Generalization on Differential Subordination with a Subclass of Harmonic Univalent Functions

Nakeertha Nagalaxmi [1] , Prof. V.Srinivas[2]
[1]Research scholar, Dept. of Mathematics, Dr. Braou, Hyderabad,
[2]Professor in Mathematics
Dr.Braou, Hyderabad

Article Info
Page Number: 13519 - 13528
Publication Issue:
Vol 71 No. 4 (2022)

Article History
Article Received: *15October2022*
Revised: *24November 2022*
Accepted: *18December 2022*

Abstract
The new class of harmonic functions that are defined in this work through the usage of subordination is being introduced. For this class of functions we find compactness, convexity, and radii of starlikeness, together with necessary and sufficient requirements. We may also derive distortion theorems and coefficient estimates for this class of functions by employing extreme point theory.
Key words: harmonic function, univalent function, modified operators, subordination, starlike functions.

Introduction

By studying the class SH and its geometric subclasses in 1984, Clunie and Sheil-Small [1] and Jahangiri and Silverman [5] were able to determine certain coefficient limitations. Afterwards, numerous papers on SH and its subclasses have been published. The current study, which is a follow-up to previous research, examines how sense-preserving, univalent, and nearly convex harmonic functions can be constructed by using the Alexander integral transforms of specific analytic functions (that is, those that are starlike or convex of positive order) as starting points. The co-analytic portion of f must be identically zero for SH to reduce to the class S of normalized analytic univalent functions in U.

Let H represent as follows: the family of continuous complex-valued harmonic functions

1. in the open unit disk, which are harmonic

$$U = \{z : z \in C \text{ and } |z| < 1\}$$

2. Assume that A is the subclass of H that consists of functions that have analytic properties within U. Since h and g are members of A, a function harmonic in U can be expressed as f = h + $\bar{g}$.
3. For f to be locally univalent and sense-preserving in U,$|h'(z)| > |g'(z)|$.
4. Without losing generality, we can write

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=2}^{\infty} b_k z^k . \qquad (1.1)$$

5. Define SH as the family of harmonic, univalent, and sense-preserving functions $f = h + \bar{g}$ in U, where $f(0) = f_z(0) - 1 = 0$

SH^0 is a subclass of SH that includes all functions with the condition

$$f_{\bar{z}}(0) = b_1 = 0 .$$

Assume SH reduces to the class S of normalized analytic univalent functions in U, if the Co-analytic portion of f is exactly zero.

I. For f ∈ S, the differential operator D^n $(n \in \mathbb{N}_0 = N \cup \{0\})$ of f was introduced by Salagean [9].

II. For $f = h + \bar{g}$ given by (1.1), Jahangiri et al. [8] defined the modified Salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)} \quad (1.2)$$

Where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k ,$$

And

$$D^n g(z) = \sum_{k=2}^{\infty} k^n b_k z^k. \quad (n=1,2\ldots) \quad (1.3)$$

One can write f(z) ≺ g(z), where g: U → C is the subordinate function of

f: U → C.

I. If a complex-valued function w exists that maps U into itself and has w (0) = 0, such that

f(z) = g(w(z)) for every z in U. (1.4)

II. if the function g is univalent in U, then we have the following equivalence:

f (z) ≺ g (z) ⇔ f (0) = g (0) and f(U) ⊂ g(U). (1.5)

III. The Hadamard product (or convolution) of functions f_1 and f_2 of the form

$$f_t(z) = z + \sum_{n=2}^{\infty} a_{t,n} z^n + \sum_{n=2}^{\infty} \overline{b_{t,n} z^n}. \ (z \in U, t \in \{1, 2\}) \quad (1.6)$$

is defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{1,n} a_{2,n} z^n + \sum_{n=2}^{\infty} \overline{b_{1,n} b_{2,n} z^n}. (z \in \mathbb{U}). \quad (1.7)$$

Denote by $SH_\delta^0(n, A, B)$ the subclass of SH^0 consisting of functions f of the form (4.1) that satisfy the situation

$$\frac{D_{\mathcal{H}} f(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (-B \le A < B \le 1) \quad (1.8)$$

where $D_H f(z)$ is defined by (4.3).

2. Main results

A necessary and sufficient convolution condition for the harmonic functions in $SH_\delta^0(n, A, B)$ is given by the first theorem.

