Some Families of Bilinear and Bilateral Generating Functions in Function Spaces Associated with Hypergeometric Polynomials

S. K. Sahani[1*], A. K. Thakur[1] and Yashwant Jaiswal [2]

[1*]Department of Mathematics, MIT Campus, (T.U.), Janakpurdham, Nepal

[1]Department of Mathematics, Dr. C.V. Raman University, Bilaspur (C.G.)

[2]Department of Mathematics, Research Scholar, Dr. C.V. Raman University, Bilaspur (C.G.)

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Abstract:

In this research paper we have obtained some specific theorems which one then applied to obtain a set of bilinear and bilateral generating functions in function spaces in terms of Laplace, beta and some other integrals. These theorems include as special cases many known results of the authors [1].

Keywords: Bilinear and Bilateral generating functions, Hypergeometric functions, Hypergeometric polynomials, Laplace transform, etc.

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Introduction:

The Appell's functions F_1 to F_4 (see [26]; p. 224) are defined in the following way:

$$F_1\left(\alpha;\beta,c;r;\ a,b\right)=$$

$$\sum_{f,g=0}^{\infty}\frac{(\alpha)_{f+g}(\beta)_f(\gamma)_g}{(r)_{f+g}}.\frac{a^f}{f!}.\frac{b^g}{g!} \tag{1}$$

$$F_2\left(\alpha;\beta,\gamma;r,s;\ a,b\right)=$$

$$\sum_{f,g=0}^{\infty}\frac{(\alpha)_{f+g}(\beta)_f(\gamma)_g}{(r)_f(s)_g}.\frac{a^f}{f!}.\frac{b^g}{g!} \tag{2}$$

and

$$F_3\left(\alpha;\beta;\gamma,r;x;\ a,b\right)=$$

$$\sum_{f,g=0}^{\infty} \frac{(\alpha)_f (\beta)_g (\gamma)_f (r)_g}{(s)_{f+g}} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!} \tag{3}$$

$F_4(\alpha;\beta;\gamma,r;\, a,b) =$

$$\sum_{f,g=0}^{\infty} \frac{(\alpha)_{f+g} (\beta)_{f+g}}{(r)_f (r)_g} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!} \tag{4}$$

The convergence conditions of Appell series are

(i) the series F_1 and F_3 converges for $|a|<|,|b|<|$

(ii) the series F_2 converges for $|a|+|b|<|$,

and (iii) the series F_4 converges when $|a^{\frac{1}{2}}|+|b^{\frac{1}{2}}|<|$,

In 1967, Srivastava [27] defined the triple hypergeometric series $F^{(3)}$ in the following way

$$F^{(3)}\left[\begin{matrix}(\alpha)::(\beta);(\beta^1);(\beta^{11}):(\gamma);(\gamma^1);(\gamma^{11}); \\ (r)::(s);(s^1);(s^{11}):(w);(w^1);(w^{11});\end{matrix}\; a,b,c\right]$$

$$= \sum_{f,g,\ell=0}^{\infty} \frac{(\alpha)_{f+g+\ell} (\beta)_{f+g} (\beta^1)_{g+\ell} (\beta^{11})_{\ell+f}}{(r)_{f+g+\ell} (s)_{f+g} (s^1)_{g+\ell} (s^{11})_{\ell+f}} \cdot \frac{(\gamma)_g (\gamma^1)_g (\gamma^{11})_\ell}{(w)_g (w^1)_g (w^{11})_\ell} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!} \cdot \frac{c^\ell}{\ell!} \tag{5}$$

here $(\alpha)_f$ is interpreted as

$$(\alpha)_f = \prod_{\delta=1}^{A} (\alpha_\delta)_f$$

$$= \prod_{\delta=1}^{A} \frac{\Gamma(\alpha_\delta + f)}{\Gamma(\alpha_\delta)} \tag{6}$$

$$(\beta)_f = \prod_{\delta=1}^{B} \frac{\Gamma(\beta_\delta + f)}{\Gamma(\beta_\delta)} \tag{7}$$

$$(\gamma)_f = \prod_{\delta=1}^{C} \frac{\Gamma(\gamma_\delta + f)}{\Gamma(\gamma_\delta)} \tag{8}$$

$$(r)_f = \prod_{\delta=1}^{D} \frac{\Gamma(r_\delta + f)}{\Gamma(r_\delta)} \tag{9}$$

