Mathematical Statistician and Engineering Applications
ISSN:2094-0343
2326-9865

Detour Polynomials of Some Certain Graphs

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Article Info
Page Number: 151-160
Publication Issue:
Vol. 72 No. 2 (2023)

Article History
Article Received: *15 February 2023*
Revised: *20 April 2023*
Accepted: *10 May 2023*

Abstract

The length of a longest u–v path in a connected graph G between two distinct vertices u and v is called detour distance and denoted by $D(u,v)$, the detour distance is an important type of distances types in graph theory because of its importance in chemical applications and other sciences. In this paper, we find detour polynomials, detour index and study some properties for some certain graphs such as: Wagner graph, friendship graph, double cycles and double paths graphs.

***Keywords**: - detour polynomial, detour index, Wagner graph, friendship graph, double cycles graph, double paths graph.*

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1. Introduction:

The initial notation and terminology used in this paper sourced from the references **[1,2]**. The graphs examined in this paper are finite and simple. Let G be a connected graph of order p and size q, between any two distinct vertices u and v in G the detour distance is the maximum length of $u-v$ paths, it is denoted by $D_G(u,v)$ or simply $D(u,v)$, $D(u,v)=1$ if and only if uv is a bridge of G, furthermore $D(u,v)=d(u,v)$ if and only if G is a tree. From clearly that $D(u,v)=p-1$ if and only if G contains a Hamiltonian $u-v$ path. The ***detour index*** $D(G)$ of G is defined as **[3,4]**:

$$D(G)=\sum_{\{u,v\}\subseteq V(G)} D(u,v).$$

The average detour distance of a graph G is defined by:

$$\mu_D(G)=\frac{2D(G)}{p(p-1)}.$$

Since the **eccentricity** of the vertex v is defined by

$$e(v)= max\{d(v,u): u\in V(G)\}$$

where $d(v,u)$ is the length of a shortest $u-v$ path in G.

The **diameter** and the **radius** of a connected graph G are defined respectively as:

$$diam(G)=max\,\{e(v): v\in V(G)\}$$

and

$$rad(G)=min\{e(v): v\in V(G)\},$$ **[1]**.

The ***detour eccentricity*** $e_D(v)$ of a vertex v in G is defined as the maximum of $\{D(u,v): u\in V(G)\}$. So the ***detour diameter*** of G denoted by $\delta_D(G)$, (or $diam_D(G)$) and the ***detour radius*** of G denoted by $r_D(G)$, (or $rad_D(G)$) are given respectively by

$$\delta_D(G)=\max.\{e_D(v): v\in V(G)\},$$

and, $$r_D(G)=\min.\{e_D(v): v\in V(G)\}.$$

Obviously $e(v) \leq e_D(v)$ for every vertex v in G, since $d(u,v) \leq D(u,v)$, for u and v in G. Therefore, $diam(G) \leq diam_D(G)$ and $rad(G) \leq rad_D(G)$, (see **[5,6]**).

The vertex v is a **detour peripheral** vertex of G if $e_D(v) = \delta_D(G)$ and the **detour peripheral** of G is the set of all peripheral vertices of G which is denoted $P_D(G)$, while the **detour center** vertex in G is v if $e_D(v) = r_D(G)$, for every v in G, also the **detour central** of G is the set of all detour center vertices of G and it is denoted $C_D(G)$. The **minimum detour distance** is defined by:

$m_D(G) = \min\{D(u,v): \{u,v\} \subseteq V(G)\}$.