Theorem 2.1:

A function f belongs to the class $SH_\delta^0(n, A, B)$if and only if f $f \in S_H^0$ and

$$f(z) * \varphi(z ; \xi) \neq 0 \ (\xi \in \mathbb{C}, |\xi| = 1) \quad (1.9)$$

where

$$\varphi(z; \xi) = \frac{(B-A)\xi z + (1+A\xi)z^2}{(1-z)^2} - \frac{\{2+(A+B)\xi\}\bar{z} - (1+A\xi)\bar{z}^2}{(1-\bar{z})^2} \quad z \in \mathbb{U} \quad (1.10)$$

Proof:

Let $f \in S_H^0$. Then $SH_\delta^0(n, A, B)$ if and only if $\frac{D_{\mathcal{H}} f(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}$, (1.11)

Or equivalently

$$\frac{D_{\mathcal{H}} f(z)}{f(z)} \neq \frac{1 + A\xi}{1 + B\xi}, \ (\xi \in \mathbb{C}, |\xi| = 1).$$

Since

$f = h + \overline{g}$ for some analytic functions h, g and

$$zh'(z) = h(z) * \frac{z}{(1-z)^2}, h(z) = h(z) * \frac{z}{1-z} \quad (1.12)$$

We have

$(1 + B\xi)\, D_{\mathcal{H}} f(z) - (1 + A\xi) f(z)$

$= (1 + B\xi)\, zh'(z) - (1 + A\xi)h(z) - [\,(1 + B\xi)\,\overline{zg'(z)} + (1 + A\xi)\overline{g(z)}$

$$= h(z) * \left(\frac{(1 + B\xi)z}{(1-z)^2} - \frac{(1 + A\xi)z}{1-z}\right) - \overline{g(z)} * \left(\frac{(1 + B\xi)\bar{z}}{(1-z)^2} + \frac{(1 + A\xi)\bar{z}}{1-z}\right) \quad (1.13)$$

$= f(z) * \varphi(z\,;\,\xi)$

Thus, (1.9) and (1.11) are equal hence the proof is completed

After that, we provide the adequate coefficient bounded for the function in $SH_{\delta}^{0}(n, A, B)$. The following corollary results from solving Theorem 1 for B= -A= 1

Corollary 2.2:

A function f belongs to the class $SH_{\delta}^{0}(n, A, B)$if and only if $f \in SH^{o}$ and

$$f(z) * \varphi(z\,;\,\xi) \neq 0 \ (\xi \in C, |\xi| = 1) \quad (1.14)$$

where

$$\varphi(z;\xi) = \frac{2\xi z + (1-\xi)z^2}{(1-z)^2} - \frac{2\bar{z} - (1+\xi)\bar{z}^2}{(1-\bar{z})^2},\ z \in \mathbb{U} \quad (1.15)$$

Next we give the sufficient coefficient bounded for function in SH_{δ}^{0}(n, A, B).

Theorem 2.3:

A function $f = h + \bar{g}$ be so that h and g are given by

$$f(z) = \sum_{n=0}^{\infty} (a_n z^n + \overline{b_n z^n})\ (a_1 = 1, |b_1| < 1, \quad z \in \mathbb{U})$$

Then f $\in SH_{\delta}^{0}(n, A, B)$ if

$$\sum_{n=1}^{\infty} (|\gamma_n|\,|a_n| + |\delta_n|\,|b_n|) \leq 2(B - A) \quad (1.16)$$

Where

$$\gamma_n = (n(1 + B) - (1 + A))$$
$$\delta_n = (n(1 + B) + (1 + A)) \quad (1.17)$$

Proof :

For $f(z) = z$, the theorem's validity is readily apparent. Accordingly, for n≥ 2, we assume that $a_n \neq 0, b_n \neq 0$. Since

$$|h'(z)| > |g'(z)|\ (z \in \mathbb{U}).$$

By $|u_n| \geq 1, |v_n| \geq 1$
we obtain

$$\frac{|\gamma_n|}{B-A} \geq n, \frac{|\delta_n|}{B-A} \geq n \ (n = 2, 3, ...)$$

