Comprehensive Study of Eulerian and Hamiltonian Graphs

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Abstract

This paper presents an extensive study of Eulerian and Hamiltonian graphs, exploring their definitions, properties, and characterizations. We research into various algorithms designed to identify Eulerian and Hamiltonian paths and cycles within graphs, providing detailed analyses and comparisons of their performance. This study aims to provide a holistic understanding of Eulerian and Hamiltonian graphs and their significance in both theoretical and applied contexts.

Keywords: Eulerian Graphs, Hamiltonian Graphs, Graph Theory, Algorithms, Transportation Planning, Network Analysis.

Introduction

Graph theory is a fundamental area of discrete mathematics with extensive applications across computer science, engineering, biology, and social sciences. Among the various concepts in graph theory, Eulerian and Hamiltonian graphs hold significant importance due to their theoretical richness and practical applicability.

An **Eulerian graph** is one in which there exists a trail that traverses each edge exactly once, known as an Eulerian trail or circuit. This concept traces back to Leonhard Euler's solution to the famous Königsberg bridge problem in 1736, which laid the foundation for graph theory.

A **Hamiltonian graph**, on the other hand, contains a cycle that visits each vertex exactly once, termed a Hamiltonian cycle. This concept is named after Sir William Rowan Hamilton, who studied such cycles in the 19th century through the "Icosian Game."

Understanding the properties and characteristics of these graphs is crucial for solving complex problems in various domains such as route optimization, network design, and DNA sequencing. Efficient algorithms for detecting Eulerian and Hamiltonian paths and cycles enable practical solutions to these problems.

This paper aims to:

1. Explore the definitions, properties, and characterizations of Eulerian and Hamiltonian graphs.
2. Investigate algorithms for finding Eulerian and Hamiltonian paths and cycles.
3. Examine real-world applications of these graphs, particularly in transportation planning and network analysis.

4. Compare and analyze the performance of different algorithms concerning efficiency and computational complexity.

Preliminaries

2.1 Definitions

A **graph** G=(V,E) consists of a finite set of vertices V and a set of edges EEE, where each edge connects two vertices.

Definition 2.1.2 (Trail and Path):

- A **trail** in a graph is a sequence of edges where no edge is repeated.
- A **path** is a trail where all vertices (and thus edges) are distinct.

Definition 2.1.3 (Cycle and Circuit):

- A **cycle** is a path that starts and ends at the same vertex.
- A **circuit** is a trail that starts and ends at the same vertex

Definition 2.1.4 (Degree of a Vertex): The **degree** *deg* (*v*) of a vertex *v* is the number of edges incident to *v*.

Definition 2.1.5 (Connected Graph): A graph is **connected** if there is a path between every pair of vertices.

2.2 Eulerian and Hamiltonian Graphs

Definition 2.2.1 (Eulerian Trail and Circuit):

- An **Eulerian trail** is a trail that uses every edge of the graph exactly once.
- An **Eulerian circuit** is an Eulerian trail that starts and ends at the same vertex.

Definition 2.2.2 (Eulerian Graph):

A graph is called **Eulerian** if it contains an Eulerian circuit.

- **Definition 2.2.3 (Hamiltonian Path and Cycle):**
- A **Hamiltonian path** is a path that visits each vertex of the graph exactly once.
- A **Hamiltonian cycle** is a Hamiltonian path that starts and ends at the same vertex.

Definition 2.2.4 (Hamiltonian Graph):

A graph is called **Hamiltonian** if it contains a Hamiltonian cycle.

Eulerian Graphs: Properties and Characterizations

Necessary and Sufficient Conditions

One of the fundamental results in graph theory provides a simple characterization for Eulerian graphs.

Theorem 3.1 (Euler's Theorem): *A connected undirected graph G is Eulerian if and only if every vertex has an even degree.*

Proof:

Assume *G* is Eulerian. Then there exists an Eulerian circuit traversing every edge exactly qonce and starting and ending at the same vertex.

Consider any vertex *v*. Every time the circuit enters *v*, it must also leave *v*. Thus, edges contribute in pairs to the degree of *v*. Therefore, the degree of v must be even.

If *v* is a vertex in an Eulerian circuit, then deg (*v*) = 2k, where k∈ N.

Assume every vertex of *G* has an even degree. We need to show that *G* contains an Eulerian circuit.

We use induction on the number of edges.

Base case: If GGG has no edges, the theorem trivially holds.

Inductive step:

Choose any cycle *C* in *G*.