The distance polynomial **[6]** of a connected graph G based on detour distance is called ***detour polynomial*** $D(G;x)$. Let $C_D(G,k)$ be the number of unordered pairs u and v such that $D(u,v) = k$, $k \geq m_D(G)$, then the detour polynomial of G is defined by:

$$D(G;x) = \sum_{k \geq m_D(G)}^{\delta_D(G)} C_D(G,k)x^k,$$

also, the detour polynomial can be defined as:

$$D(G;x) = \sum_{\{u,v\} \subseteq V(G)} x^{D(u,v)}.$$

The detour polynomial of a vertex v in G is define as:

$$D(v,G;x) = \sum_{k \geq m_D(G)}^{e_D(v)} C_D(v,G,k)x^k,$$

where $C_D(v,G,k)$ is the number of vertices u , $(u \neq v)$ such that $D(u,v) = k$, $k \geq m_D(G)$.
It is clear that:

- $D(G;x) = \frac{1}{2}\sum_{v \in V(G)} D(v,G;x)$.
- $D(G) = \frac{d}{dx} D(G;x) \Big|_{x=1}$.
- $D(v,G;1) = p - 11$.

In the mathematical field of the graph theory, the graph G is a **symmetric** (or **arc-transitive**) if, given any two pairs of adjacent vertices u_1v_1 and u_2v_2 of G, there is an automorphism:

$f: V(G) \to V(G)$, such that: $f(u_1) = u_2$ and $f(v_1) = v_2$, **[7]**.

Since the year 2000, detour distance and detour index work have become very wide and many researchers and graduate students have been interested in this type of distance because of its increasing importance. Its importance enters into chemistry and the rest of the sciences **[8, 9, 10]**.

Some researchers studied the detour distance such as: Chartrand and others introduced the concept of detour distance **[3,4]**, but the concept of detour distance polynomial of a connected graph G was introduced in 2011 in **[5]** and Mohammed also found polynomials of detour for special graphs and operations defined on graphs in 2013, see **[6]**, some work has been done on detour indices. Several authors had obtained detour polynomials and detour indices for structure chemical **[10-12]**, and Ahmed M. A. recently worked about the connected detour numbers of special graphs **[13, 14]**.

2. Detour Polynomials of Some Certain Graphs

2.1. Wagner Graph G_{2p}:

Definition 2.1.1: Let C_{2p} be a cycle of order $2p$, $p \geq 2$ such that $C_{2p} : v_1, v_2, \ldots, v_{2p}, v_1$. A **Wagner graph** denoted by G_{2p} is the graph obtained from C_{2p} by adding p edges $v_i v_{i+p}$, $i = 1,2,\ldots,p$, **[15]**, as shown in Fig.1.

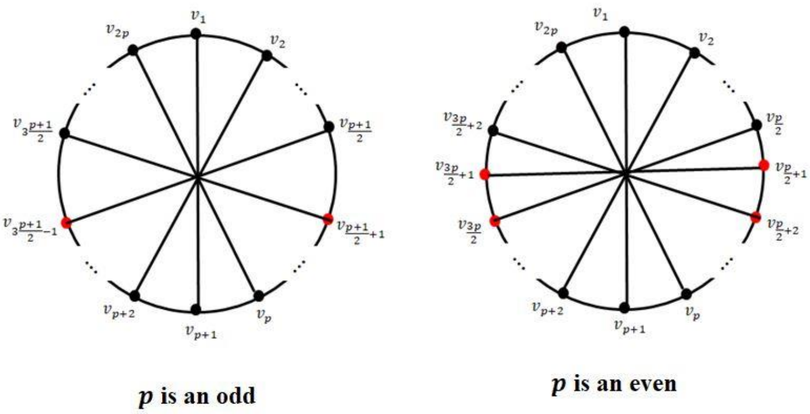

Fig.1. A Wagner graph G_{2p}

Some Properties of the Wagner Graph G_{2p}:

1. ***Order and size:*** The order of G_{2p} is $2p$ and the size of G_{2p} is $3p$, $p \geq 2$.
2. ***Diameter and radius of detour distance:*** $\delta_D(G_{2p}) = r_D(G_{2p}) = 2p - 1$.
3. ***Minimum detour distance :*** $m_D(G_{2p}) = \begin{cases} 2p-1\,; if\ p\ is\ even \\ 2p-2\,; if\ p\ is\ odd \end{cases}$
4. ***Symmetric property:*** The graph G_{2p} is a symmetric graph.
5. ***Degree vertices:*** Since Wagner graph G_{2p} is $3-$regular graph, then each vertices v_i have degree three ($degv_i = 3$), for all $i = 1,2,\ldots,2p$, $p \geq 2$.
6. ***Detour distance:***

