On Approximation of Functions by Algebraic Polynomials in the Weighted Quasi Normed Space

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Issue: *Special Issue on Mathematical Computation in Combinatorics and Graph Theory in Mathematical Statistician and Engineering Applications*

Article Info
Page Number: 01 - 24
Publication Issue:
Vol 71 No. 3s3 (2022)

Article Received: *30 April 2022*
Revised: *22 May 2022*
Accepted: *25 June 2022*
Publication: *02 August 2022*

Abstract

In this paper , we discuss the approximation of weighted function in the weighted quasi normed space by algebraic polynomials. This approximation of the functions is found to the weight quasi normed space by using Jacobi function ,and the zero value is excluded , because it gives the space without weighted .

Two results were obtained in this research, each of which depends on a constant with certain values, One of these constants, it is restricted to several values, while the other result showed another restriction in the constant.

Keywords: -Approximation Theory, Algebraic polynomial, Jacobi weight, Heaviside function, quasi normed space.

Subject Classification: Mathematics subject classification 2010:41A10,41A50

1.Introduction and Main Result

In this paper we are study approximation of function F in the weighted quasi normed space $L_{\omega,p}(I), 1 \leq p < \infty$. Where we will study space in an interval $I = [-\mathfrak{b}, \mathfrak{b}]$, $\mathfrak{b}$ =positive integer and the positive weighted function $\omega(x)$ on I, its called Jacobi weight as in [1-3] ,$\omega(x) = (1-x)^{a_1}(1+x)^{a_2}, a_1, a_2 > -b, x \in I \subset R$.

Characteristic for $L_{\omega,p}(I)$, are the inequalities of Holder and Minkowski as in [4,5]. The study of approximation in this paper will be by sing algebraic polynomial in $\prod_m, 0 < m \leq \frac{1}{h}, \frac{1}{h} \in R^+$(positive integer).The space $L_{\omega,p}(I)$, consisting of all functions F such that $F: I \to R$ and the function F is defined as

$\|F\|_{L_{\omega,p}(I)} = \left(\int_I |F(x)|^p \omega(x) dx\right)^{\frac{1}{p}} < \infty, x \in I, 1 \leq p < \infty$. If $a_1 = a_2 = 0$ then $L_{\omega,p}(I) = L_p(I)$, and $\|F\|_{L_p(I)} = \left(\int_I |F(x)|^p dx\right)^{\frac{1}{p}} < \infty, x \in I, 1 \leq p < \infty$.

So we will exclude zero values to a_1 and a_2.Let the space $W_{\omega,p}^{(k+v)}(I), k > 0, v \in (0, b]$, it is the set of all functions F is defined as $\left\{F: F \in L_p;\ \exists F^{(k)} \in L_p, \forall h \in (0,b) \ni \left\|F^{(k)}(x+h) - F^{(k)}(x)\right\|_{L_{\omega,p}[-b,b-h]} \leq h^v\right\}$. Let $a_1, a_2 > -b, 1 \leq p < \infty, k > 0, v \in (0,b]$, we will know the class of function as $EW_{\omega,p}^{(k+v)}(I)=\left\{f\in L_{\omega,p}(I):\exists F^{(k)}\in L_{\omega,p}(I),\forall h\in(0,b)\ni\left(\int_{-b+h}^{b}\left|F^{(k)}(x)-F^{(k)}(x-h)\right|^{p}\omega(x)dx\right)^{\frac{1}{p}}\leq Eh^{v}\right\}$ (1)

The Heaviside function as in [6,7], was used in the search, may be defined as:

1) A piecewise function $H(x)=\begin{cases}1, & x>0\\ 0, & x\leq 0\end{cases}$.
2) An indicator function $H(x)=1_{x>0}$.

The Dirac delta function as in [8-10] is the derivative of the Heaviside function

$\delta_a(x)=\frac{d}{dx}H(x)$, hence the Heaviside function can be considered to be the integral of the Dirac delta function. This is some time written as $H(x)=\int_{-\infty}^{x}\delta_a(u)du$.

Theorem A : Let $F\in W_{\omega,p}^{(k+v)}(I)$, then there exist a polynomial Q_m of degree $\leq m, 0<m\leq\frac{1}{h}$, m depends on (a_1,a_2,p,v,b), such that :

1) $\forall\frac{1}{h}\leq\frac{1}{h_1}$ there exist a constant $\mathcal{C}=\mathcal{C}(a_1,a_2,p,b,v,k,\alpha,\alpha_1)$ satisfy

$$\left\|\frac{F-Q_m}{(\varphi+h)^{k+v}}\right\|_{L_{\omega,p}(I)}\leq\mathcal{C}h^{k+v+\frac{2}{p}}Ln^{\frac{1}{p}}\frac{1}{h}.$$

Such that $a_1,a_2>-b, 1\leq p<\infty, v\in(0,\mathfrak{b}], k>0, \alpha\in(0,1), \alpha_1=Ln\alpha\in R^{+}$ and $\mathfrak{b}$ positive integer.

2) $\forall\frac{1}{h}\leq\frac{1}{h_1}$, there exist a constant $\mathcal{C}$ depends on a_1,a_2,p,b,v,k,μ satisfy

$$\left\|\frac{F-Q_m}{(\varphi+h)^{k+\mu}}\right\|_{L_{\omega,p}(I)}\leq\mathcal{C}h^{k+v}$$

$\mathcal{C}=\mathcal{C}(a_1,a_2,p,b,v,k,\mu,s)$,Such that $a_1,a_2>-b, 1\leq p<\infty, v\in(0,\mathfrak{b}], k>0, \mathfrak{b}$ positive integer and s is the integer function of the greatest integer.