Remove the edges of *C* from *G*. The degrees of vertices remain even, and the remaining graph may have multiple connected components.

Apply the induction hypothesis to each connected component, obtaining Eulerian circuits.

Combine these circuits with *C* to form an Eulerian circuit for *G*.

Thus, *G* has an Eulerian circuit.

Since all vertices have even degrees, *G* is Eulerian

Corollary 3.1.2: *A connected undirected graph has an Eulerian trail (but not necessarily a circuit) if and only if exactly two vertices have odd degrees, and all other vertices have even degrees.*

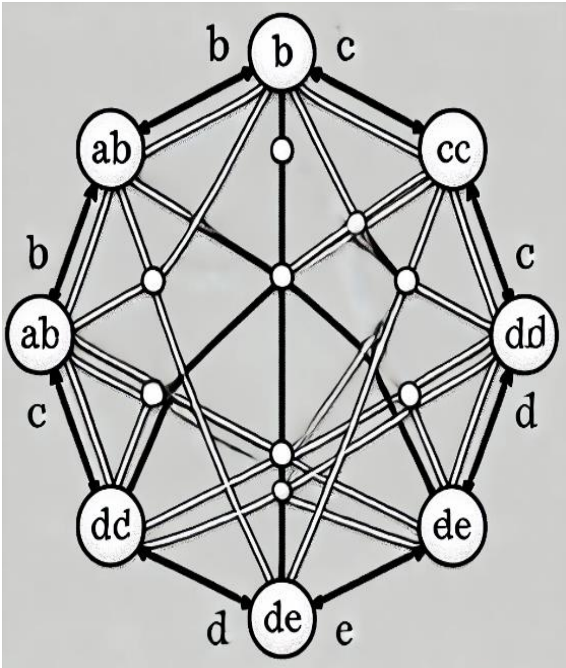

Figure 1. Eulerian Circuit

Proof

If exactly two vertices have odd degrees, we can start an Eulerian trail at one of the odd-degree vertices and end at the other. The proof follows similar reasoning as in Euler's theorem, adjusting for the trail not being a circuit.

Eulerian trail exists if $\sum_{v\in V} \deg(v)$ is even, and exactly two vertices have odd degrees.

3.2. Directed Eulerian Graphs

For directed graphs, the characterization is slightly different.

Definition 3.2.1: In a directed graph, the in-degree $\deg^{-}(v)$ is the number of edges entering v, and the out-degree $\deg^{+}(v)$ is the number of edges leaving v.

Theorem 3.2 (Directed Eulerian Graphs): A connected directed graph G has an Eulerian circuit if and only if for every vertex $v, \deg^{-}(v) = \deg^{+}(v)$.

Proof:

Assume G has an Eulerian circuit. Every time the circuit arrives at v_t it must also depart from v, implying $\deg^{-}(v) = \deg^{+}(v)$.

If v is a vertex in an Eulerian circuit, then $\deg^{-}(v) = \deg^{+}(v)$.

Assume $\deg^{-}(v) = \deg^{+}(v)$ for all v in G. Similar to the undirected case, we can construct an Eulerian circuit by traversing edges while ensuring all are covered and connectivity is maintained.

Since $\deg^{-}(v) = \deg^{+}(v)$ for all ΣV, an Eulerian circuit exists in G.

4. Hamiltonian Graphs: Properties and Characterizations

Unlike Eulerian graphs, there is no simple necessary and sufficient condition for a graph to be Hamiltonian. However, several sufficient conditions exist.

4.1. Dirac's Theorem

Theorem 4.1 (Dirac's Theorem): If G is a simple graph with $n \geq 3$ vertices such that the degree of each vertex $\deg(v) \geq \frac{n}{2}$, then G is Hamiltonian.

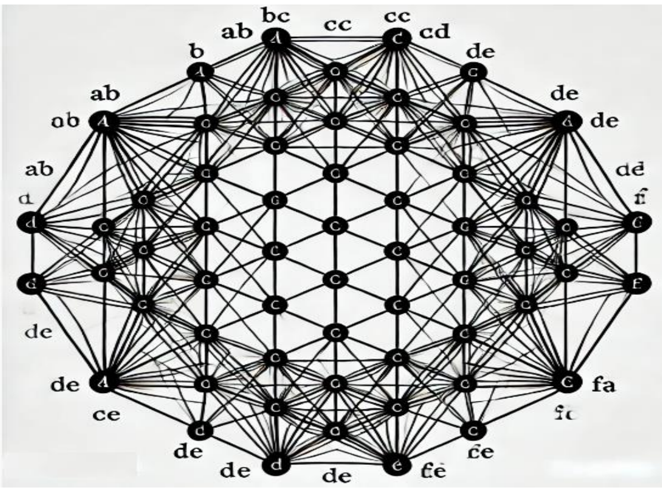

Figure 2. Hamiltonian Cycle

Proof:
Assume G is not Hamiltonian. Consider the longest possible path P in G with distinct vertices $v_1, v_2, \dots, v_k$.