- If p is even, then $D(v_i, v_j) = 2p - 1$, for all $i, j = 1,2,\ldots,2p$, $i \neq j$.
- If p is odd, then
  - $D(v_{2i+1}, v_{2j}) = 2p - 1$, for all $i = 0,1,\ldots,p-1$ and $j = i+1,\ldots,p$.
  - $D(v_{2i+2}, v_{2j+1}) = 2p - 1$for all $i = 0,1,\ldots,p-2$ and $j = i+1,\ldots,p-1$.
  - $D(v_{2i+1}, v_{2j+1}) = D(v_{2i+2}, v_{2j+2}) = 2p - 2$, for all $i = 0,1,\ldots,p-2$ $and\ j = i+1,\ldots,p-1$.

7. ***Detour peripheral of G_{2p}:*** $P_D(G_{2p}) = V(G_{2p})$.
8. ***Detour central of G_{2p}:*** $C_D(G_{2p}) = V(G_{2p})$.

The detour polynomial of the **Wagner graph G_{2p}** is sought out in the next theorems:

Theorem 2.1.2: For $p \geq 2$ and p is even, then

$$D(G_{2p}; x) = p(2p-1)x^{2p-1}.$$

Proof: Since for any pairs of two vertices v_i, v_j in G_{2p} for even p and for all $i, j = 1,\ldots,2p, i \neq$

j, have detour distance $(2p-1)$, then $D(v_i,v_j)=2p-1$,

and, $\sum_{i=1}^{2p-1}\sum_{j=i+1}^{2p} D(v_i,v_j,G_{2p};x)=p(2p-1)x^{2p-1}$.

Since $D(G_{2p};x)=\sum_{i=1}^{2p-1}\sum_{j=i+1}^{2p} D(v_i,v_j,G_{2p};x)$

then, $D(G_{2p};x)=p(2p-1)x^{2p-1}$. ■

Theorem 2.1.3: For $p \geq 2$ and p is odd, then

$$D(G_{2p};x)=p^2x^{2p-1}+(p^2-p)x^{2p-2}.$$

Proof: Since any pairs of two vertices v_i, v_j in G_{2p} for all $i,j=1,\dots,2p, i\neq j$, and for odd p, have detour distance $2p-1$ or $2p-2$.

If $D(v_i,v_j)=2p-1$, then

$$\sum_{i=0}^{p-1}\sum_{j=i+1}^{p} D(v_{2i+1},v_{2j},G_{2p};x)+\sum_{i=0}^{p-2}\sum_{j=i+1}^{p-1} D(v_{2i+2},v_{2j+1},G_{2p};x)$$
$$=\frac{p(p+1)}{2}x^{2p-1}+\frac{p(p-1)}{2}x^{2p-1}.$$

If $D(v_i,v_j)=2p-2$, then

$$\sum_{i=0}^{p-2}\sum_{j=i+1}^{p-1} D(v_{2i+1},v_{2j+1},G_{2p};x)+\sum_{i=0}^{p-2}\sum_{j=i+1}^{p-1} D(v_{2i+2},v_{2j+2},G_{2p};x)$$
$$=\frac{p(p-1)}{2}x^{2p-2}+\frac{p(p-1)}{2}x^{2p-2}.$$