Classes of harmonic functions that are defined by convolution are obtained by (1.16)

$$\sum_{n=2}^{\infty} (n|a_n| + n|b_n|) \leq 1 - b_1 \quad (1.18)$$

Moreover

$|h'(z)| - |g'(z)|$

$\geq 1 - b_1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} + \sum_{n=2}^{\infty} n|b_n||z|^{n-1}$

$$\geq 1 - b_1 - \sum_{n=2}^{\infty} n(|a_n| + |b_n|)|z|^{n-1}$$

$\geq 1 - b_1 - |z| \sum_{n=2}^{\infty}(n|a_n| + n|b_n|)(1 - b_1)(1 - |z|)$ (1.19)
$\geq 0 \; (z \in \mathbb{U})$

Thus, by $|h'(z)| > |g'(z)| \; (z \in \mathbb{U})$. The function f is locally univalent as well as sense preserving in $\mathbb{U}$. Also, for $z_1, z_2 \epsilon \mathbb{U}, z_1 \neq z_2$ we have

$\left|\frac{z_1^n - z_2^n}{z_1 - z_z}\right| = \left|\sum_{k=1}^{n} z_1^{k-1} z_2^{n-k}\right|$

$\leq \sum_{k=1}^{n} |z_1|^{k-1} |z_2|^{n-k} < n \; (n = 2,3, \ldots)$. (1.20)
Hence, by (1.18) we get
$|f(z_1) - f(z_2)|$
$\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)|$
$$\geq \left|z_1 - z_2 + \sum_{n=2}^{\infty} a_n (z_1^n - z_2^n)\right| - \left|\sum_{n=2}^{\infty} \overline{b_n (z_1^n - z_2^n)}\right|$$
$$\geq |z_1 - z_2| - \sum_{n=2}^{\infty} |a_n| |z_1^n - z_2^n|) - \sum_{n=2}^{\infty} |b_n| \, |z_1^n - z_2^n|$$
$= |z_1 - z_2| \left(1 - b_1 - \sum_{n=2}^{\infty} |a_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| - \sum_{n=2}^{\alpha} |b_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_z}\right|\right)$ (1.21)
$> |z_1 - z_2|(1 - b_1 - \sum_{n=2}^{\infty}(n|a_n| + n|b_n|))$
≥ 0

Which proves univalent. On the other $f \in SH_\delta^0(n, A, B)$ if and only if there exists a complex-valued function

$$\omega, \omega(0) = 0, |\omega(z)| < 1 \; (z \in \mathbb{U})$$

such that

$\frac{D_{\mathcal{H}} f(z)}{f(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \; (z \in \mathbb{U})$ (1.22)

or equivalently

$\left|\frac{D_{\mathcal{H}} f(z) - f(z)}{BD_{\mathcal{H}} f(z) - Af(z)}\right| < 1 \; (z \in \mathbb{U}).$ (1.23)

Thus, it suffices to demonstrate that

$$|D_{\mathcal{H}} f(z) - f(z)| - |BD_{\mathcal{H}} f(z) - Af(z)| < 0$$

$(z \in \mathbb{U} \setminus \{0\})$.

Indeed, letting $|z| = r \; (0 < r < 1)$ we have

$|D_{\mathcal{H}} f(z) - f(z)| - |BD_{\mathcal{H}} f(z) - Af(z)|$

$= \left|\sum_{n=2}^{\infty}(n-1) a_n z^n - \sum_{n=1}^{\infty}(n+1) \bar{b}_n \bar{z}^n\right| - \left|(B - A)z + \sum_{n=2}^{\infty}(Bn - A) a_n z^n - \sum_{n=1}^{\infty}(Bn + A)) \bar{b}_n \bar{z}^n\right|$

$\leq \sum_{n=2}^{\infty}(n-1)|a_n| r^n + \sum_{n=1}^{\infty}(n+1)|b_n| r^n - (B-A) r + \sum_{n=2}^{\infty}(Bn - A)|a_n| r^n + \sum_{n=1}^{\infty}(Bn + A)|b_n| r^n$

$\leq r \left\{ \sum_{n=1}^{\infty}(|\gamma_n| \, |a_n| + |\delta_n| \, |b_n|) \, r^{n-1} - 2(B - A)\right\} < 0.$

hence $f \in SH_\delta^0(n, A, B)$ and so the proof is complete.