Since P is the longest path, no edges exist between v_1 and v_k. Consider the degrees:

Since deg $(v_1) \geq \frac{n}{2}$, v_1 is connected to at least $\frac{n}{2}$ other vertices.

Similarly, deg $(v_k) \geq \frac{n}{2}$ vertices.

Since there are only n vertices and v_1 and v_k are not connected, there must be a vertex v connected to both v_1 and v_k, contradicting the maximality of P.

Therefore, G must be Hamiltonian.

4.2. Ore's Theorem

Theorem 4.2 (Ore's Theorem): If G is a simple graph with $n \geq 3$ vertices such that for every pair of non-adjacent vertices u and $v,$ deg (u) + deg $(v) \geq n$, then G is Hamiltonian.

Proof:
Suppose, for contradiction, that G is not Hamiltonian. Consider adding edges between non-adjacent vertices where deg (u) + deg $(v) \geq n$ until we optain a graph G' that satisfies the condition but still is not Hamiltonian.

However, according to the closure properties and maximal path arguments similar to those used in Dirac's theorem, this leads to a contradiction, implying that G must be Hamiltonian.

Algorithm for Finding Eulerian and Hamiltonian Paths and Cycles

5.1. Eulerian Path and Circuit Algorithms

5.1.1. Hierholzer's Algorithm

Description: Hierholzer's algorithm efficiently finds an Eulerian circuit in a graph if one exists.

Algorithm Steps:

1. **Initialization**: Start at any vertex *v* with non-zero degree.
2. **Path Construction**:

- Follow edges from *v* to form a path, ensuring each edge is used exactly once.
- When returning to *v*, if all edges have been used, the path is an Eulerian circuit.

3. **Path Extension**:

- If not all edges are used, find a vertex www on the current path with unused edges.
- Construct a new path from www using unused edges and merge it into the existing path.

4. **Repeat** until all edges are used.

Code

function Hierholzer(G):

```
select starting vertex v
circuit = [v]
while there are unused edges in G:
    for vertex u in circuit:
        if u has unused edges:
            sub_circuit = []
            current = u
            do:
                select an unused edge (current, w)
                remove edge (current, w) from G
                sub_circuit.append(w)
                current = w
            while current != u
            insert sub_circuit into circuit at position after u
            break
return circuit
```

Complexity:

- Time: $O(E)$, where E is the number of edges.
- Space: $O(V+E)$, where V is the number of vertices.

Correctness Proof: The algorithm ensures that all edges are used exactly once and constructs a circuit that satisfies the Eulerian property.

5.2. Hamiltonian Path and Cycle Algorithms

Finding Hamiltonian paths and cycles is computationally challenging.

5.2.1. Backtracking Algorithm

Description: A simple approach using backtracking explores all possible paths.

Algorithm Steps:

1. **Start at any vertex**.
2. **Recursively explore** all adjacent unvisited vertices.
3. **Check for completion** when all vertices are visited and return to the start for a cycle.
4. **Backtrack** when a dead end is reached.

Code

```
function HamiltonianPath(G, path, visited):
```

```
if length(path) == number of vertices in G:
    if path forms a cycle:
        return path
    else:
        return failure
for vertex v adjacent to last(path):
    if v not in visited:
        visited.add(v)
        path.append(v)
        result = HamiltonianPath(G, path, visited)
        if result != failure:
            return result
        visited.remove(v)
        path.pop()
return failure
```

Complexity:

- Time: Exponential $O(n!)$, impractical for large graphs.

Conclusion

This study provides a comprehensive study of Eulerian and Hamiltonian graphs, covering foundational definitions, critical properties, and effective algorithms for path and cycle detection. Eulerian paths and circuits are efficiently solvable and have direct applications in various logistical and network tasks. In contrast, Hamiltonian paths and cycles present computational challenges, often requiring heuristic approaches for practical solutions. Understanding these concepts and algorithms enables efficient problem-solving in transportation planning, network analysis, and other domains where optimal traversal and coverage are essential.

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