2. Notation

The following notation is used in this paper:

Let $x_i=b\cos\gamma_i, i=1,\dots,Log_{\alpha}\frac{1}{h},\frac{1}{h}$ positive integer $0<\alpha<1$, $x_i<x_{i+1}$, it's clear to see that $b\sin\gamma_i=\sqrt{1-\cos^2\gamma_i}=\sqrt{1-x_i^2}$, Then $|I|=x_{i+1}-x_i=b\cos\gamma_{i+1}-b\cos\gamma_i=bh_i=\frac{bhK_i}{\alpha^i}$, ($K_i$ is a maximum of i). Let $\mathbb{J}_N$ is a partition to the an interval $I=[-\mathfrak{b},\mathfrak{b}]$ such that: $\mathbb{J}_N=\{\jmath_{\mathscr{s}},\mathscr{s}=0,\dots,N\mid-\mathfrak{b}=\jmath_0<\jmath_1<\cdots<\jmath_{N-1}<\jmath_N=\mathfrak{b}\}$, and $\Delta_{\frac{1}{h}}(x)=h\varphi(x)+h$ when $\varphi(x)=\sqrt{1-x^2}, x\in I$. Suppose t_i the sets of points which is belong to an interval I and satisfy the followings :

1. $t_i=0, t_{i+1}=t_i+\frac{bhK_i}{\alpha_i}$

2. $\alpha^{-i} < \sqrt{1 - t_i^2}$
3. $\sqrt{1 - t_{i+1}^2} = \sqrt{1 - \left(t_i + \frac{bhK_i}{\alpha^i}\right)^2} < \alpha^{-i}, i = 1, \dots, \mathrm{Log}_\alpha \frac{1}{h}$

It can be pointed out that $t_{k+1} = t_{\mathrm{Log}_\alpha \frac{1}{h}+1} = b, t_{-i} = -t_i$, and let

$A_k = [t_k, t_{k+1}] \cup [t_{-(k+1)}, t_{-k}], k \geq 0, \forall x \in A_k.$

3.Auxiliary Result

Lemma 3.1. $\left\| F_{\frac{1}{h}} - F_h(F) \right\|_{L_{\omega,p(I_1)}} \leq \mathcal{C} E h^v, \mathcal{C} = \mathcal{C}(a_1, a_2, b),$

such that $a_1, a_2 > -b, b =$positive integer, and $I_1 = [-(b-h), b-h]$

Proof: Let $A = \left\| F_{\frac{1}{h}} - F_h(F) \right\|_{L_{\omega,p(I_1)}} = \left(\int_{-(b-h)}^{b-h} \left| F_{\frac{1}{h}}(x) - F_h(F,x) \right|^p \omega(x) dx \right)^{\frac{1}{p}}$

Let $\forall F_h(F,x) \in EW_{\omega,p}^{(k+v)}(I)$, defined

$$F_h(F,x) = h \int_{kbh}^{kbh+h} F(x^*) dx^*, x \in \left[-\frac{1}{h}, \frac{1}{h} - 1\right)$$

Then $A = \left(\sum_{k=\frac{h-b}{hb}}^{\frac{b-2h}{bh}} \int_{kbh}^{kbh+h} \left| h \int_x^{x+h} F(x^*) dx^* - h \int_{kbh}^{kbh+h} F(x^*) dx^* \right|^p \omega(x) dx \right)^{\frac{1}{p}}$

$A^p \leq \mathcal{C} \sum_{k=\frac{h-b}{hb}}^{\frac{b-2h}{bh}} \int_{kbh}^{kbh+h} \left(h^p \int_{kbh}^{kbh+h} |F(x^* + h) - F(x^*)|^p \omega(x) dx^* dx \right)$

$$\leq \mathcal{C} \sum_{k=\frac{h-b}{hb}}^{\frac{b-2h}{bh}} h^p \int_{kbh}^{kbh+h} |F(x^* + h) - F(x^*)|^p dx^* \int_{kbh}^{kbh+h} \omega(x) dx$$

Let $L_1^p = \sum_{k=\frac{h-b}{hb}}^{\frac{-b}{bh}} h^p \int_{kbh}^{kbh+h} |F(x^* + h) - F(x^*)|^p dx^* \int_{kbh}^{kbh+h} \omega(x) dx$

And $L_2^p = \sum_{k=0}^{\frac{b-2h}{bh}} h^p \int_{kbh}^{kbh+h} |F(x^* + h) - F(x^*)|^p dx^* \int_{kbh}^{kbh+h} \omega(x) dx$. We have

$$(1-x)^{a_1} min\{B_1^{-a_2}, B_1^{a_2}\} \leq \omega(x) \leq (1-x)^{a_1} Max\{B_1^{-a_2}, B_1^{a_2}\}, x \in [0, b)$$

If $a_1 < 0$ then we get

$L_2^p \leq \mathcal{C}(a_2)h^p \sum_{k=0}^{\frac{b-2h}{bh}} h^p \int_{kbh}^{kbh+h} |F(x^*+h)-F(x^*)|^p\,(1-x^*)^{a_1}dx^* \left(\int_{kbh}^{kbh+h}(1-bkh)^{-a_1}(1-x)^{a_1}\,dx\right)$. Let $T_k^2 = \int_{kbh}^{kbh+h}(1-bkh)^{-a_1}(1-x)^{a_1}\,dx$. Hence$T_k^2 \leq \mathcal{C}(a_1,b)$, then $L_2^p \leq \mathcal{C}(a_1,a_2,b)h^p \sum_{k=0}^{\frac{b-2h}{bh}} \int_{kbh}^{kbh+h} |F(x^*+h)-F(x^*)|^p\,\omega(x^*)dx^*$ (2)