Since $D(G_{2p};x)=\sum_{i=1}^{2p-1}\sum_{j=i+1}^{2p} D(v_i,v_j,G_{2p};x)$, then

$$\begin{aligned}D(G_{2p};x)&=\sum_{i=0}^{p-1}\sum_{j=i+1}^{p} D(v_{2i+1},v_{2j},G_{2p};x)\\&+\sum_{i=0}^{p-2}\sum_{j=i+1}^{p-1} D(v_{2i+2},v_{2j+1},G_{2p};x)\\&+\sum_{i=0}^{p-2}\sum_{j=i+1}^{p-1} D(v_{2i+1},v_{2j+1},G_{2p};x)\\&+\sum_{i=0}^{p-2}\sum_{j=i+1}^{p-1} D(v_{2i+2},v_{2j+2},G_{2p};x)\\&=\frac{p(p+1)}{2}x^{2p-1}+\frac{p(p-1)}{2}x^{2p-1}+\frac{p(p-1)}{2}x^{2p-2}+\frac{p(p-1)}{2}x^{2p-2}\\&=p^2x^{2p-1}+(p^2-p)x^{2p-2}.\end{aligned}$$ ■

Corollary 2.1.4: For $p \geq 2$, then

1. $D(G_{2p})=\begin{cases} p(2p-1)^2, & \text{for even } p, \\ p^2(2p-1)+2p(p-1)^2, & \text{for odd } p.\end{cases}$
2. $\mu_D(G_{2p})=\begin{cases} 2p-1, & \text{for even } p, \\ p+\frac{2(p-1)^2}{(2p-1)}, & \text{for odd } p.\end{cases}$

Proof: We get $D(G_{2p})$ by derivative $D(G_{2p};x)$ with respect to x and then $x=1$. ■

2.2. Friendship Graph Fr_p:

Definition 2.2.1:[16] A Friendship Graph Fr_p is the graph obtained by taking m ($m=\frac{p-1}{3}$) copies of the cycle graph C_4 with a vertex in common, the Friendship graph Fr_p is also called the Flower graph F_4^m, and has the vertex set $\{v_1,v_2,\dots,v_{2m},u_1,u_2,\dots,u_m,c\}$. We will rename friendship graph vertices as shown in Fig. 2.

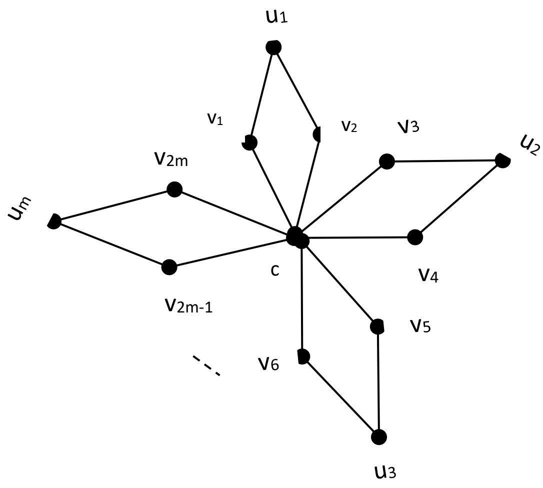

Fig. 2. Friendship Graph Fr_p

Some Properties of the Friendship Graph Fr_p :

1. ***The order and size:*** $p(Fr_p) = 3m + 1$ and $q(Fr_p) = 4m$, $m \geq 2$.
2. ***Diameter and radius property:*** The graph Fr_p has $\delta_D(Fr_p) = 2r_D(\mathbf{Fr_p}) = 6$.
3. ***Minimum detour distance :*** $m_D(Fr_p) = 2$.
4. ***Symmetric property:*** The graph Fr_p is not symmetric.
5. ***Degree vertices:*** All vertices of Fr_p have degree two $(deg(v_i) = deg(u_j) = 2, for\ i = 1, \dots, 2m\ and\ j = 1, \dots, m)$ except the vertex c has degree $2m$.

6. The Detour distance:

- $D(c, v_i) = 3$ for $i = 1, \dots, 2m$ and $D(c, u_j) = 2$ for $j = 1, \dots, m$.
- $D(v_{2i+1}, v_{2i+2}) = 2\ for\ i = 0, \dots, m-1$ and $D(v_{2i+1}, v_j) = D(v_{2i+2}, v_j) = 6\ for\ i = 0, \dots, m-2\ and\ j = 2i+3, \dots, 2m$.
- $D(u_i, u_j) = 4\ for\ i = 1, \dots, m-1\ and\ j = 1, \dots, m$.
- $D(u_i, v_j) = 3\ for\ i = 1, \dots, m\ and\ j = 2i-1, 2i$ and $D(u_i, v_j) = 5\ for\ i = 1, \dots, m\ and\ j = 1, \dots, 2m$.

