Lower Order of Entire Functions of Several Complex Variables

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Abstract: In the present paper, we study the growth properties of entire functions of several complex variables. The characterizations of generalized lower order of entire functions of several complex variables have been obtained in terms of their Taylor's series coefficients. Also we have obtained the characterization of generalized lower order of entire functions of several complex variables in terms of approximation and interpolation errors.

Key words and phrases: Entire function, Maximum term, Central index, generalized lower order, Approximation errors, Interpolation errors

Introduction

We denote the complex N−space by C^N. Thus, $z \in C^N$ means that $z = (z_1, z_2 ... z_N)$, where z_1, z_2, ..., z_N are complex numbers. A function $g(z)$, $z \in C^N$ is said to be analytic at a point $\xi \in C^N$ if it can be expanded in some neighborhood of ξ as an absolutely convergent power series. If we assume

$\xi = (0, 0, ..., 0)$, then $g(z)$ has representation.

$$g(z) = \sum_{|k|=0}^{\infty} a_{k_1,k_2,...,k_N} z_1^{k_1} z_2^{k_2} ... z_N^{k_N} = \sum_{n=0}^{\infty} a_k z^k$$

Where $k = (k_1, k_2 ... , k_N) \in N_0^N$ and $n = |k| = k_1 + k_2 + ... + k_N$. For $r > 0$, the maximum modulus $S(r, g)$ of entire function $g(z)$ is given by (see [3])

$S(r, g) = \sup\{|g(z)| : |z_1|^2 + |z_2|^2 + ... + |z_N|^2 = r^2\}$. For $r > 0$, the maximum term $\mu(r)$ of entire function $g(z)$ is defined as (see [4] and [5])

$$\mu(r) = \mu(r, g) = \max_{n \geq 0}\{\|a_k\| r^n\}$$

Also the index k with maximal length n for which maximum term is achieved is called the central index and is denoted by $v(r) = v(r, g) = k$.

For generalization of the classical characterizations of growth of entire functions, Seremeta [7] introduced the concept of the generalized order and generalized type using the general growth functions as follows:

Let L^0 denote the class of functions $h(x)$ satisfying the following conditions:

i. $h(x)$ is defined on $[a, \infty)$ and is positive, strictly increasing, differentiable and tends

to ∞ as x $\rightarrow \infty$,

ii. $\lim\limits_{x\to\infty} \dfrac{h[\{1+1/\psi(x)\}x]}{h(x)} = 1$ for every function $\psi(x)$ such that $\psi(x) \rightarrow \infty$ as x $\rightarrow \infty$.

Let Λ denote the class of functions h (x) satisfying conditions (i) and

iii. $\lim\limits_{x\to\infty} \dfrac{h(cx)}{h(x)} = 1$, $h(x) = 1$, for every $c > 0$, that is $h(x)$ is slowly increasing. If   g (z) is an

entire function and functions $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$, then the generalized order $\rho(\alpha, \beta, g)$ of g (z) is defined as (see [2])

$$\rho(\alpha,\beta,g) = \lim\limits_{r\to\infty} \sup \frac{\alpha[\log S(r,g)]}{\beta(\log r)}$$

For an entire function g (z) and functions $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$, we define the generalized lower order $\lambda(\alpha, \beta, g)$ of $g(z)$ as

$$\lambda(\alpha,\beta,g) = \lim\limits_{r\to\infty} \inf \frac{\alpha[\log S(r,g)]}{\beta(\log r)} \tag{1.2}$$

Following Bose and Sharma ([1], p. 219-220) we can easily show that the generalized lower order $\lambda(\alpha, \beta, g)$ of g(z) can be expressed in terms of central index as

$$\lambda(\alpha,\beta,g) = \lim\limits_{r\to\infty} \inf \frac{\alpha\{|v(r)|\}}{\beta(\log r)} \tag{1.3}$$

Let K be a compact set in C^N and let $\|.\|_K$ denote the supremum norm on K. The function

$$\Phi_K(z) = sup\big[|p(z)|^{1/n} : p - \text{ polynomial, deg } p \le n, \| p \|_K \le 1\big],$$