If $a_1 \geq 0$, then by the same way we get the same estimate. Now, for L_1^p

$$(1+x)^{a_2}min\{B_2^{-a_1},B_2^{a_1}\} \leq \omega(x) \leq (1+x)^{a_2}Max\{B_2^{-a_1},B_2^{a_1}\}.\forall a_2 \geq 0$$

$$L_1^p \leq \mathcal{C}(a_1)h^p \sum_{k=\frac{b-2h}{bh}}^{-b} \int_{kbh}^{kbh+h} |F(x^*+h)-F(x^*)|^p\,(1+x^*)^{a_2}dx^* \left(\int_{kbh}^{kbh+h}(1+bkh+h)^{-a_2}(1+x)^{a_2}\,dx\right)$$

Let $T_k^1 = \int_{kbh}^{kbh+h}(1+bkh+h)^{-a_2}(1+x)^{a_2}\,d$.Hence $T_k^1 \leq \mathcal{C}(a_2,b)$, then

$L_1^p \leq \mathcal{C}(a_1,a_2,b)h^p \sum_{k=\frac{h-b}{hb}}^{-b} \int_{kbh}^{kbh+h} |F(x^*+h)-F(x^*)|^p\,\omega(x^*)dx^*$ (3)

There for we get from (2) and (3) and by using definition (1)

$$A^p \leq L_1^p + L_2^p$$

$$\left\|F_{\frac{1}{h}} - F_h(F)\right\|_{L_{\omega,p}(I_1)}^p \leq \mathcal{C}(a_1,a_2,b)(Eh^v)^p$$

$$\left\|F_{\frac{1}{h}} - F_h(F)\right\|_{L_{\omega,p}(I_1)} \leq \mathcal{C}Eh^v, \mathcal{C} = \mathcal{C}(a_1,a_2,b),$$

such that $a_1, a_2 > -b, b =$positive integer.

Lemma 3.2. $\left\|F - F_{\frac{1}{h}}\right\|_{L_{\omega,p}(I_2)} \leq \mathcal{C}Eh^v, \mathcal{C} = \mathcal{C}(a_1,a_2,b),$

such that $a_1, a_2 > -b, b =$positive integer, and $I_2 = [-(b-h), b-h]$

Proof: Let $B = \left\|F - F_{\frac{1}{h}}\right\|_{L_{\omega,p}(I_2)} = \left(\int_{-(b-h)}^{b-h} \left|F - F_{\frac{1}{h}}(x)\right|^p \omega(x)dx\right)^{\frac{1}{p}}$

By using the same method in Lemma 3.1. then

$$B^p \le \mathcal{C}(a_2)h^p \sum_{k=\frac{h-b}{bh}}^{\frac{b-2h}{bh}} h^p \int_{kbh}^{kbh+h} |F(x^*) - F(x^* - h)|^p \, dx^* \left(\int_{kbh}^{kbh+h} \omega(x) \, dx \right)$$

Hence $\left\| F - F_{\frac{1}{h}} \right\|_{L_{\omega,p}(I_2)} \le \mathcal{C}Eh^v, \mathcal{C} = \mathcal{C}(a_1, a_2, b),$

such that $a_1, a_2 > -b, b =$positive integer.

Lemma 3.3.$\int_{b-h}^{b} \int_{b-h}^{b} |F(x) - F(y)|^p \omega(y) dx dy \le \mathcal{C}E^p h^{\left(v+\frac{1}{p}\right)p}$

$$\mathcal{C} = \mathcal{C}(v, p), 1 \le p < \infty, v \in (0, b]$$

Proof: Let $T = \int_{b-h}^{b} \int_{b-h}^{b} |F(x) - F(y)|^p \omega(y) dx dy$

We will now make a s substitution for the internal integration, put $t = x - y$, then we get

$$T = \int_{y=b-h}^{b} \int_{t=b-h-y}^{b-y} |F(t+y) - F(y)|^p \omega(y) dy dt + \int_{0}^{h} \int_{b-h}^{b-t} |F(t+y) - F(y)|^p \omega(y) dy dt$$

Let $v = -t, dv = -dt$

$$T \le \int_{0}^{h} \int_{b-h+v}^{b} |F(y) - F(y-v)|^p \omega(y) dy dv + \int_{0}^{h} \int_{-b}^{b-t} |F(t+y) - F(y)|^p \omega(y) dy dt$$

By definition (1) when $t \in (0, b)$

$T \le \int_{0}^{h} \int_{-b+v}^{b} |F(y) - F(y-v)|^p \omega(y) dy dv + \int_{0}^{h} (Et^v)^p dt$

Also by definition (1) when $v \in (0, v), -b + v \le y \le b, (k = 0)$, then

$$\int_{-b+v}^{b} |F(y) - F(y-v)|^p \omega(y) dy \ \le (Et^v)^p$$

Hence $T \le 2E^p \int_{0}^{h} t^{vp} dt$, hence

$\int_{b-h}^{h} \int_{b-h}^{b} |F(x) - F(y)|^p dx \, \omega(y) dy \ \le \frac{2}{vp+1} E^p \, h^{\left(v+\frac{1}{p}\right)p}$

$=\mathcal{C}(v, p) E^p h^{\left(v+\frac{1}{p}\right)p}, 1 \le p < \infty, v \in (0, b]$.

Lemma 3.4. $\int_{-b}^{-b+h} \int_{-b}^{-b+h} |F(x) - F(y)|^p dx \omega(y) dy \le \mathcal{C}E^p h^{\left(v+\frac{1}{p}\right)p}$,$\mathcal{C} = \mathcal{C}(v, p), 1 \le p < \infty, v \in (0, b]$

Proof: By using the same way in Lemma (3.3) we get the above result.