$$(s)_f = \prod_{\delta=1}^{F} \frac{\Gamma(s_\delta + f)}{\Gamma(s_\delta)} \qquad (10)$$

and

$$(w)_f = \prod_{\delta=1}^{F} \frac{\Gamma(w_\delta + f)}{\Gamma(w_\delta)} \qquad (11)$$

where

$A+B+B^{11}+C \le D+E+E^{11}+F$,

$A+B+B^{1}+C^{1} \le D+E+E^{1}+F^{1}$,

$A+B^{1}+B^{11}+C^{11} \le D+E^{1}+E^{11}+F^{11}$

and $A, B, B^{1}, B^{11}, C, C^{1}, C^{11}, D, E, E^{1}$,

E^{11}, F, F^{1} and F^{11} are non-negative integers and $|a|<|,|b|<|,|c|<|$;

but

$A+B+B^{11}+C = D+E+E^{11}+F+1$,

$A+B^{1}+B^{11}+C^{11} = D+E^{1}+E^{11}+F^{11}+1$,

$A+B+B^{1}+C^{1} = D+E+E^{1}+F^{1}+1$.

In 1920, Humbert (see [28]) defined seven functions in which some of them are the limiting form of Appell' functions and them are:

$\varphi_1(\alpha;\beta;\gamma;\, a,b) =$

$$\sum_{f,g=0}^{\infty} \frac{(\alpha)_{f+g}(\beta)_f}{(r)_{f+g}} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!} \qquad (12)$$

$\varphi_2(\alpha;\beta;\gamma;\, a,b) =$

$$\sum_{f,g=0}^{\infty} \frac{(\alpha)_f(\beta)_g}{(r)_{f+g}} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!} \qquad (13)$$

$\varphi_3(\alpha;\beta;\, a,b) =$

$$\sum_{f,g=0}^{\infty} \frac{(\alpha)_f}{(\beta)_{f+g}} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!}$$

(14)

$\zeta_1(\alpha;\beta;\gamma,r;\ a,b) =$

$$\sum_{f,g=0}^{\infty} \frac{(\alpha)_{f+g}(\beta)_f}{(r)_f(r)_g} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!}$$

(15)

$\zeta_2(\alpha;\beta,\gamma;\ a,b) =$

$$\sum_{f,g=0}^{\infty} \frac{(\alpha)_{f+g}}{(\beta)_f(\gamma)_g} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!}$$

(16)

and

$\chi_1(\alpha;\beta,\gamma,r;\ a,b) =$

$$\sum_{f,g=0}^{\infty} \frac{(\alpha)_f(\beta)_g(\gamma)_f}{(r)_{f+g}} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!}$$

$|a|<|$

(17)

Kampe' de Feriet function (see [29]) is denoted by

$F\begin{matrix} A:B;D \\ F:G;H \end{matrix}$ and defined in the following way

$$F\begin{matrix} A:B;D \\ F:G;H \end{matrix}\left[\begin{matrix} (\alpha):(\beta);(r); \\ (w):(u);(v); \end{matrix}\ a,b\right] =$$

$$\sum_{f,g=0}^{\infty} \frac{(\alpha)_{f+g}(\beta)_f(\gamma)_g}{(w)_{f+g}(u)_f(v)_g} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!}$$

(18)

where for convergence

(i) $A+B \le E+G,\ A+D \le E+H$ for

$\{|a|,|b|<\infty\}$

(ii) $A+B=E+G+1, A+D=E+H+1$,

$$\begin{cases} |a|^{\frac{1}{A-E}}+|b|^{\frac{1}{A-E}}<|, \, if \; A>E \\ \max\{|a|,|b|\}<|, \, if \; A\leq E \end{cases}.$$

In 1893, Lauricella (see [30]) further generalized the four Appell functions $F_1,...,F_4$ to functions of g variables and defined his functions as follows:

$$F_A^{(g)}\left[\alpha,\beta_1,...,\beta_g;\gamma_1,...,\gamma_g;a_1,...,a_g\right]=$$

$$\sum_{f_1,...,f_n=0}^{\infty}\frac{(\alpha)_{f_1+\cdots+f_g}(\beta_1)_{f_1}\cdots(\beta_g)_{f_g}}{(\gamma_1)_{f_1}\cdots(\gamma_g)_{f_g}}\cdot\frac{a_1^{f_1}}{f_1!}\cdots\frac{b_g^{f_g}}{f_g!}$$