7. *Detour peripheral of Fr_p:* $P_D(Fr_p) = \{v_1, v_2, \dots, v_{2m}\}$.

8. *Detour central of Fr_p:* $C_D(Fr_p) = \{c\}$.

The detour polynomial of the Friendship graph Fr_p is sought out in the next theorem:

Theorem 2.2.2: For $m \geq 2$, $m = \frac{p-1}{3}$, then

$D(\mathbf{Fr_p}; x) = 2mx^2 + 4mx^3 + \frac{m(m-1)}{2}x^4 + 2m(m-1)x^5 + 2m(m-1)x^6$.

Proof: Let U be the subset of vertices of $V(\mathbf{Fr_p})$ such that $(U = \{u_1, u_2, \dots, u_m\})$ and $V = V(\mathbf{Fr_p}) - U - \{c\} = \{v_1, v_2, \dots, v_{2m}\}$, m≥ 2, then there are four cases to find $D(\mathbf{Fr_p}; x)$:

Case1: If $u_i, u_j \in U$, then $D(u_i, u_j) = 4$, for all $i = 1,2, \dots, m-1\ and\ j = i+1, \dots, m$, hence, $\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} D(u_i, u_j, \mathbf{Fr_p}; x) = \frac{m(m-1)}{2}x^4$.

Case2: If $v_i, v_j \in V$, for all $i = 1, \dots, 2m-1$ *and* $j = i+1, \dots, 2m$, then there are two subcase:

Subcase I: If v_i and v_j are in the same cycle, then $D(v_{2i+1}, v_{2i+2}) = 2$, for all $i = 0,1, \dots, m-1$.

Subcase II: If v_i and v_j are not in the same cycle, then $D(v_{2i+1}, v_j) = D(v_{2i+2}, v_j) = 6$, for all $i = 0,1, \dots, m-2$ *and* $j = 2i+3, \dots, 2m$.

Hence,

$$\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} D(v_i, v_j, \mathbf{Fr_p}; x) = \sum_{i=0}^{m-1} D(v_{2i+1}, v_{2i+2}, \mathbf{Fr_p}; x) + \sum_{i=0}^{m-2}\sum_{j=2i+3}^{2m}\{D(v_{2i+1}, v_j, \mathbf{Fr_p}; x) + D(v_{2i+2}, v_j, \mathbf{Fr_p}; x)\} = mx^2 + 2m(m-1)x^6.$$

Case3: If $u_i \in U$ and $v_j \in V$, for all $i = 1,2, \dots, m$ and $j = 1,2, \dots, 2m$, then there are two subcases:

Subcase I: $D(u_i, v_j) = 3$ If u_i *is adjacent to* v_j, for all $i = 1, \dots, m$ *and* $j = 2i-1, 2i$, then

$$\sum_{i=1}^{m}\sum_{j=2i-1}^{2i} D(u_i, v_j, \boldsymbol{Fr_p}; x) = 2mx^3$$

Subcase II: $D(u_i, v_j) = 5$ If u_i not adjacent to v_j, for all $i = 1, \dots, m$ *and* $j = 1, \dots, 2m$, then

$$\sum_{i=1}^{m}\sum_{j=1}^{2m} D(u_i, v_j, \mathbf{Fr_p}; x) = 2m(m-1)x^5.$$

Hence,

$$\sum_{i=1}^{m}\sum_{j=i+1}^{2m} D(u_i, v_j, \mathbf{Fr_p}; x) = \sum_{i=1}^{m}\sum_{j=2i-1}^{2i} D(u_i, v_j, \mathbf{Fr_p}; x) + \sum_{i=1}^{m}\sum_{j=1}^{2m} D(u_i, v_j, \mathbf{Fr_p}; x) = 2mx^3 + 2m(m-1)x^5.$$