Theorem 3.5. Let $F \in EW^{(v)}_{\omega,p}(I), 1 < p < \infty$, then

$$\|F - F_h(F)\|_{L_{\omega,p}(I)} \le \mathcal{C}Eh^v, \mathcal{C} = \mathcal{C}(a_1, a_2, v, p)$$

Such that $a_1, a_2 > -b, v \in (0, b], 1 < p < \infty$.

Proof: Since the an interval $I = [-\mathfrak{b}, \mathfrak{b}]$, can be written as

$[-\mathfrak{b}, \mathfrak{b}] = [-\mathfrak{b}, -\mathfrak{b} + h] \cup [-\mathfrak{b} + h, \mathfrak{b} - h] \cup [\mathfrak{b} - h, \mathfrak{b}]$, and we have

$$|F(x) - F_h(F, x)| \le \mathcal{C}\left|F(x) - F_{\frac{1}{h}}(x)\right| + \mathcal{C}\left|F_{\frac{1}{h}}(x) - F_h(F, x)\right|$$

Now by using Holder inequality if $p \ge 1$, [10],and by take $L_{\omega,p}$- of both sides, we get

$$\|F - F_h(F)\|_{L_{\omega,p}(I)}$$
$$\le \mathcal{C}\int_{-b}^{-b+h}\int_{-b}^{-b+h} |F(x) - F(y)|^p \omega(y)dxdy + \left(\int_{-(b-h)}^{b-h} \left|F_{\frac{1}{h}}(x) - F_h(F, x)\right|^p \omega(x)dx\right)^{\frac{1}{p}}$$
$$+ \left(\int_{\mathfrak{b}-h}^{b}\int_{\mathfrak{b}-h}^{\mathfrak{b}} |F(x) - F(y)|^p \omega(y)dx\,dy\right)$$

Now, by using Lemmas (3.1),(3.3) and (3.4) we get

$$\|F - F_h(F)\|_{L_{\omega,p}(I)} \le \mathcal{C}(v, p)E^p h^{\left(v+\frac{1}{\mathrm{p}}\right)p} + \mathcal{C}(a_1, a_2, \mathfrak{b})Eh^v + \mathcal{C}(v, p)E^p h^{\left(v+\frac{1}{p}\right)p}$$

$=\mathcal{C}(a_1, a_2, p, \mathfrak{b}, v)Eh^v$, $\mathcal{C} = \mathcal{C}(a_1, a_2, v, p)$

Such that $a_1, a_2 > -b, v \in (0, b], 1 < p < \infty, b =$positive integer.

Theorem 3.6. Let $F \in EW^{(k+v)}_{\omega,p}(I), k = 0, 1 < p < \infty$, then

$$\left\|\frac{F - F_h(F)}{\Delta_{\frac{1}{h}}}\right\|_{L_{\omega,p}(I)} \le \mathcal{C}h^{\frac{2}{p}}\left(Ln\frac{1}{h}\right)^{\frac{1}{p}}, \mathcal{C} = \mathcal{C}(\alpha, \alpha_1, v, p, \mathfrak{b})$$

Such that $\alpha \in (0,1), \alpha_1 \in R^+, v \in (0, b], 1 \le p < \infty, b =$positive integer.

Proof: We have $A_k \subset [-\mathfrak{b}, \mathfrak{b}], k \ge 0$ then

$$\left\|\frac{F - F_h(F)}{\Delta_{\frac{1}{h}}}\right\|^p_{L_{\omega,p}(I)} = \int_{-\mathfrak{b}}^{\mathfrak{b}} \left|\frac{F(x) - F_h(F, x)}{\Delta_{\frac{1}{h}}(x)}\right|^p \omega(x)dx$$

$$= \sum_{k=0}^{Log_\alpha \frac{1}{h}} \int_{A_k} \left| \frac{F(x) - F_h(F,x)}{\Delta_{\frac{1}{h}}(x)} \right|^p \omega(x)dx$$

By using Theorem 3.5 and $\forall\, F \in W_{\omega,p}^{(v)}(I), F_{h_i}(F,x) = F_{\frac{h}{\alpha^k}}(F,x)$

$$\int_{A_k} \left| F(x) - F_{\frac{h}{\alpha^k}}(F,x) \right|^p \omega(x)dx \le \int_I |F(x) - F_h(F,x)|^p \omega(x)dx$$

$\le \mathcal{C}(a_1, a_2, v, p)\left(\frac{h}{\alpha^k}\right)^{vp}$.

Now,for $k < Log_\alpha \frac{1}{h}$, on an interval A_k, we have $\alpha^k < \alpha^{k+1}$, hence by property (2) we get $\alpha^{-(k+1)} < \alpha^{-k} < \sqrt{1-t_k^2}$

$$< \sqrt{1-t_{k+1}^2} < \sqrt{1-x^2}$$

$$(\alpha^{k+1})^v \left(\sqrt{1-x^2} + h\right)^v > 1$$

$$\frac{\frac{1}{h^v}}{\left(\sqrt{1-x^2}+h\right)} < \frac{(\alpha^{k+1})^v}{h^v} \Rightarrow \frac{1}{(h\varphi(x)+h^2)^v} < \left(\frac{\alpha^{k+1}}{h}\right)^v$$

Hence for $k = Log_\alpha \frac{1}{h}$, we get

$$\left\| \frac{F - F_h(F)}{\Delta_{\frac{1}{h}}} \right\|_{L_{\omega,p}(I)}^p \le \mathcal{C}(v,p,\alpha)|I| = \mathcal{C}(\alpha, v, p)h_i = \mathcal{C}\left(\frac{bhk_i}{\alpha^i}\right)$$

$=\mathcal{C}\left(\frac{hLog_\alpha \frac{1}{h}}{\alpha^{Log_\alpha \frac{1}{h}}}\right) = \mathcal{C}(\alpha, p, v, \mathfrak{b})h^2\left(\frac{\mathrm{Ln}\frac{1}{h}}{Ln\alpha}\right)$, let $Ln\alpha = \alpha_1 \in R^+$

$$\left\| \frac{F - F_h(F,.)}{\Delta_{\frac{1}{h}}(.)} \right\|_{L_{\omega,p}(I)} \le \mathcal{C}h^{\frac{2}{p}}\left(Ln\frac{1}{h}\right)^{\frac{1}{p}}, \mathcal{C} = \mathcal{C}(\alpha, \alpha_1,, v, p, \mathfrak{b})$$

Such that $\alpha \in (0,1), \alpha_1 \in R^+, v \in (0,b], 1 \le p < \infty, b$ =positive integer.