Where $n = 1, 2, ...$ and $z \in C^N$, is called the Siciak extremal function of the compact set K (see [2] and [3]). Given a function f defined and bounded on K, for $n = 1, 2, ...$, we put

$$E^1 n \ (f, K) = \|f - t_n\|_K \, ;$$
$$E^2 n \ (f, K) = \|f - l_n\|_K \, ;$$
$$E^3 n{+}1 \ (f, K) = \|l_{n+1} - l_n\|_K \, ;$$

Where t_n denotes the nth Chebyshev polynomial of the best approximation to f on K and l_n denotes the nth Lagrange interpolation for f with nodes at extremal points of K (see [2] and [3]). Kumar and Srivastava ([6], Thm. 2.1) have obtained coefficient characterizations of lower order of entire functions of several complex variables in terms of their Taylor's series coefficients. In the present paper we have obtained the characterizations of generalized lower order of entire functions of several complex variables in terms of their Taylor's series coefficients. Also we have obtained the characterizations of generalized order of entire functions of several complex variables in terms of approximation and interpolation errors.

2. Main Results

Now we prove Theorem 2.1. Let $g(z)$ be an entire function whose Taylor's series representation is given by (1.1). If $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$, then the generalized lower order λ of $g(z)$ satisfies

$$\lambda = \lambda(\alpha,\beta,g) \ge \lim\limits_{n\to\infty} \inf \frac{\alpha(n)}{\beta\{\log\|a_k\|^{-1/n}\}} \tag{2.1}$$

Further, if

$$\psi(n) = \max_{|k|=n}\left\{\frac{\|a_k\|}{\|a_{k'}\|}, \|k'\| = \| k \| + 1\right\}$$

is a non-decreasing function of n then equality holds in   (2.1)

Proof. Write

$$\Phi = \lim_{n\to\infty} \inf \frac{\alpha(n)}{\beta\left\{\log\|a_k\|^{-1/n}\right\}}$$

First we prove that $\lambda \geq \Phi$. The coefficients of an entire Taylor's series satisfy Cauchy's inequality, that is

$$\| ak \| \leq \mathrm{S}(r, g)\, r^{-\mathrm{n}}, |k| = n. \qquad (2.2)$$

Also from (1.2), for arbitrary $\varepsilon > 0$ and a sequence $r = r_s \to \infty$ as $s \to \infty$, we have

$$\|a_k\| \leq r^{-n}\exp\left[\alpha^{-1}\{\bar\lambda\beta(\log r)\}\right]$$

Putting $r = \exp[\beta^{-1}\{\alpha\,(n)\overline{/\lambda}\}]$ in the above inequality we get

$$\|a_k\| \leq \exp\left[n - n\,\beta^{-1}\{\alpha\,(n)\overline{/\lambda}\}\right],$$

or

$$\beta^{-1}\left[\frac{\alpha(n)}{\bar\lambda}\right] \leq 1 - \frac{1}{n}\{\log\|a_k\|\},$$

or

$$\frac{\alpha(n)}{\beta\left\{1+\log\|a_k\|^{-1/n}\right\}} \leq \bar\lambda.$$

Since $\beta(1+x)' \beta(x)$ as $x \to \infty$, proceeding to limits as $n \to \infty$, we get

$$\Phi = \lim_{n\to\infty}\inf \frac{\alpha(n)}{\beta\left\{\log\|a_k\|^{-1/n}\right\}} \leq \bar\lambda$$

