac{\varepsilon}{2}} \frac{b^v}{v!} \right) \left(\sum_{u=0}^{\infty} \frac{\cos\frac{\lambda}{2} - \cos\left(u+\frac{1}{2}\right)\lambda}{\sin\frac{\lambda}{2}} \frac{a^u}{u!} \right) d\lambda\, d\varepsilon$$

$$+\int_{\frac{\pi}{a^m}}^{\pi} \int_{\frac{\pi}{b^n}}^{\pi} \chi_1(\lambda, \varepsilon)\, e^{-(a+b)} \left(\sum_{u=0}^{\infty} \frac{\cos\left(u+\frac{1}{2}\right)\lambda}{\sin\frac{\lambda}{2}} \frac{a^u}{u!} \right) \left(\sum_{v=0}^{\infty} \frac{\cos\left(v+\frac{1}{2}\right)\varepsilon}{\sin\frac{\varepsilon}{2}} \frac{b^v}{v!} \right) d\lambda\, d\varepsilon$$

$$= \int_0^{\frac{\pi}{a^m}} \int_0^{\frac{\pi}{b^n}} \chi_1(\lambda, \varepsilon) \cot\frac{\lambda}{2} \cot\frac{\varepsilon}{2} \left(1 - \frac{\cos\left(a\sin\lambda + \frac{\lambda}{2}\right)}{e^{a(1-\cos\lambda)}} \frac{1}{\cos\frac{\lambda}{2}} \right) \left(1 - \frac{\cos\left(b\sin\varepsilon + \frac{\varepsilon}{2}\right)}{e^{b(1-\cos\varepsilon)}} \frac{1}{\cos\frac{\varepsilon}{2}} \right) d\lambda\, d\varepsilon$$

$$-\int_{\frac{\pi}{a^m}}^{\pi} \int_0^{\frac{1}{a^n}} \frac{\chi_1(\lambda, \varepsilon)}{\sin\frac{\lambda}{2}} \cot\frac{\varepsilon}{2} \frac{\cos\left(a\sin\lambda + \frac{\lambda}{2}\right)}{e^{a(1-\cos\lambda)}} \left(1 - \frac{\cos\left(b\sin\in + \frac{\varepsilon}{2}\right)}{e^{b(1-\cos\varepsilon)}} \frac{1}{\cos\frac{\varepsilon}{2}} \right) d\lambda\, d\varepsilon -$$

$$\int_0^{\frac{\pi}{a^m}} \int_{\frac{\pi}{b^n}}^{\pi} \frac{\chi_1(\lambda, \varepsilon)}{\sin\frac{\varepsilon}{2}} \cot\frac{\in}{2} \frac{\cos\left(b\sin\varepsilon + \frac{\varepsilon}{2}\right)}{e^{b(1-\cos\varepsilon)}} \frac{1}{\cos\frac{\lambda}{2}} \Bigg) d\lambda\, d\varepsilon$$

$$+\int_{\frac{\pi}{a^m}}^{\pi}\int_{\frac{\pi}{b^n}}^{\pi}\frac{\chi_1(\lambda,\varepsilon)}{\sin\frac{\lambda}{2}\sin\frac{\varepsilon}{2}}\frac{\cos\left(a\sin\lambda+\frac{\lambda}{2}\right)}{e^{a(1-\cos\lambda)}}\frac{\cos\left(b\sin\varepsilon+\frac{\varepsilon}{2}\right)}{e^{b(1-\cos\varepsilon)}}\,d\lambda\,d\varepsilon,$$

where $0<a<\frac{1}{2},\ 0<b<\frac{1}{2}$

$= I_1+I_2+I_3+I_4,$ say … (41)

For

$$I_4=\frac{a^m\ b^n}{a\,2\sin^2\frac{1}{2a^m}+b\,2\sin^2\frac{1}{2b^n}}\times$$

$$\left\{\int_{\frac{\pi}{a^m}}^{\tau}\int_{\frac{\pi}{b^n}}^{\delta}\left\{\chi_1(\lambda,\varepsilon)\cos\left(a\sin\lambda+\frac{\lambda}{2}\right)\cos\left(b\sin\varepsilon+\frac{\varepsilon}{2}\right)\right\}d\lambda\,d\varepsilon\right\}+O(1)$$

where $\frac{\pi}{a^m}<\tau<\pi,\frac{\pi}{b^n}<\delta<\pi$

thus,

$$\left|I_n\right|=\frac{a^m\ b^n}{e^{\left(a^{1-2m}+b^{1-2n}\right)}}\int_{\frac{\pi}{a^m}}^{\tau}\int_{\frac{\pi}{b^n}}^{\delta}\left\{\chi_1(\lambda,\varepsilon)\right\}d\lambda\,d\varepsilon$$

$=O(1)$, as $a, b\to\infty$.

If we assume the existence of the conjugate integral (10) and follow the theorem's lines of reasoning in order, the three remaining integrals on the right-hand side disappear.

The theorem has now been proven.

Similar theorems (3) and (4) can also be proved for the allied series (11) and (12).

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