Lemma 3.7. For all $x \in I = [-\mathfrak{b}, \mathfrak{b}]$, and $[x_i, x_{i+1}] \subset [t_{k-1}, t_k]$ we have

$$\int_{x_i}^{x_{i+1}} \omega(x)dx \le \mathcal{C}hsin^{2a_1+1}\gamma_{i,\frac{1}{h}}$$

Where $\omega(x)$ the Jacobi weights, and $\mathcal{C}=\mathcal{C}(a_1,a_2,\mathfrak{b})$, such that $a_1,a_2>-b,\mathfrak{b}$ =positive integer.

Proof: i) If $a_1<0$, the proof will be discussed in case $a_1<0$, since $[x_i,x_{i+1}]\subset[t_{k-1},t_k]$, hence

$$\int_{x_i}^{x_{i+1}}(1-x)^{a_1}(1+x)^{a_2}dx \ \le \mathcal{C}(a_2)\int_{t_k-\frac{h}{\alpha^{i-1}}}^{t_k}(1-x)^{a_1}dx$$

1) If $t_k<b$ ($k\le Log_\alpha\frac{1}{h}$), then

$\mathcal{C}(a_2)\int_{t_k-h\alpha^{-(i-1)}}^{t_k}(1-x)^{a_1}\,dx\ \le \mathcal{C}(a_1,a_2,\mathfrak{b})\,(t_k-t_{k-1})\,\left(\sqrt{1-t_{k-1}^2}\right)^{2a_1}$

$\Delta t_{k-1}=t_k-t_{k-1}=h\alpha^{-(k-1)}$, and by property (2) we have $\alpha^{-(k-1)}<\sqrt{1-t_{k-1}^2}$, hence
$\mathcal{C}(a_2)\int_{t_k-h\alpha^{-(i-1)}}^{t_k}(1-x)^{a_1}dx\le\mathcal{C}(a_1,a_2,\mathfrak{b})h\,\left(\sqrt{1-t_{k-1}^2}\right)^{2a_1+1}$.

It is clear that $cos\,\gamma_{i,\frac{1}{h}}=t_{k-1}$, hence $sin^2\gamma_{i,\frac{1}{h}}=1-cos^2\gamma_{i,\frac{1}{h}}$, hence $\gamma_{i,\frac{1}{h}}=\sqrt{1-t_{k-1}^2}$, then we get $\int_{x_i}^{x_{i+1}}\omega(x)dx\le\mathcal{C}(a_1,a_2,b)h\,sin^{2a_1+1}\gamma_{i,\frac{1}{h}}$.

2) If $t_k=b$ ($k=Log_\alpha\frac{1}{h}+1$), then

$$\int_{x_i}^{x_{i+1}}\omega(x)dx\le\mathcal{C}(a_1)\int_{\mathfrak{b}-\frac{h}{\alpha^{Log_\alpha\frac{1}{h}}}}^{\mathfrak{b}}(1-x)^{a_1}dx$$

Let $1-x=y,dy=-dx$

$$\int_{x_i}^{x_{i+1}}\omega(x)dx\le-\mathcal{C}(a_2,\mathfrak{b})\int_{-h^2}^{0}y^{a_1}\,dy\le\ \mathcal{C}(a_1,a_2,\mathfrak{b})hh^{2a_1+1}$$

We have from property (1) and (2)

$$\frac{1}{\alpha^{(i-1)}}<\sqrt{1-t_{i-1}^2}=\sqrt{1-\left(t_i-\frac{h}{\alpha^{i-1}}\right)^2},(t_k=\mathfrak{b})$$

$\frac{1}{\alpha^{Log_\alpha\frac{1}{h}}}<\sqrt{1-\left(\mathfrak{b}-\frac{h}{\alpha^{Log_\alpha\frac{1}{h}}}\right)^2},\left(k=Log_\alpha\frac{1}{h}+1\right)$hence $h\le sin\gamma_{i,\frac{1}{h}}$ (4)

$$\int_{x_i}^{x_{i+1}}\omega(x)dx\ \le\mathcal{C}(a_1,a_2,\mathfrak{b})hsin^{2a_1+1}\gamma_{i,\frac{1}{h}}$$

ii) If $a_1\ge0$, we take this case if $t_k<b$ and if $t_k=\mathfrak{b}$, and by the same way in (i) we get

$\int_{x_i}^{x_{i+1}} \omega(x)dx \ \le \mathcal{C}(a_1, a_2, \mathfrak{b}) h sin^{2a_1+1} \gamma_{i,\frac{1}{h}}.$

Lemma 3.8. For $x \in [x_i, x_{i+1}]$ and $\gamma_i = cos^{-1} x_i$, we have

$$\left(sup_{x_i \le x \le x_{i+1}} \left|\aleph_j(x) - P_j(x)\right|\right)^p \le 1 + \mathcal{C}(m, \alpha, p, q)\alpha^{-kp} h^{\frac{2m-1}{2}+1+p}$$

Where $\chi_j(x) = \begin{cases} 1 & , x \ge x_i \\ 0 & x < x_i \end{cases}, \alpha \in (0,1), m \ge 2, 1 < p < \infty$