(19)

$$F_B^{(g)}\left[\alpha_1,\cdots,\alpha_g,\beta_1,\cdots,\beta_g;\gamma_1, \, a_1,a_{2,},\cdots,a_g\right]=$$

$$\sum_{f_1,\cdots,f_g=0}^{\infty}\frac{(\alpha_1)_{f_1}\cdots(\alpha_g)_{f_g}(\beta_1)_{f_1}\cdots(\beta_g)_{f_g}}{(\gamma_1)_{f_1+\cdots+f_g}}\cdot\frac{a_1^{f_1}}{f_1!}\cdots\frac{a_g^{f_g}}{f_g!}$$

(20)

$$F_C^{(g)}\left[\alpha,\beta;\gamma_1,\gamma_2,\cdots,\gamma_g;a_1,a_2,\cdots,a_g\right]=$$

$$\sum_{f_1,\cdots,f_g=0}^{\infty}\frac{(\alpha)_{f_1+\cdots+f_g}(\beta)_{f_1+\cdots+f_g}}{(\gamma_1)_{f_1}\cdots(\gamma_g)_{f_g}}\cdot\frac{a_1^{f_1}}{f_1!}\cdots\frac{a_g^{f_g}}{f_g!}$$

(21)

and

$$F_D^{(g)}\left[\alpha,\beta_1,\cdots,\beta_g;\gamma;a_1,\cdots,a_g\right]=$$

$$\sum_{f_1,\cdots,f_g=0}^{\infty}\frac{(\alpha)_{f_1+\cdots+f_g}(\beta_1)_{f_1}\cdots(\beta_g)_{f_g}}{(\gamma_1)_{f_1+\cdots+f_g}}\cdot\frac{a_1^{f_1}}{f_1!}\cdots\frac{a_g^{f_g}}{f_g!}$$

(22)

(i) the series $F_B^{(g)}$ and $F_D^{(g)}$ converges when max. $\{|a_1|,\cdots,|a_g|\}<|$

(ii) the series $F_A^{(g)}$ converges when $|a_1|+\cdots+|a_g|<|$

and the series $F_C^{(g)}$ converges when $|\sqrt{a_1}|+\cdots+|\sqrt{a_g}|<|$.

Hermite Polynomials:

Hermite polynomials (see [31]) are defined by means of generating relation

$$e^{\left(2ax-x^2\right)}=\sum_{g=0}^{\infty}H_g\left(a\right)\frac{x^g}{g!}$$
(23)

It follows from (23) that

$$H_g\left(a\right)=\sum_{\delta=0}^{\left[\frac{g}{2}\right]}\frac{\left(-1\right)^{\delta}.\delta!\left(2a\right)^{g-2\delta}}{\delta!\left(g-2\delta\right)!}$$
(24)

(or) equilatently

$$H_g\left(a\right)=\left(2a\right)^g\,2^F0\left[\frac{-g}{2},\frac{-g+1}{2},-;\frac{-1}{a^2}\right]$$
(25)

Associated Laguerre Polynomials:

The associated Laguerre polynomials (see [31])

$$\sum_{g=0}^{\infty}L_g^{(m)}\left(a\right)x^g=\frac{1}{\left(1-x\right)^{m+1}.e^{at}}$$
(26)

where $L_g^{(m)}\left(a\right)=\sum_{\delta=0}^{g}\binom{g+m}{g-\delta}\frac{\left(-a\right)^{\delta}}{\delta!}$
(27)

The hypergeometric form of the Laguerre polynomials given in [31; p. 200 (1)] is

$$L_g^{(m)}\left(a\right)=\frac{\left(1+m\right)_g}{g!}1^F\,1\left[-g;1+\delta;a\right]$$
(28)

where (Re (m) > – 1).