Motivated by Silverman [10] we denote by $\mathfrak{J}^\lambda$ $\{\lambda \in (0,1)\}$ the class of functions $f \in \mathcal{H}_0$ of form

$$f(z) = \sum_{n=1}^{\infty}(a_n z^n + \overline{b_n z^n})$$
$$(a_1 = 1, |b_1| < 1, \; z \in \mathbb{U})$$

such that $a_n = -|a_n|$, $b_n = (-1)^\lambda|b_n|$ ($n = 2, 3, \ldots$); that is,

$$f = h + \bar{g},$$

$$h(z) = z - \sum_{n=2}^{\infty}|a_n|z^n$$

$$g(z) = (-1)^\lambda \sum_{n=1}^{\infty}|b_n|\bar{z}^n, (z \in \mathbb{U}). \qquad (1.24)$$

Or

$$f = z - \sum_{n=2}^{\infty}|a_n|z^n + (-1)^\lambda \sum_{n=1}^{\infty}|b_n|\bar{z}^n,$$

Define $R_{\mathrm{H}}(n, A, B)$ which denotes the class of functions $f \in$SH that

$$\frac{D_{\mathcal{H}}f(z)}{z} = \frac{1 + A\omega(z)}{1 + B\omega(z)}$$

Moreover, let us define

$\mathcal{S}_{\mathcal{J}}^{0}$(n, A, B) = $\mathfrak{I}^0 \cap \mathcal{S}_{\mathcal{H}}^{0}$ (n, A, B),
$\mathcal{S}_{\mathcal{J}}^{*}$(n, A, B) = $\mathfrak{I}^0 \cap \mathcal{S}_{\mathcal{H}}^{*}$ (n, A, B),
$\mathcal{S}_{\mathcal{J}}^{c}$(n, A, B) = $\mathfrak{I}^1 \cap \mathcal{S}_{\mathcal{H}}^{c}$ (n, A, B),

Now, we show that condition (1.16) is also the sufficient condition for a function $f \in \mathfrak{I}^0$ to be in class $S_{\mathfrak{I}}^{0}(n, A, B)$.

Theorem 2.4

A function $f = h + \bar{g}$ be so that h and g are given by

$$f(z) = \sum_{n=0}^{\infty}(a_n z^n + \overline{b_n z^n}) \ (a_1 = 1, |b_1| < 1 \ z \in \mathbb{U})$$

Then $f \in SH_{\delta}^{0}(n, A, B)$ if

$\sum_{n=1}^{\infty}(|\gamma_n||a_n| + |\delta_n||b_n|) \leq 2(B - A)$

Where $\gamma_n = (n(1 + B) - (1 + A))$

$\delta_n = (n(1 + B) + (1 + A))$

Proof :

With respect to Theorem 2.3,

We have to prove “only if” part, i.e., each function from the class $SH_{\delta}^{0}(n, A, B)$ satisfies the condition (1.23). If $f \in \mathcal{SH}_{\delta}^{0}(n, A, B)$, then it is of the form (1.1) with (1.7) and satisfies (1.16) or equivalently

$$\left|\frac{\sum_{n=1}^{\infty}\{(n-1)|a_n|z^n+(n+1)|b_n|\bar{z}^n\}}{2(B-A)z-\sum_{n=1}^{\infty}\{(Bn-A)|a_n|z^n+(Bn+A)|b_n|\bar{z}^n\}}\right| < 1, (z \in \mathbb{U})$$

For z = r (0 ≤ r < 1), we have

$$\frac{\sum_{n=1}^{\infty}\{(n-1)|a_n|+(n+1)|b_n|r^{n-1}\}}{2(B-A)-\sum_{n=1}^{\infty}\{(Bn-A)|a_n|+(Bn+A)|b_n|r^{n-1}\}} < 1 \qquad (1.25)$$

It is clear that the denominator of the left-hand side cannot vanish for r ∈ (0, 1). Moreover, it is positive for r = 0, and in consequence for r ∈ (0, 1).

we have $\{\sum_{n=2}^{\alpha}(|\gamma_n||a_n| + |\gamma_n||b_n|)\} \leq 2(B - A)$ (0 < r < 1). (1.26)

The sequence of partial sums $\{S_n\}$ associated with the series

$\sum_{n=2}^{\infty}(|\gamma_n|\,|a_n| + |\delta_n|\,|b_n|)$ is a nondecreasing sequence. Moreover, by (1.26) it is bounded by B − A. Hence, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=2}^{\infty}(|\gamma_n|\,|a_n| + |\delta_n|\,|b_n|)$$

$$= \lim_{n\to\infty} s_n$$

$$\leq B - A \qquad (1.27)$$

which yields the assertion $\sum_{n=1}^{\infty}(|\gamma_n|\,|a_n| + |\delta_n|\,|b_n|) \leq 2(B - A)$

The following result may be proved in the same way as Theorem 2.4.