Proof: By reference [11], then we get when $\aleph_j(x) = 1, x \ge x_i$

$$\left(\sum_{j=`}^{N} sup_{x_i \le x \le x_{i+1}} \left|\aleph_j(x) - P_j(x)\right|\right)^p \le \left(\sum_{j=`}^{N} \left(1 + \frac{\mathcal{C}}{2m-1} h_{j,\frac{1}{h}} \Psi_{j,\frac{1}{h}}^{2m-1}(x)\right)\right)^p$$

$$\le \left(\sum_{j=`}^{N} \left(1 + \mathcal{C}(m) h_{j,\frac{1}{h}} \left(\frac{\Delta_{\frac{1}{h}}(x_j)}{\Delta_{\frac{1}{h}}(x_j) + \left|cos^{-1}x - \ cos^{-1}x_j\right|}\right)^{2m-1}\right)\right)^p$$

Since $\Delta_{\frac{1}{h}}(x_j) = x_{j+1} - x_j = h_j = \frac{h}{\alpha^k}$

$$\left(\sum_{j=`}^{N} sup_{x_i \le x \le x_{i+1}} \left|\aleph_j(x) - P_j(x)\right|\right)^p$$

$$\le \left(\sum_{j=`}^{N} \left(1 + \mathcal{C}(m) h\alpha^{-k} \left(\frac{1}{1 + \frac{\left|cos^{-1}x - \ cos^{-1}x_j\right|}{h\alpha^{-k}}}\right)^{2m-1}\right)\right)^p$$

Let $\gamma = cos^{-1}x$ and $\gamma_{j,\frac{1}{h}} = cos^{-1}x_j$, and we have $\gamma \in \left[\gamma_{i+1,\frac{1}{h}}, \gamma_{i,\frac{1}{h}}\right]$, since

$$\left|\gamma_{i,\frac{1}{h}} - \gamma_{j,\frac{1}{h}}\right| \le \left|\gamma_{i,\frac{1}{h}} - \gamma\right| + \left|\gamma - \gamma_{j,\frac{1}{h}}\right|$$

$\le \left(1 + \frac{1}{h}\left|\gamma_{i,\frac{1}{h}} - \gamma_{i+1,\frac{1}{h}}\right| (1 + \frac{1}{h}\left|\gamma - \gamma_{j,\frac{1}{h}}\right|\right)$, then

$$\frac{1 + \frac{1}{h}\left|\gamma_{i,\frac{1}{h}} - \gamma_{j,\frac{1}{h}}\right|}{1 + \frac{1}{h}\left|\gamma - \gamma_{j,\frac{1}{h}}\right|} \le 1 + \frac{\frac{1}{h}\left|\gamma_{i,\frac{1}{h}} - \gamma\right|}{1 + \frac{1}{h}\left|\gamma - \gamma_{j,\frac{1}{h}}\right|} \le 1 + \frac{1}{h}\left|\gamma_{i,\frac{1}{h}}, \gamma_{i+1,\frac{1}{h}}\right|$$

Therefor

$$\left(\sum_{j=`}^{N} sup_{x_i \le x \le x_{i+1}} \left|\aleph_j(x) - P_j(x)\right|\right)^p \le \left(\sum_{j=1}^{N}\left(1 + \mathcal{C}(m)h\alpha^{-k}\left(\frac{1}{1+\frac{\alpha^k}{h}\left|\gamma_{i,\frac{1}{h}} - \gamma_{j,\frac{1}{h}}\right|}\right)^{2m-1}\right)\right)^p$$

By Holder inequality

$$\left(\sum_{j=1}^{N}\left(\frac{1}{1+\frac{\alpha^k}{h}\left|\gamma_{i,\frac{1}{h}}-\gamma_{j,\frac{1}{h}}\right|}\right)^{2m-1}\right)^p = \left(\sum_{j=1}^{N}\frac{1}{\left(1+\frac{\alpha^k}{h}\left|\gamma_{i,\frac{1}{h}}-\gamma_{j,\frac{1}{h}}\right|\right)^{\frac{2m-1}{2}}}\cdot\frac{1}{\left(1+\frac{\alpha^k}{h}\left|\gamma_{i,\frac{1}{h}}-\gamma_{j,\frac{1}{h}}\right|\right)^{\frac{2m-1}{2}}}\right)^p$$

$$\le \left(\sum_{j=1}^{N}\frac{1}{\left(1+\frac{\alpha^k}{h}\left|\gamma_{i,\frac{1}{h}}-\gamma_{j,\frac{1}{h}}\right|\right)^{\frac{2m-1}{2}p}}\right)\cdot\left(\left(\sum_{j=1}^{N}\frac{1}{\left(1+\frac{\alpha^k}{h}\left|\gamma_{i,\frac{1}{h}}-\gamma_{j,\frac{1}{h}}\right|\right)^{\frac{2m-1}{2}q}}\right)\right)^{\frac{p}{q}}$$

$\forall j \in \mathbb{I}_k = \{j: [x_j, x_{j+1}] \subset A_k\}$, hence

$$\left(\sum_{j=1}^{N}\frac{1}{\left(1+\frac{\alpha^k}{h}\left|\gamma_{i,\frac{1}{h}}-\gamma_{j,\frac{1}{h}}\right|\right)^{\frac{2m-1}{2}q}}\right) = \sum_{k=0}^{Log_\alpha \frac{1}{h}}\sum_{j\in\mathbb{I}_k}\frac{1}{\left(1+\frac{\alpha^k}{h}\left|\gamma_{i,\frac{1}{h}}-\gamma_{j,\frac{1}{h}}\right|\right)^{\frac{2m-1}{2}q}}$$