A generating functions involving confluent hypergeometric function for Laguerre polynomials is in the form

$$\sum_{g=0}^{\infty}\frac{(\alpha)_g L_g^{\delta}(a)x^g}{(1+\delta)_g}=$$

$$(1-x)^{-x}{}_1F_1\left[\alpha;1+\delta;\frac{-ax}{1-x}\right]$$

(29)

and $L_g^{(0)}(a)=L_g(a)={}_1F_1[-g;1;a]$

(30)

Generalized Rice and Related Polynomials:

Investigation of Rice [32]; were continued by Khandeker [33] who in 1964 defined the generalized Rice polynomials

$$H_g^{(\in,\theta_1)}(\ell,i,a)=\frac{(1+\in)_g}{g!}.$$

$${}_3F_2\left[\begin{matrix}-g,g+\in+\theta_1+1,\ell; \\ 1+\in \quad ,i;\end{matrix} a\right],$$

$\text{Re}(\in)>-1,$ and $\text{Re}(\theta_1)>-1$

(31)

and

$$H_g(\ell,i,a)={}_3F_2\left[\begin{matrix}-g,g+1,\ell; \\ 1 \ , \ i \ , \ ;\end{matrix} a\right]$$

(32)

Jacobi Polynomials:

The Jacobi polynomials (see [31]) $P_g^{(\in,\theta_1)}(a)$ are defined by the generating relation

$$\sum_{g=0}^{\infty}P_g^{(\in,\theta_1)}(a)x^g=$$

$$\left[1+\frac{(a+1)x}{2}\right]^{\in}\left[1+\frac{(a-1)}{2}\right]^{\theta_1}$$

(33)

In case when we put $\ell=i$ and $a=\frac{1-a}{2}$ in equation (30) reduces to Jacobi polynomial $P_g^{(\in,\theta_1)}(a)$ (see [31; p. 254]);

$$P_g^{(\in,\theta_1)}(a)=\frac{(1+\in)_g}{g!}\,{}_2F_1\left[\begin{matrix}-g,1+\in+\theta_1+g\,;\ \frac{1-a}{2}\\ 1+\in\,;\end{matrix}\right]$$

(34)

$$=\frac{(1+\in)_g}{g!}\cdot\left\{\frac{(a+1)}{2}\right\}^g\,{}_2F_1\left[\begin{matrix}-g,-\theta_1-g\,;\ \frac{a-1}{a+1}\\ 1+\in\,;\end{matrix}\right]$$

$\mathrm{Re}(\in)>-1$, $\mathrm{Re}(\theta_1)>-1$, and

$g\geq 0$

(35)

An equivalent form of (34), given in Rainville [31; p. 255 (8)] is

$$P_g^{(\in,\theta_1)}(a)=$$

$$\frac{(1+\theta_1)_g}{g!}\left(\frac{a-1}{2}\right)^g\,{}_2F_1\left[\begin{matrix}-g,-\in-g\,;\ \frac{a+1}{a-1}\\ 1+\theta_1\,;\end{matrix}\right]$$

(36)

Legendre Polynomials:

Put $\in=\theta_1=0$ in equation (33), we get the Legendre polynomials $P_g(a)$ (see [31; p. 166 (2)] which is defined in the following way

$$P_g(a)=\begin{cases}=\dfrac{\left(\frac{1}{2}\right)_g(2a)^g}{g!}\,{}_2F_1\left[\dfrac{-g}{2},\dfrac{1-g}{2};\dfrac{-g}{2};\dfrac{1}{a^2}\right]\\[2ex] =\left(\dfrac{a-1}{2}\right)^g\,{}_2F_1\left[-g;-g;1;\dfrac{a+1}{a-1}\right]\\[2ex] =a^g\,{}_2F_1\left[\dfrac{-g}{2},\dfrac{-g+1}{2};1;\dfrac{a^2-1}{a^2}\right]\end{cases}$$

(36)

The Legendre polynomial $P_g(a)$ (see [31]) of order *m* is generated by means of relation;

$$\sum_{g=0}^{\infty}P_g(g)x^g=\left(1-2ax+x^2\right)^{\frac{-1}{2}}$$

(37)

and has its series representation as follows:

$$P_g(a)=\sum_{g=0}^{\left[\frac{g}{2}\right]}\frac{(-1)^{\delta}(2g-2\delta)!a^{g-2\delta}}{2^g\delta!(g-\delta)!(g-2\delta)!}$$
(38)

where

$$\left[\frac{g}{2}\right]=\begin{cases}\frac{g}{2},\ if\ \ g\ \ is\ even\\ \frac{g-1}{2}.\ of\ \ g\ \ os\ odd\end{cases}$$
(39)

Gagenbauer Polynomials:

Put $\theta_1=\in$ in equation (33), then Gagenbauer Polynomials $C_g^{(s)}(a)$, given by $C_g^{(\in)}(a)=$

$$\frac{(1+\in)_g}{g!}{}_2F_1\left[\begin{matrix}-g,1+2\in+g; & \frac{1-a}{2}\\ 1+\in; & \end{matrix}\right]$$
(40)

or equivalently (see [31; p. 279 (15)])

$$P_g^{(\in,\in)}(a)=C_g^{(\in)}(a)=\frac{(1+\in)_g C_g^{\in+\frac{1}{2}}(a)}{(1+2\in)_g}$$
(41)

$$C_g^{\delta}(a)=\frac{(2\delta)_g}{g!}{}_2F_1\left[\begin{matrix}-g,2\delta+g; & \frac{1-a}{2}\\ \delta+\frac{1}{2}; & \end{matrix}\right]$$
(42)

The Gegenbauer polynomial is generated by means of generating function;

$$\sum_{g=0}^{\infty}C_g^{\delta}(a)x^g=\left(1-2ax+x^2\right)^{-\delta}$$
(43)

and

$$C_g^{\frac{1}{2}}(a)=P_g(a)$$
(44)

The ultraspherical (or Gegenbauer) polynomial (see [34; p. 81]) defined by

$$P_g^{\delta}(a)=\frac{(2\delta)_g}{\left(\delta+\frac{1}{2}\right)_g}\ P_g^{\left(\delta-\frac{1}{2},\,\delta-\frac{1}{2}\right)}(a)=$$

$$C_g^{\delta}(a),\delta>-\frac{1}{2}$$
(45)

Generalized Sylvester Polynomials:

We consider the polynomial $\psi_g(a;\alpha)$ by means of generating relation

$$\sum_{g=0}^{\infty}\psi_g(a;\alpha)x^g=(1-a)^{-a}.e^{\alpha a x}$$
(46)

The generalization of the Sylvestor polynomial (see [26, p. 302])

$$\varphi_g(a)=\psi_g(a;1)$$
(47)

and is defined by

$$\psi_g(a;\alpha)=\frac{(\alpha a)^g}{g!}\,{}_2F_1\left[-g,a;-;\frac{1}{\alpha a}\right]$$
(48)

Bateman's Polynomials:

In 1936, Bateman (see [35; p. 574]) defined a polynomial, called Bateman polynomial denoted by $J_g^{(\eta_1,\eta_2)}(a)$ and is given as follows:

$$J_g^{(\eta_1,\eta_2)}(a)=\binom{\frac{1}{2}\eta_1+\eta_2+g}{g}.\frac{a^{\eta_1}}{\Gamma(\eta_1+1)}$$

$${}_1F_2\left[\begin{matrix}-g\ , & ; \\ \eta_1+1,\frac{\eta_1}{2}+\eta_2+1; & \end{matrix}\ a^2\right]$$
(49)

The above polynomial is derived from the generating function (see [35; p. 575])

$$\sum_{g=0}^{\infty}J_g^{(\eta_1,\eta_2)}(a)\ x^{2g+\eta_1}=$$

$$\left(1-x^2\right)^{-\eta_{2-1}}.J_{\eta_1}\left(2ax\left(1-x^2\right)^{\frac{-1}{2}}\right), 1x|<|$$

(50)

or equivalent to

$$\sum_{g=0}^{\infty} J_g^{(\eta_1,\eta_2)}(a)\ x^g =$$

$$x^{\frac{-\eta_1}{2}}.(1-x)^{-\eta_{2-1}}.J_{\eta_1}\left(\frac{2a\sqrt{x}}{\sqrt{1-x}}\right)$$

(51)

where

$$J_g(a)=\frac{\left(\frac{a}{2}\right)^g}{\Gamma(1+g)}{}_0F_1\left[-;1+g;\frac{-a^2}{4}\right]$$

(52)

where $J_g(a)$ is the Bessel function of first kind.