Corollary 2.5

A function f belongs to the class $SH_\delta^0(n, A, B)$ if and only if

$$D_{SH_\delta^0(n,A,B)} f(z) * \varphi(z\,;\,\xi) \neq 0\ (\xi \in C, |\xi| = 1) \qquad (1.28)$$

Where $\varphi(z\,;\,\xi)$ is defined by 1.10

Corollary 2.6

A function $f \in SH_\delta^0(n, A, B)$ of form

$$f(z) = \sum_{n=0}^{\infty}(a_n z^n + \overline{b_n z^n})\ (a_1 = 1, |b_1| < 1,\ Z \in \mathbb{U})$$

Satisfies the condition $\sum_{n=2}^{\alpha}(|\gamma_n|\,|a_n| + |\delta_n|\,|b_n|) \leq 2(B - A)$ $\quad Z \in \mathbb{U}$
(1.29)

Corollary 2.7

A function $f \in \mathcal{I}^o$ belongs to the class $SH_\delta^0(n, A, B)$ if and only if $f \in S_H^0$ and

$$f(z) * \varphi(z\,;\,\xi) \neq 0\ (\xi \in \mathbb{C}, |\xi| = 1) \qquad (1.30)$$

where

$$\sum_{n=2}^{\infty}(|\gamma_n|\,|a_n| + |\delta_n|\,|b_n|) \leq 2(B - A) \qquad z \in \mathbb{U} \qquad (1.31)$$

Corollary 2.8

A function $f \in \mathcal{I}^o$ and $f \in \mathcal{I}'$ be a function of form $f = h + \bar{g}$ if and only if $f \in SH^o$ and

$$f = h + \bar{g},$$

$$h(z) = z - \sum_{n=2}^{\infty}|a^n|z^n$$

$$g(z) = (-1)^{\lambda}\sum_{n=2}^{\infty}|b^n|\bar{z}^n, (z \in \mathbb{U}). \qquad (1.32)$$

Then (1.1) $f \in \mathcal{R}_{\mathcal{I}}(A, B)$ if and only

I. if $\sum_{n=1}^{\alpha} n(|a_n| + |b_n|) \leq \frac{2B - A + 1}{1 + B}$

II. $\sum_{n=1}^{\alpha} n^2(|a_n| + |b_n|) \leq \frac{2B - A + 1}{1 + B}$

3. Extreme points

In $S\mathcal{H}_\delta^0$(n, A, B), the standard topology is represented by a metric that states that a sequence $\{f_n\}$ converges to f if and only if it does so uniformly on every compact subset of U. Weierstrass and Montel's theorems imply that this topological space is complete.

Definition 3.1:

Let $\mathcal{F}$ be a subclass of the class $S\mathcal{H}T_\delta^0$. A function $f \in \mathcal{F}$ is called an extreme point of F if the condition

$$f = \gamma f_1 + (1 - \gamma) f_2\ (f_1, f_2 \in \mathcal{F}, 0 < \gamma < 1) \qquad (1.33)$$

implies $f_1 = f_2 = f$. We shall use the notation $E\,\mathcal{F}$ to denote the set of all extreme points of $\mathcal{F}$. It is clear that $E\,\mathcal{F} \subset \mathcal{F}$

We say that $\mathcal{F}$ is locally uniformly bounded if for each r, 0 < r < 1, there is a real constant M = M (r) so that

$$|f(z)| \leq M\ (f \in \mathcal{F}, |z| \leq r). \qquad (1.34)$$

Definition 3.2:

A class $\mathcal{F}$ is deemed convex if

$$\gamma f + (1-\gamma)\, g \in \mathcal{F}\ (f, g \in \mathcal{F}, 0 \leq \gamma \leq 1). \quad (1.35)$$

Additionally, the intersection of every closed convex subset of H that contains $\mathcal{F}$ is defined as the closed convex hull of $\mathcal{F}$. The closed convex hull of $\mathcal{F}$ is indicated by $\overline{co}\mathcal{F}$.