$\le \mathcal{C}(m,q,\alpha), \alpha \in (0,1), m \ge 2, q > 1$

By using the property (3) for k

$1 - \left(t_k + \frac{1}{h\alpha^{k-1}}\right)^2 < \frac{1}{\alpha^{2k}}$

$1 - t_k^2 < \alpha^{-2k}\left(1 + \frac{2\alpha\alpha^k t_k}{h} + \frac{\alpha^2}{h^2}\right)$

$1 - t_k^2 < \alpha^{-2k+2}$ (5)

We have $\forall j \in \mathbb{I}_k, \gamma_k = cos^{-1}x_k$ by inequality (5) then

$$sin\gamma_{j,\frac{1}{h}} < sin\gamma_{k,\frac{1}{h}} = \sqrt{1-t_k^2} \le \sqrt{\alpha^2\alpha^{-2k}} = \alpha\alpha^{-k}$$

$sin\,\gamma_{j,\frac{1}{h}} \le \mathcal{C}_2(\alpha)\alpha^{-k}, \mathcal{C}_2 > 0$ (6)

And from property (2) for $k \ge 0$

$\alpha^{-k} < \sqrt{1-t_k^2} = sin\gamma_{j,\frac{1}{h}}, \exists\, \mathcal{C}_1 > 0 \ \ni\ \mathcal{C}_1\, \alpha^{-k} < sin\, \gamma_{j,\frac{1}{h}}$ (7)

From the inequality (6) and (7) we get

$\mathcal{C}_1\, \alpha^{-k} < sin\, \gamma_{j,\frac{1}{h}} \le \mathcal{C}_2(\alpha)\alpha^{-k}, \forall \mathcal{C}_1, \mathcal{C}_2 > 0$ (8)

Now from the inequality (4) we have $h \le sin\gamma_{i,\frac{1}{h}}$, and

$h \le sin\gamma_{i,\frac{1}{h}} \le sin\gamma_{i,\frac{1}{h}} + \left|\gamma_{i,\frac{1}{h}} - \gamma_{j,\frac{1}{h}}\right|$

$h - sin\gamma_{j,\frac{1}{h}} \le \left|\gamma_{\mathrm{i},\frac{1}{h}} - \gamma_{j,\frac{1}{h}}\right|$

$1 + \frac{\alpha^k}{h}\left(h - sin\gamma_{j,\frac{1}{h}}\right) \le 1 + \frac{\alpha^k}{h}\left|\gamma_{i,\frac{1}{h}} - \gamma_{j,\frac{1}{h}}\right|$ (9)

From the inequality (8) we have $sin\gamma_{j,\frac{1}{h}} \le \mathcal{C}\alpha^{-k}$

$$\frac{\alpha^k}{h} sin\gamma_{j,\frac{1}{h}} \le \mathcal{C}\ \frac{\alpha^k}{h}\ \alpha^{-k}$$

$1 + \alpha^k - \frac{\alpha^k}{h} sin\gamma_{j,\frac{1}{h}} > 1 + \alpha^k - \frac{\mathcal{C}}{h}$ (10)

By substantiating (10) in (9) we get $1 + \alpha^k - \frac{\mathcal{C}}{h} \le 1 + \frac{\alpha^k}{h}\left|\gamma_{i,\frac{1}{h}} - \gamma_{j,\frac{1}{h}}\right|, \forall\, i, j$

$$\sum_{k=0}^{\infty}\sum_{j\in\mathbb{I}_k} \frac{1}{\left(1 + \frac{\alpha^k}{h}\left|\gamma_{i,\frac{1}{h}} - \gamma_{j,\frac{1}{h}}\right|\right)^{\frac{2m-1}{2}p}} < \sum_{k=0}^{\infty} \frac{1}{\left(1 + \alpha^k - \frac{\mathcal{C}}{h}\right)^{\frac{2m-1}{2}p}}$$

Hence

$$\left(\sum_{j=1}^{N} sup_{x_i \le x \le x_{i+1}}\left|\aleph_j(x) - P_j(x)\right|\right)^p$$
$$\le \left[1 + \mathcal{C}(m)h^p\alpha^{-kp}\left(\mathcal{C}(m,\alpha,q)\sum_{k=0}^{\infty}\left(1 + \alpha^k - \frac{\mathcal{C}}{h}\right)^{\frac{-(2m-1)}{2}p}\right)\right]$$

$$\sum_{k=0}^{\infty}\left(1 + \alpha^k - \frac{\mathcal{C}}{h}\right)^{\frac{-(2m-1)}{2}p} = \sum_{k=0}^{\infty}\left(\frac{h + h\alpha^k - \mathcal{C}}{h}\right)^{\frac{-(2m-1)}{2}p}$$

$$= \left(\frac{1}{h}\right)^{\frac{-(2m-1)}{2}p} \sum_{k=0}^{\infty} (h + h\alpha^k - \mathcal{C})^{\frac{-(2m-1)}{2}p}$$

$$= (h)^{\frac{(2m-1)}{2}p} \left(\mathcal{C}(m,p,\alpha)h\right)$$

$$= \mathcal{C}(m,p,\alpha)h^{\frac{2m-1}{2}p+1}, \alpha \in (0,1), m \geq 2, 1 \leq p < \infty$$

$$\left(\sum_{j=1}^{N} sup_{x_i \leq x \leq x_{i+1}} \left|\aleph_j(x) - P_j(x)\right|\right)^p \leq 1 + \mathcal{C}(m)h^p\alpha^{-kp}\left(\mathcal{C}(m,p,q,\alpha)h^{\frac{2m-1}{2}p+1}\right)$$

$$\left(\sum_{j=1}^{N} sup_{x_i \leq x \leq x_{i+1}} \left|\aleph_j(x) - P_j(x)\right|\right)^p \leq 1 + \mathcal{C}(m,p,q,\alpha)h^{\frac{2m-1}{2}p+1+p}\alpha^{-kp}$$