Known theorems:

In 1969, Chatterjee [36] was first mathematician who proved the following theorem on ultraspherical polynomials:

Theorem A: If $X(a-x)=\sum_{f=0}^{\infty}\alpha_f x^f P_f^{\delta}(a)$

then

$$P^{-2\delta}X\left(\frac{a-x}{P},\frac{xb}{P}\right)=\sum_{j=0}^{\infty}x^j\beta_j(b).P_j(a)$$

where $\beta_j(b)=\sum_{f=0}^{\infty}\binom{j}{f}\alpha_f b^f$ and

$P=\left(1-2ax+x^2\right)^{\frac{1}{2}}$ and $P_f^{\delta}(a)$ is the ultraspherical polynomial defined by the equation (45).

In 1970, Saran [1] gave three theorems on bilinear generating functions which of one is given as follows:

Theorem B: If $\psi_g(a)=\mu(g).G(a)D^g\{u(a)\}$ where $u(a)\ G(a)$ are independent of g, and $X(a,x)=\sum_{f=0}^{\infty}\alpha_f x^f \psi_f(a)$ then

$$\frac{G(a)X(a-x,xb)}{G(a-x)}=\sum_{j=0}^{\infty}\frac{(-x)^j}{\mu(j)\,j!}.\beta_j(b).\psi_j(a)$$

where

$$\beta_j(b)=\sum_{f=0}^{\infty}(-j)_f\ \mu(f)\alpha_f b^f \text{ and } D=\frac{d}{dx}.$$

In 1972, Saran [36] gave remaining two theorems which are as follows:

Theorem C: Let $X(a,x)=\sum_{g=0}^{\infty}\psi_g(a)x^g$

where $\psi_g(a)$ is a polynomial of degree g in a, then

$$\frac{1}{\Gamma(\beta)}\int_0^{\infty}e^{-\ell}\ell^{\beta-1}1^{F}1\left(\gamma;\beta;\frac{b\ell i}{b-1}\right).$$

$$X(a,x\ell)d\ell=$$

$(1-b)^{\gamma}\sum_{g=0}^{\infty}(\beta)_g\,2^{F}1(-g,r;\beta;b)\psi_g(a)x^g$, provided the integral is convergent.

Theorem D: Let $G(a,x)=\sum_{g=0}^{\infty}u_g(a)x^g$

where $u_g(a)$ is a polynomial of degree g in a, then

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\infty}\int_0^{\infty}e^{-(\ell+i)}\ell^{\alpha-1}i^{(\beta-1)}$$

$$0^{F}1\left(\alpha,\frac{b\ell i}{b-1}\right)G\left(a,\frac{x\ell i}{1-b}\right)d\ell di=$$

$$=(1-b)^{\beta}\sum_{g=0}^{\infty}(\alpha)_g(\beta)_g\,2^{F}1(-g,\beta+g;\alpha;b).u_g(a)x^g$$

provided the integral is convergent.

In this research note we obtain three theorems which are then applied to obtain a set of bilinear and bilateral generating functions in terms of Laplace and beta integrals. These theorems include as special cases many known results given above.

Main theorems:

Theorem 1: Let $\gamma\left(\delta_1,\delta_2,...,\delta_f\right)$ denotes suitable bounded multiple sequences of arbitrary complex numbers for all positive integer *f*. Let $X\left(a,x\right)$ be a special type of function having formal power series expansion in *x* such that

$$X\left(a,x\right)=\sum_{g=0}^{\infty}\gamma_g\psi_g\left(a\right)x^g \qquad (53)$$

where $\left\{\gamma_g\right\}_{g=0}^{\infty}$ is a sequence of parameters, independent of *a* and *x*. and $\psi_g\left(a\right)$ are polynomial functions of degree *g* in *a*. Then $\text{Re}\left(i\right)>0,\ \text{Re}\left(\ell\right)>0$ where ℓ and x such that the triple hypergeometric series of Srivastava and $X\left(a,\frac{xc}{c-1}\right)$ remain uniformly convergent for $0<C<1$,

$$\sum_{g=0}^{\infty}\frac{\left(\ell\right)_g}{\left(1+\ell-i\right)_g}\psi_g\left(a\right).$$

$$F^{(3)}\left[\begin{matrix}\ell+g,(\alpha)::(\beta);\left(\beta^1\right);\left(\beta^{11}\right);(r);\left(r^1\right);\left(r^{11}\right); \\ i\quad,(r)::(s);\ \left(s^1\right);\left(s^{11}\right);(w);\left(w^1\right);\left(w^{11}\right);\end{matrix}\ a_1,a_2,a_3\right]x^g=$$