A real-valued functional $\mathcal{J}: \mathcal{H} \to \mathcal{R}$ is called convex on a convex class $\mathcal{F} \subset \mathcal{H}$ if

$$\mathcal{J}\,(\gamma f + (1-\gamma) g) \leq \gamma \mathcal{J}(f) + (1-\gamma)\, \mathcal{J}(g)\ (f, g \in \mathcal{F}, 0 \leq \gamma \leq 1). \quad (1.36)$$

The Krein–Milman theorem is fundamental in the theory of extreme points. In particular, it implies the following lemma.

Lemma 3.1

Let $\mathcal{F}$ be a non-empty compact convex subclass of the class $\mathcal{H}$ and $\mathcal{J}: \mathcal{H} \to \mathcal{R}$ be a real-valued, continuous and convex functional on $\mathcal{F}$. Then

$$max\,\{\mathcal{J}\,(f) : f \in \mathcal{F}\} = max\,\{\mathcal{J}\,(f) : f \in E\,\mathcal{F}\}. \quad (1.37)$$

Since $\mathcal{H}$ is a complete metric space, Montel's theorem implies the following lemma.

Lemma 3.2

A class $\mathcal{F} \subset H$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded.

Lemma 3.3:

A class $\mathcal{SH}\mathrm{T}_{\delta}^{0}$is non empty compact subset of the class $\mathcal{H}$ then $\mathcal{SH}\mathrm{T}_{\delta}^{0}$ (n, A, B) is non –empty and $\overline{co}\mathcal{SH}\mathrm{T}_{\delta}^{0} = \overline{co}\ \mathcal{SH}_{\delta}^{0}$. (1.38)

4. Radii of starlikeness and convexity

We say that a function $f \in \mathcal{H}_0$ is starlike of order α in $\mathbb{U}$(r) if

$$\frac{\partial}{\partial t}\left(arg f(\rho e^{it})\right) > \alpha,\ 0 \leq t \leq 2\pi,\ 0 < \rho < r < 1. \quad (1.39)$$

Analogously, we say that a function $f \in \mathcal{H}_0$ is convex of order α in $\mathbb{U}$(r) if

$$\frac{\partial}{\partial t}\left(arg f(\rho e^{it})\right) > \alpha,\ 0 \leq t \leq 2\pi,\ 0 < \rho < r < 1.$$

It is easy to verify that for a function $f \in T\,(\varphi)$ the condition (1.39) is equivalent to the following

$$Re\ \frac{D_{\mathcal{H}} f(z)}{f(z)} > \alpha\ (z \in \mathbb{U}\,(r)) \quad (1.40)$$

or equivalently

$$\left|\frac{D_{\mathcal{H}}f(z) - (1+\alpha)\, f(z)}{D_{\mathcal{H}}f(z) + (1-\alpha)\, f(z)}\right| < 1\ (z \in \mathbb{U}\,(r)) \quad (1.41)$$

Let $\mathcal{B}$ be a subclass of the class H_0. We define the radius of starlikeness R_{α}^{*} ($\mathcal{B}$)and the radius of convexity R_{α}^{c} ($\mathcal{B}$)for the class $\mathcal{B}$ by

$$R_{\alpha}^{*}(\mathcal{B}) := \inf_{f\in\mathcal{B}}(\sup\{r \in (0,1] : f \text{ is starlike of order in } \mathbb{U}(r)\}),$$

$$R_{\alpha}^{c}(\mathcal{B}) := \inf_{f\in\mathcal{B}}(\sup\{r \in (0,1]\ f \text{ is convex of order in } \mathbb{U}(r)\}),$$

Theorem 4.1

The radius of starlikeness of order α for the class $S_{\mathcal{H}}^{o}$(n; A, B) is given by

$$R_{\alpha}^{*}(S_{\mathcal{H}}^{0}(n; A, B)) = \inf_{n\geq 2}\left\{\frac{1-\alpha}{B-A} min\left\{\frac{|\gamma_n|}{n-\alpha}, \frac{|\delta_n|}{n+\alpha}\right\}\right\}^{\frac{1}{n-1}} \quad (1.42)$$

where γ_n and δ_n are defined by (1.17).