Case 4: If $u_i \in U$ and $v_j \in V$, for all $i = 1,2, \dots, m$ and $j = 1,2, \dots, 2m$, then there are two subcases:

Subcase I: $D(c, u_i) = 2$, and $\sum_{i=1}^{m} D(c, u_i, \mathbf{Fr_p}; x) = mx^2$

Subcase II: $D(c, v_j) = 3$, and $\sum_{j=1}^{2m} D(c, v_j, \mathbf{Fr_p}; x) = 2mx^3$

Hence, $\sum_{i=1}^{m} D(c, u_i, \mathbf{Fr_p}; x) + \sum_{j=1}^{2m} D(c, v_j, \mathbf{Fr_p}; x) = mx^2 + 2mx^3$

Since $D(\mathbf{Fr_p}; x) = \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} D(u_i, u_j, \mathbf{Fr_p}; x) + \sum_{i=1}^{2m-1}\sum_{j=i+1}^{2m} D(v_i, v_j, \mathbf{Fr_p}; x) + \sum_{i=1}^{m}\sum_{j=1}^{2m} D(u_i, v_j, \mathbf{Fr_p}; x) + \sum_{i=1}^{m} D(c, u_i, \mathbf{Fr_p}; x) + \sum_{j=1}^{2m} D(c, v_j, \mathbf{Fr_p}; x)$,

then, $D(\mathbf{Fr_p}; x) = \frac{m(m-1)}{2}x^4 + mx^2 + 2m(m-1)x^6 + 2mx^3 + 2m(m-1)x^5 + mx^2 + 2mx^3$

$= 2mx^2 + 4mx^3 + \frac{m(m-1)}{2}x^4 + 2m(m-1)x^5 + 2m(m-1)x^6$. ■

Corollary 2.2.3: For m≥ 2, then

1. $D(Fr_p) = 24m^2 - 8m$.

2. $\mu_D(\boldsymbol{Fr_p}) = \frac{16m(3m-1)}{3m(3m+1)}$. ■

2.3 Double Cycles Graph:

Definition 2.3.1: A **Double Cycles Graph** $\boldsymbol{DC_{2p}}$, p≥ 3, is a graph construct from two cycles C_p^i, $i = 1,2$ such that $V(C_p^1) = \{v_2, v_4, v_6, \dots, v_{2p}\}$ and $V(C_p^2) = \{v_1, v_3, v_5, \dots, v_{2p-1}\}$, with $2p$ additional edges $\{v_i v_{i+1}, v_i v_{i-1}: i = 3,4, \dots, 2p-1\} \cup \{v_1 v_{2p}, v_1 v_2\}$, as depicted in Fig.3.

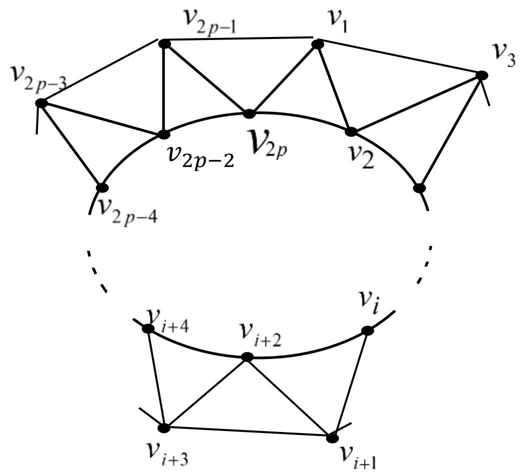

Fig.3. A graph $\boldsymbol{DC_{2p}}$.