Since $\varepsilon > 0$ is arbitrarily small so finally we get $\Phi \leq \lambda$. Now we prove the reverse inequality i.e., $\lambda \leq \Phi$. From the assumption on ψ, $\psi(n) \to \infty$ as $n \to \infty$. By the definition given in section 1, if $\mu(r) = \| a_k \| r^{|k|}$ is the maximum term then for $|k_1| \leq |k| < |k_2|,$

$$\|a_{k1}\|r^{|k1|} \leq \|a_k\|\,r^{|k|} > \|a_{k2}\|\,r^{|k2|}$$

and for $|k| = n, \psi(n-1) \leq r < \psi(n).$

Now suppose that $\|a_{k1}\|\,r^{|k1|}$ and $\|a_{k2}\|\,r^{|k2|}$ are two consecutive maximum terms. Then $|k^1| \leq |k^2| - 1$. Let $|k^1| \leq n \leq |k^2|$. Then for $\psi(|k^{1*}|) \leq r < \psi(|k^1|)$, we have $|v(r)| = |k^1|$ where $|k^{1*}| = |k^1| - 1$. Hence from (1.3), for arbitrary $\varepsilon > 0$ and all $r > r_0(\varepsilon)$, we have

$$|k^1| = |v(r)| > \alpha^{-1}\{\lambda'\beta(\log r)\}, \lambda' = \lambda - \varepsilon,$$

or

$$|k^1| = |v(r)| \geq \alpha^{-1}\{\lambda'\beta[\log\{\psi(|k^1|) - q\}]\},$$

Or

$$\log\psi(|k^1|) \leq O(1) + \beta^{-1}\{\alpha(|k^1|)/\lambda'\}$$

where q is a constant such that

$$0 < q < min\{1, [\psi(|k^1|) - \psi(|k^{1*}|)]/2\}$$

Further we have

$$\psi(|k^1|) = \psi(|k^1| + 1) = \cdots = \psi(n - 1)$$

Now we can write

$$\psi(|k^0|) \ldots \psi(|k^*|) = \frac{\|a_{k^0}\|}{\|a_k\|} \leq [\psi(|k^*|]^{n-|k^0|}$$

where $|k^*| = n - 1$ and $n \gg |k^0|$ or

$$\log\|a_k\|^{-1} \leq n\log\psi(|k^1|) + O(1) \leq n\beta^{-1}\{\alpha(|k^1|)/\lambda'\} + O(1)$$

or

$$-\frac{1}{n}\log\|a_k\| \leq [\beta^{-1}\{\alpha(|k^1|)/\lambda'\}][1 + o(1)],$$

or

$$-\frac{1}{n}\log\|a_k\| \leq [\beta^{-1}\{\alpha(n)/\lambda'\}][1 + o(1)],$$

or

$$\lambda' \leq \frac{\alpha(n)}{\beta\{\log\|a_k\|^{-1/n}\}}[1 + o(1)].$$

Now taking limits as $n \to \infty$, we get $\lambda \leq \Phi$. Hence the Theorem 2.1 is proved. ——————

Next we prove

Theorem 2.2.

Let K $\subseteq C^N$ be a compact set such that Φ_K is locally bounded in C^N. If $\alpha(x) \in \Lambda$ and $\beta(x) \in$ L^0 then the function f, defined and bounded on K, is a restriction to K of an entire function g of generalized lower order $\lambda(\alpha, \beta, g)$ if and only if

$$\lambda = \lambda(\alpha, \beta, g) \geq \lim_{n\to\infty}\sup\frac{\alpha(n)}{\beta[\log\{E_n^S(f,K)\}^{-1/n}]}; s = 1,2,3 \qquad (2.3)$$

Also if E^s_n (f, K)/E$^s_{n+1}$ (f, K) is a non-decreasing function of n, then equality holds in (2.3).

Proof. First we assume that f has an entire function extension g which is of generalized order $\rho = \rho(\alpha, \beta, g)$. We write

$$\theta_s = \lim_{n\to\infty}\sup\frac{\alpha(n)}{\beta[\log\{E_n^S\}^{-1/n}]}; s = 1,2,3$$

Here E^s_n stands for E^s_n ($g|_K$, K), s = 1, 2, 3. Following Winiarski [8], we have

$$E^1_n \leq E^2_n \leq (n_* + 2)E^1_n \quad n \geq 0, \qquad (2.4)$$

and

$$E^3_{n,} \leq 2(n_* + 2)E^1_{n-1} \qquad n \geq 1, \qquad (2.5)$$

Where $= \begin{pmatrix} n + N \\ n \end{pmatrix}$. Using Stirling formula for the approximate value of n!