Proof :

Let $f \in S_{\mathcal{H}}^{0}(n; A, B)$ be of the form (1.1) with (1.7). Then, for

|z| = r < 1 we have

$$\left|\frac{D_{\mathcal{H}}f(z)-(1+\alpha)f(z)}{D_{\mathcal{H}}f(z)+(1-\alpha)f(z)}\right| = \left|\frac{-\alpha z+\sum_{n=2}^{\infty}\{(n-1-\alpha)|a_n|z^n-(n+1+\alpha)|b_n|\bar{z}^n\}}{(2-\alpha)z+\sum_{n=2}^{\infty}\{(n+1-\alpha)|a_n|z^n-(n+1+\alpha)|b_n|\bar{z}^n}\right|$$

$$\leq \frac{\alpha+\sum_{n=2}^{\infty}\{(n-1-\alpha)|a_n|r^n+(n+1+\alpha)|b_n|r^{n-1}\}}{2-\alpha-\sum_{n=2}^{\infty}\{(n+1-\alpha)|a_n|r^n+(n+1+\alpha)|b_n|r^{n-1}\}}.$$

Thus, the condition (1.40) is true if and only if

$$\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n|+\frac{n+\alpha}{1-\alpha}|b_n|\right)r^{n-1}\leq 1, \tag{1.43}$$

Theorem 1, we have

$$\sum_{n=2}^{\infty}\left(\frac{|\gamma_n|}{B-A}|a_n|+\frac{|\delta_n|}{B-A}|b_n|\right)\leq 1, \tag{1.44}$$

where γ_n and δ_n are defined by (1.17). Thus, the condition (1.43) is true if

$\frac{n-\alpha}{1-\alpha} r^{n-1} \leq \frac{|\gamma_n|}{B-A}, \frac{n+\alpha}{1-\alpha} r^{n-1} \leq \frac{|\delta_n|}{B-A}$ (n = 2, 3 . . .),

that is, if

$$r \leq \left\{\frac{1-\alpha}{B-A}\min\left\{\frac{|\gamma_n|}{n-\alpha},\frac{|\delta_n|}{n+\alpha}\right\}\right\}^{\frac{1}{n-1}} \text{ (n = 2, 3 . . .).}$$

It follows that the function f is starlike of order α in the disk U (r^*), where r^*

$$r^* = \inf_{n\geq 2}\left\{\frac{1-\alpha}{B-A}\min\left\{\frac{|\gamma_n|}{n-\alpha},\frac{|\delta_n|}{n+\alpha}\right\}\right\}^{\frac{1}{n-1}}. \tag{1.45}$$

The functions γ_n, δ_n of the form (1.17) realize equality in (1.44), and the radius r^* cannot be larger. Thus we have (1.42). The following result may be proved in much the same way as Theorem 4.1.

5 Applications

It is clear that if the class

$F = \{f_n \in H : n \in N\}$ is locally uniformly bounded, then

$$\overline{co}F = \{\sum_{n=1}^{\infty}\gamma_n f_n : \sum_{n=1}^{\infty}\gamma_n = 1,\ \gamma_n \geq 0\ (n\in\mathbb{N})\}. \tag{1.46}$$

Corollary 5.1

$$S_c^{\infty}(n;A,B)=\left\{\sum_{n=1}^{\infty}(\gamma_n h_n+\delta_n g_n):\sum_{n=1}^{\infty}(\gamma_n+\delta_n)=1,\delta_1=0,\delta_n,\gamma_n\geq 0,n\in\mathbb{N}\right\}$$

where h_n, g_n are defined by (1.22).

$$h_n = z-\frac{B-A}{\eta^{n-1}\gamma_n}z^n,$$

$$g_n = z+\frac{B-A}{\eta^{-n-1}\gamma_n}\bar{z}^n \qquad (z\in\mathbb{U}) \tag{1.47}$$

For each fixed value of n ∈ ℕ, z ∈𝕌, the following real-valued functionals are continuous and convex on $\mathcal{H}$:

$\mathcal{J}$ (f) = $|a_n|$, $\mathcal{J}$ (f) = $|b_n|$,

$\mathcal{J}$ (f) = | f (z)| $\mathcal{J}$ (f) = | $D_{\mathcal{H}}$ f (z)| (f ∈ $\mathcal{H}$). (1.48)

Moreover, for γ ≥ 1, 0 < r < 1, the real-valued functional

$$\mathcal{J}(f) = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})^{\gamma}\right| d\theta\right)^{\frac{1}{\gamma}} (f\in\mathcal{H}) \tag{1.49}$$

is also continuous and convex on $\mathcal{H}$.