Some Properties of the graph $\boldsymbol{DC_{2p}}$:

1. ***The order and size property:*** $p(\boldsymbol{DC_{2p}}) = q(\boldsymbol{DC_{2p}}) = 2p, p \geq 3$.
2. ***Diameter and radius property:*** The graph $\boldsymbol{DC_{2p}}$ has $\delta_D(\boldsymbol{DC_{2p}}) = r_D(\boldsymbol{DC_{2p}}) = 2p - 1$.
3. ***Minimum detour distance property:*** $m_D(\boldsymbol{DC_{2p}}) = 2p - 1$.
4. ***Symmetric graph property:*** The graph $\boldsymbol{DC_{2p}}$ is a symmetric graph.
5. ***The vertices degree:*** Each vertex in the graph $\boldsymbol{DC_{2p}}$ has degree 4.
6. ***The detour distance property:*** $D(v_i, v_j) = 2p - 1$, for all $i, j = 1,2, \dots, p, i \neq j$.
7. ***Detour peripheral property:*** $P_D(\boldsymbol{DC_{2p}}) = V(\boldsymbol{DC_{2p}})$.
8. ***Detour central property:*** $C_D(\boldsymbol{DC_{2p}}) = V(\boldsymbol{DC_{2p}})$.

The detour polynomials of graph $\boldsymbol{DC_{2p}}$ are sought out in the next theorem:

Theorem 2.3.2: The detour polynomial of $\boldsymbol{C_{2p}}$, $p \geq 3$, is:

$$D(\boldsymbol{DC_{2p}}; x) = p(2p-1)x^{2p-1} .$$

Proof: For p≥ 3, since the graph $\boldsymbol{DC_{2p}}$ is 4 −regular and the detour distance between any two vertices $v_i, v_j \in V(\boldsymbol{DC_{2p}})$ is $2p - 1$, for all $i, j = 1,2, \dots, p, i \neq j$, this means that $D(v_i, v_j) = 2p - 1$.

Since, $(\boldsymbol{DC_{2p}}; x) = \sum_{i=1}^{2p-1} \sum_{j=i+1}^{2p} D(v_i, v_j, \boldsymbol{DC_{2p}}; \boldsymbol{x})$,

then, $D(\boldsymbol{DC_{2p}}; x) = \sum_{i=1}^{2p-1} \sum_{j=i+1}^{2p} x^{2p-1} = p(2p-1)x^{2p-1}$.

Hence, the proof is completed. ∎

Corollary 2.3.3: For all $p \geq 3$, we have:

1. $D(\boldsymbol{DC_{2p}}) = p(2p-1)^2$.
2. $\mu_D(\boldsymbol{DC_{2p}}) = 2p-1$. ∎

2.4. Double Paths Graph:

Definition 2.4.1: A **Double Paths Graph** $\boldsymbol{DP_{2p}}$, $p \geq 3$, is a graph construct from two paths P_p^i, $i = 1,2$ such that $V(P_p^1) = \{v_2, v_4, v_6, \dots, v_{2p}\}$ and $V(P_p^2) = \{v_1, v_3, v_5, \dots, v_{2p-1}\}$, with $2p-2$ additional edges $\{v_i v_{i+1}: i = 1,2,\dots,2p-1\}$, as depicted in Fig. 4.

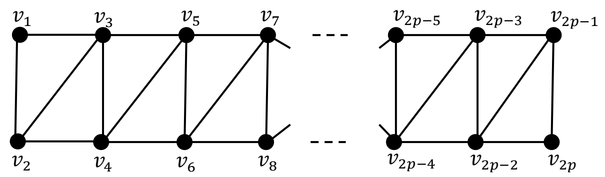

Fig. 4. A double paths graph $\boldsymbol{DP_{2p}}$

Some Properties of the Double Paths Graph $\boldsymbol{DP_{2p}}$:

1. ***The order and size:*** $p(\boldsymbol{DP_{2p}}) = 2p$ and $q(\boldsymbol{DP_{2p}}) = 4p-3$, $p \geq 2$.
2. ***Diameter and radius property:*** The graph $\boldsymbol{DP_{2p}}$ has $\delta_D(\boldsymbol{DP_{2p}}) = r_D(\boldsymbol{DP_{2p}}) =$ 2p − 1.
3. ***Minimum detour distance :*** $m_D(\boldsymbol{DP_{2p}}) = p$.
4. ***Symmetric property:*** The graph $\boldsymbol{DP_{2p}}$ is not symmetric.
5. ***Degree vertices:***

- $deg(v_i) = 2, i = 1,2p$.
- $deg(v_i) = 3, i = 2,2p-1$.
- $deg(v_i) = 4, for\ all\ i = 3,\dots,2p-2$.