We get $n_* \approx \frac{n^N}{N!}$ for all large values of n. Hence for all large values of n, we have

$$E_n^1 \leq E_n^2 \leq \frac{n^N}{N!}\{1 + o(1)\}E_n^1$$

and

$$E_n^3 \leq 2\frac{n^N}{N!}\{1 + o(1)\}E_n^1.$$

Thus $\theta 3 \leq \theta 2 = \theta 1$. First we prove that $\theta_s \leq \lambda$. Without any loss of generality, we may suppose that

$$K \subset B = \{ z \in C^N : |z_1|^2 + |z_2|^2 + ... + |z_N|^2 \leq 1.$$

Then

$$E^1_n \leq E^1_n (g, B).$$

Now following Janik ([3], p. 324), we have

$$E^1_n (g, B) \leq r^{-n} s(r, g) \qquad r \geq 2, \quad n \geq 0,$$

or

$$E_n^1 \leq r^{-n}\exp\{\alpha^{-1}[\bar\lambda\beta(\log r)]\}$$

Putting r = exp{ $\beta -1$ [α(n)/λ]} in the above inequality, we get

$$E_n^1 \leq \exp\{n - n\beta^{-1}[\alpha(n)/\bar\lambda]\}$$

or

$$\frac{\alpha(n)}{\beta\left(1-\frac{1}{n}[\log\{E_n^1\}]\right)} \leq \bar\lambda$$

Since $\beta(1 + x) \simeq \beta(x)$ as $x \to \infty$, proceeding to limits as $n \to \infty$, we get

$$\theta_1 \le \lambda,$$

or

$$\theta_s \le \lambda,$$

Now we will prove that $\lambda \le \theta_s$. Let $\psi(n) = E^s_{|k|} / E^s_{|k1|}$. Then $\psi(n) \to \infty$ as $n \to \infty$.
Now as in the proof of Theorem 2.1, here we have

$$\log [E^s_n]^{-1} \le n\log \psi(|k^1|) + O(1) \le n\beta^{-1}\{\alpha(|k^1|)/\lambda'\} + O(1),$$

or

$$-\frac{1}{n}\log E^s_n \le [\beta^{-1}\{\alpha(|k^1|)/\lambda'\}][1 + o(1)],$$

or

$$-\frac{1}{n}\log E^s_n \le [\beta^{-1}\{\alpha(n)/\lambda'\}][1 + o(1)]$$

or

$$\lambda' \le \frac{\alpha(n)}{\beta\{\log[E^s_n]^{-1/n}\}}[1 + o(1)].$$

Now taking limits as $n \to \infty$, we get $\lambda \le \theta_s$

Now let f be a bounded function defined on K and such that for s = 1, 2, 3

$$\theta_s = \lim_{n\to\infty} \sup \frac{\alpha(n)}{\beta[\log\{E^s_n\}^{-1/n}]}$$

Then for every $d_1 > \theta_s$ and for sufficiently large value of n , we have

$$\frac{\alpha(n)}{\beta[\log\{E^s_n\}^{-1/n}]} \le d_1 ,$$

or

$$0 \le E^s_n \le \exp\left[-n\beta^{-1}\left\{\frac{1}{d_1}\alpha(n)\right\}\right].$$

Proceeding to limits as n $\to \infty$, we get

$$\lim_{n\to\infty} [E^s_n]^{1/n} = 0.$$

So by the result of Janik ([2], Prop. 3.1), we infer that the function f can be continuously extended to an entire function. Let us put

$$g = l_0 + \sum_{n=1}^{\infty} (l_n - l_{n-1}),$$

where $\{l_n\}$ is the sequence of Lagrange interpolation polynomials of *f* as defined earlier. Now we claim that *g* is the required continuation of *f* and $\rho(\alpha, \beta, g) = \theta_s$. As in the proof of this Theorem given above, we have

$$\lambda \le \theta_s .$$

Now using the inequalities (2.4), (2.5) and the proof of first part given above, we have $\lambda(\alpha, \beta, g) = \theta_s$, as claimed. This completes the proof of the Theorem 2.2.

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