$$\frac{\Gamma i}{\Gamma\ell.\Gamma\left(i-\ell\right)}\int_0^1 c^{\ell-1}.(1-c)^{i-\ell-1}.$$

$$F^{(3)}\left[\begin{matrix}(\alpha)::(\beta);\left(\beta^1\right);\left(\beta^{11}\right);(\gamma);\left(\gamma^1\right);\left(\gamma^{11}\right); \\ (r)::(s);\ \left(s^1\right);\ \left(s^{11}\right);(w);\left(w^1\right);\left(w^{11}\right);\end{matrix}\ a_1\ c,\ a_2c,\ a_3c\right]\times\left(a,\frac{xc}{c-1}\right)dc,$$

where $\left(\alpha\right)_g=\frac{\Gamma\left(\alpha+g\right)}{\Gamma\left(\alpha\right)}$ (54)

and $F^{(3)}\left(a,b,c\right)$ is the triple hypergeometric series of Srivastava, defined by equation (s).

Theorem 2: Let $X\left(a,x\right)=\sum_{g=0}^{\infty}\psi_g\left(a\right)x^g$ where $\psi_g\left(a\right)$ is a polynomial of degree *g* in *a*, then for equation (2),

$$\Gamma(\beta)\int_0^\infty e^{-\ell}.\ell^{\beta-1}{}_1F_1(\gamma;\beta;b\ell){}_1F_1(\alpha;r;c\ell).$$

$$X(a,x\ell)d\ell =$$

$$\sum_{g=0}^{\infty}(\beta)_g F_2(\beta+g,\alpha,\gamma;r,\beta;c,b)\psi_g(a).x^g$$

(55)

provided the integral is convergent.

Theorem 3: Let $G(a,x)=\sum_{g=0}^{\infty}u_g(a)x^g$

(56)

where $u_g(a)$ is a polynomial of degree g in a, then for the equation (4),

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty\int_0^\infty e^{-(\ell+i)}.\ell^{\alpha-1}.\,i^{\beta-1}.$$

$${}_0F_1(-;\gamma;b\ell i){}_0F_1(-;r;c\ell i).$$

$$G(a,x\ell i)d\ell di =$$

$$\sum_{g=0}^{\infty}(\alpha)_g(\beta)_g F_4(\alpha+g,\beta+g;\gamma,r;b,c)u_g(a)x^g$$

(56)

provided the integral is convergent.

Proof of the theorem 1:

To prove our theorem 1, we replace $X(a,x)$ by its equation (53) by $F^3(a,b,c)$ by its series representation (5) in the integral of (54). Changing the order of integration and summation, which is permissible due to uniform convergence of the series involved and evaluating the inner beta function integral, we arrive at the result (54).

Proof the theorem 2:

To prove our theorem 2, put $n\ell$ for x equation (55), multiply both sides by $e^{-\ell}.\ell^{\beta-1}{}_1F_1(\gamma;\beta;b\ell){}_1F_1(\alpha;r;c\ell)$ and integrate with respect to ℓ between the limits 0 to ∞ with the help of the result [31; p. 15, th. 6], we get the required result.

Proof of theorem 3:

To prove the theorem 3, we multiply equation (56) by $e^{-(\ell+i)}.\ell^{\alpha-1}.i^{\beta-1}.{}_0F_1(-;\gamma;b\ell i){}_0F_1(-;r;c\ell i)$, replace *x* by $x\ell i$ and integrate with respect to ℓ and i between the limits 0 to ∞, we obtain the theorem 3.

Corollary 1:

Put $B = B^1 = B^{11} = E = E^1 = E^{11} = 0$, $a_3 = 0$, the theorem 1 gives the result of type

$$\sum_{g=0}^{\infty}\frac{(\ell)_g}{(1+\ell-i)_g}F^{(2)}\left[\begin{matrix}\ell+g,(\alpha):(\gamma);(r^1); \\ i \quad ,(r):(w);(w^1);\end{matrix}\; a_1,a_2\right]u_g(a)x^g =$$

$$\frac{\Gamma i}{\Gamma\ell.\Gamma(i-\ell)}\int_0^1 c^{\ell-1}.(1-c)^{i-\ell-1}.F^{(2)}\left[\begin{matrix}(\alpha):(\gamma);(\gamma^1); \\ (r):(w);(w^1);\end{matrix}\; a_1,a_2c\right]\times\left(a,\frac{xc}{c-1}\right)dc$$

(57)

where $F^{(2)}(a,b)$ is the Kempè de Feriet's double hypergeometric function, given by equation (18) and $(\alpha)_g = \frac{\Gamma(\alpha+g)}{\Gamma(a)}$.