6. ***The Detour distance:***

- $D(v_i, v_{i+1}) = 2p-i$ for $i = 1,\dots,p$ and $D(v_i, v_{i+1}) = i$ for i= $p+1,\dots,2p-1$
- $D(v_i, v_j) = 2p-1\ for\ i = 1,\dots,2p-2\ and\ j = i+2,\dots,2p$.

7. ***Detour peripheral of $\boldsymbol{DP_{2p}}$:*** $P_D(\boldsymbol{DP_{2p}}) = V(\boldsymbol{DP_{2p}})$.
8. ***Detour central of $\boldsymbol{DP_{2p}}$:*** $C_D(\boldsymbol{DP_{2p}}) = V(\boldsymbol{DP_{2p}})$.

The detour polynomial of the Double Paths graph $\boldsymbol{DP_{2p}}$ is sought out in the next theorem:

Theorem 2.4.2: The detour polynomial of $\boldsymbol{DP_{2p}}$, p≥ 3, is:

$$D(\boldsymbol{DP_{2p}}; x) = x^p + (2p-1)(2p-2)x^{2p-1} + 2\sum_{i=1}^{p-1} x^{2p-i} .$$

Proof: From Fig.4 we show that $D(v_i, v_{i+1}) = 2p-i, for\ i = 1,2,\dots,p$ and $D(v_i, v_{i+1}) = i, for\ i = p+1,,\dots,2p-1$, then

$$\sum_{i=1}^{2p-1} D(v_i, v_{i+1}, \boldsymbol{DP_{2p}}; \boldsymbol{x}) = 2\sum_{i=1}^{p-1} x^{2p-i} + x^p,$$

and $D(v_i, v_j) = 2p-1, for\ i = 1,2,\dots,2p-2\ and\ j = i+2,\dots,2p$, then

$\sum_{i=1}^{2p-2}\sum_{j=i+2}^{2p} D(v_i, v_{i+1}, \boldsymbol{DP_{2p}}; \boldsymbol{x})$=$\sum_{i=1}^{2p-2}\sum_{j=i+2}^{2p} x^{2p-1}$=$(2p-1)(2p-2)x^{2p-1}$.

Since, $(\boldsymbol{DP_{2p}}; x) = \sum_{i=1}^{2p-1}\sum_{j=i+1}^{2p} D(v_i, v_j, \boldsymbol{DP_{2p}}; \boldsymbol{x})$,

hence, $D(\boldsymbol{DP_{2p}}; x) = \sum_{i=1}^{2p-1} D(v_i, v_{i+1}, \boldsymbol{DP_{2p}}; \boldsymbol{x}) + \sum_{i=1}^{2p-2}\sum_{j=i+2}^{2p} D(v_i, v_{i+1}, \boldsymbol{DP_{2p}}; \boldsymbol{x})$

$= 2\sum_{i=1}^{p-1} x^{2p-i} + x^p + (2p-1)(2p-2)x^{2p-1}$. ■

Corollary 2.3.3: For all $p \geq 3$, we have:

1. $D(\boldsymbol{DP_{2p}}) = 8p^3 - 13p^2 + 8p - 2$.
2. $\mu_D(\boldsymbol{DP_{2p}}) = \frac{8p^3-13p^2+8p-2}{p(2p-1)}$. ■

Conclusion:

From this paper, we concluded the notion of detour distance between any two vertices in a connected graph. We also studied the properties of this distance. In particular, we focused on some certain graphs to calculate the detour polynomial and detour index, and mentioned to some properties of this graphs based to the detour distance.

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