Therefore, by Lemma **3.1** Theorem 4.2 we have the following corollaries.

Corollary 5.2

Let f ∈ $S_{\mathcal{H}}^0$(n; A, B) be a function of the form (1.1). Then

$$|an| \leq \frac{B-A}{|\gamma_n|}\ |b_n| \leq \frac{B-A}{|\delta_n|}\ (n = 2, 3, . . .), \tag{1.50}$$

where γ_n, δ_n are defined by (1.17). The result is sharp. The functions hn, gn of the form (1.47) are the extremal functions.

Corollary 5.3

Let $f \in S^0_{\mathcal{H}}(n; A, B)$ $|z| = r < 1$

Then

$$r-\frac{B-A}{|u_2|(1+2B-A)} r^2 \le |f(z)| \le r + \frac{B-A}{|u_2|(1+2B-A)} r^2$$

$$r-\frac{2(B-A)}{|u_2|(1+2B-A)} r^2 \le |D_{\mathcal{H}} f(z)| \le r + \frac{2(B-A)}{|u_2|(1+2B-A)} r^2$$

where u_2 are defined by (1.2). The result is sharp. The functions h_n, g_n of the form (1.47) are the extremal functions.

Corollary 5.4

Let $0 < r < 1$, $\gamma \ge 1$. If $f \in S^{\eta}_{\mathcal{H}}(\varphi; A, B)$ then

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f((re^{i\theta}) \right|^{\gamma} d\theta \le \frac{1}{2\pi} \int_0^{2\pi} \left| h_2(re^{i\theta}) \right|^{\lambda} d\theta,$$

$$\frac{1}{2\pi} \int_0^{2\pi} |D_{\mathcal{H}} f(z)|^{\gamma} d\theta \le \frac{1}{2\pi} \int_0^{2\pi} |D_{\mathcal{H}} h_2(z))|^{\gamma} d\theta$$

where h_2 is the function defined by (1.47).

Conclusion

In the chapter new classes of univalent harmonic functions are introduced. Necessary and sufficient conditions for defined classes of functions are obtained. Some topological properties, radii of convexity and starlikeness, and extreme points of the classes are also considered. By using extreme points theory we obtained coefficients estimates, distortion theorems, and integral mean inequalities for these classes of functions.

References

[1] J. Clunie, T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A I Math., 9 (1984) 3-25.
[2] J. Dziok, Classes of harmonic functions defined by subordination, Abstr. Appl. Anal., 2015 Article ID 756928 (2015) 9 pages.
[3] J. Dziok, On Janowski harmonic functions, J. Appl. Anal., 21 (2015) 99-107 .
[4] J. Dziok, J.M. Jahangiri, H. Silverman, Harmonic functions with varying coefficients, J. Inequal. Appl., 139 (2016) 12 pages.
[5] J.M. Jahangiri and H. Silverman, Harmonic close-to-convex mappings, J. Appl. Math. Stochast. Anal., 15 (2002) 23–28.
[6] J.M. Jahangiri , Y.C. Kim and H.M. Srivastava, Construction of a certain class of harmonic close-to-convex functions associated with the Alexander integral transform, Integral Transforms Spec. Funct., 14 (2003) 237-242.
[7] J.M. Jahangiri, Harmonic functions starlike in the unit disk, J. Math. Anal. Appl., 235 (1999) 470-477.
[8] J.M. Jahangiri, G. Murugusundaramoorthy, K. Vijaya, Salagean-type harmonic univalent functions, South J. Pure Appl. Math., 2 (2002) 77-82.

[9] G.S. S˘al˘agean, Subclasses of univalent functions, (Lecture Notes in Math. Springer- Verlag Heidelberg, 1983) 1013, 362-372.

[10] H. Silverman, Harmonic univalent functions with negative coefficients, J. Math. Anal. Appl., 220 (1998) 283-289.

[11] H. Silverman, E.M. Silvia, Subclasses of harmonic univalent functions, N. Z. J. Math.,28 (1999) 275-284. 1 Department of Mathematics, Faculty of A

[12] H. Lewy, "On the non-vanishing of the Jacobian in certain one-to-one mappings," Bulletin of the American Mathematical Society, vol. 42, no. 10, pp. 689–692, 1936