Corollary 2:

Put $B = B^1 = B^{11} = E = E^1 = E^{11} = D = 0,\ C = C^1 = C^{11} = F = F^1 = F^{11} = A = 1$, replacing then α_1 by *i* in the given theorem 1, $F^{(3)}$ reduces to F_A, we get result of author [2; p. 222 (2.2)];

$$\sum_{g=0}^{\infty}\frac{(\ell)_g}{(1+\ell-i)_g}F_A[\ell+g,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3;a_1,a_2,a_3]\psi_g(a)x^g =$$

$$\frac{\Gamma i}{\Gamma(\ell).\Gamma(i-\ell)}\int_0^1 c^{\ell-1}.(1-c)^{i-\ell-1}.F_A[i,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3;a_1c,a_2c,a_3c].$$

$$X\left(a,\frac{xc}{c-1}\right),|a_1|+|a_2|+|a_3|<|$$

(58)

Similarly results for Lauricella's functions F_B, F_C and F_D (see equation (20) to equation (24)) given by author [2] appear as special case of our theorem 1 and are as follows:

$$\sum_{g=0}^{\infty}\frac{(\ell)_g}{(1+\ell-i)_g}F_C[\ell+g,\beta;\gamma_1,\gamma_2,\gamma_3;a_1,a_2,a_3]\psi_g(a)x^g =$$

$$\frac{\Gamma i}{\Gamma(\ell).\Gamma(i-\ell)}\int_0^1 c^{\ell-1}.(1-c)^{i-\ell-1}.F_C\left[i,\beta;\gamma_1,\gamma_2,\gamma_3;a_1c,a_2c,a_3c\right]$$

$$X\left(a,\frac{xc}{c-1}\right)dc,$$

$$\left|\sqrt{a_1}\right|+\left|\sqrt{a_2}\right|+\left|\sqrt{a_3}\right|<| \quad (59)$$

$$\sum_{g=0}^{\infty}\frac{(\ell)_g}{(1+\ell-i)_g}F_D\left[\ell+g,\beta_1,\beta_2,\beta_3;\gamma;a_1,a_2,a_3\right]\psi_g(a)x^g=$$

$$\frac{\Gamma i}{\Gamma(\ell).\Gamma(i-\ell)}\int_0^1 c^{\ell-1}.(1-c)^{i-\ell-1}.F_D\left[i,\beta_1,\beta_2,\beta_3;\gamma;a_1c,a_2c,a_3c\right]$$

$$X\left(a,\frac{xc}{c-1}\right)dc,$$

$$|a_1|<|,\ldots,|a_3|<|$$

(60)

and $\sum_{g=0}^{\infty}\frac{(1-\beta)_g}{(1-\beta+\alpha)_g}F_B\left[\in_1,\in_2,\in_3;\theta_1,\theta_2,\theta_3;\beta-g;a_1,a_2,a_3\right]\psi_g(a)x^g=$

$$\frac{\Gamma(\beta)}{\Gamma(\alpha).\Gamma(\beta-\alpha)}\int_0^1 c^{\alpha-1}.(1-c)^{\beta-\alpha-1}.F_B\left[\in_1,\in_2,\in_3;\theta_1,\theta_2,\theta_3;\alpha;a_1c,a_2c,a_3c\right].$$

$$X\left(a,\frac{x}{1-c}\right)dc,$$

$$|a_1|<|,\ldots,|a_3|<|$$

(61)

Corollary 3:

Putting $c=0,b=\frac{b}{b-1}$ and making use of linear transformation;

$$_2F_1\left[\begin{matrix}\alpha,\beta; \\ \gamma;\end{matrix}c\right]=$$

$$(1-c)^{-\beta}\,{}_2F_1\left[\begin{matrix}\gamma-\alpha,\beta; \\ \gamma;\end{matrix}\frac{-c}{1-c}\right],$$

$$|<|<|,\left|\frac{c}{1-c}\right|<| \qquad (62)$$

In theorem 2 and 3, we get result of author [36; pp. 12-13], gives our required result.

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