Lemma 3.9. $\forall\, j \in \mathbb{I}_k$, there exist a polynomial $\mathcal{P}_{j,\frac{1}{h}}$ of degree $\leq m$, $0 < m \leq \frac{1}{h}$ such that $\sum_{j\in\mathbb{I}_k} \left|\mathcal{P}_{j,\frac{1}{h}}\right|^p \leq \mathcal{C}\alpha^{2a_1k-(vp-1)k}h^{vp-1}$,, where

$$\mathcal{P}_{j,\frac{1}{h}} = \frac{1}{h_{j-1}} \int_{x_{j-1}}^{x_j} F(t)dt, h_{j-1} = x_j - x_{j-1}, j > 0, \mathcal{C}(p) = \mathcal{C} > 0$$

Proof: $\forall\, j \in \mathbb{I}_k$ we have $\sum_{j_k\in\mathbb{I}_k} \left|\mathcal{P}_{j_k,\frac{1}{h}}\right|^p = \sum_{j_k\in\mathbb{I}_k} \left|\frac{1}{h}\int_{x_{j-1}}^{x_j} F(t)dt\right|^p$, since

$$\left[x_{j-1}, x_j\right] \subset [t_k, t_{k+1}] \cup \left[t_{-(k+1)}, t_{-k}\right]$$

$$\sum_{j_k\in\mathbb{I}_k} \left|\mathcal{P}_{j_k,\frac{1}{h}}\right|^p \leq \mathcal{C}(p)\frac{\alpha^k}{h}\left|\int_{t_k}^{t_k+\frac{h}{\alpha^k}} F(t) + \int_{t_k-\frac{h}{\alpha^{k-1}}}^{t_k-\frac{h}{\alpha^k}} F(t) + \int_{t_k-\frac{h}{\alpha^k}}^{t_k} F(t)\right|^p dt$$

$$\leq \mathcal{C}(p)\frac{\alpha^k}{h}\left(\int_{t_k}^{t_k+\frac{h}{\alpha^k}} |F(t)| + \int_{t_k-\frac{h}{\alpha^{k-1}}}^{t_k} |F(t)|\right)^p dt$$

$$\leq \mathcal{C}(p)\left(\frac{\alpha^k}{h}\right)^p \int_{t_k-\frac{h}{\alpha^{k-1}}}^{t_k} \left|F\left(t + \frac{h}{\alpha^k}\right) - F(t)\right|^p dt$$

$$\leq \mathcal{C}(p)\left(\frac{\alpha^k}{h}\right)^p sin^{-2a_1}\gamma_{j,\frac{1}{h}} \int\limits_{t_k-\frac{h}{\alpha^{k-1}}}^{t_k} \left|F\left(t+\frac{h}{\alpha^k}\right)-F(t)\right|^p \omega(t)\,dt$$

$sin^{-2a_1}\gamma_{j,\frac{1}{h}} < \alpha^{2a_1k}$, by using inequality (8)

$$\left[t_k-\frac{h}{\alpha^{k-1}},t_k\right] \subset \left[t_k-\frac{h}{\alpha^{k-1}},t_{k+1}-\frac{h}{\alpha^k}\right]$$

$$\sum_{j_k\in\mathbb{I}_k}\left|\mathcal{P}_{j_k,\frac{1}{h}}\right|^p \leq \mathcal{C}(p)\frac{\alpha^k}{h} sin^{-2a_1}\gamma_{j,\frac{1}{h}} \int\limits_{t_k-\frac{h}{\alpha^{k-1}}}^{t_k} \left|F\left(t+\frac{h}{\alpha^k}\right)-F(t)\right|^p \omega(t)\,dt$$

$$\leq \mathcal{C}(p)\frac{\alpha^k}{h}\ \alpha^{2a_1k} \int\limits_{-\mathfrak{b}}^{\mathfrak{b}-\frac{h}{\alpha^k}} \left|F\left(t+\frac{h}{\alpha^k}\right)-F(t)\right|^p \omega(t)\,dt$$

$$\leq \mathcal{C}(p)\frac{\alpha^k}{h}\ \alpha^{2a_1k}\left(\frac{h}{\alpha^k}\right)^{vp} = \mathcal{C}(p)\alpha^{k-kvp+2ka_1}h^{vp-1}$$

$$\sum_{j\in\mathbb{I}_k}\left|\mathcal{P}_{j,\frac{1}{h}}\right|^p \leq \mathcal{C}\alpha^{2a_1k-(\mathrm{v}p-1)k}h^{vp-1}$$

Theorem 3.10. $\forall\ F \in W^{(v)}_{\omega,p}(I), k = 0, 1 < p < \infty$, there exist a polynomial $\mathcal{P}_{\frac{1}{h}}(x)$, of degree $\leq m, 0 < m \leq \frac{1}{h}\ (\frac{1}{h} \in R^+)$($m$ which is depended on a_1, a_2, v, p) such that

$$\left\|\frac{F_{\frac{1}{h}}(F)-\mathcal{P}_m}{\Delta^v_{\frac{1}{h}}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^{\frac{2}{p}}\left(Ln\frac{1}{h}\right)^{\frac{1}{p}}, \mathcal{C} = \mathcal{C}(\alpha,\alpha_1,a_2,v,p,\mathfrak{b})$$

Such that $\alpha \in (0,1), \alpha_2 \in R^+, v \in (0,b], 1 \leq p < \infty, b =$positive integer.

Proof: $\forall\ x\ \in [-\mathfrak{b},\mathfrak{b}], n \in N$, by lemma (3.9) there is a polynomial $\mathcal{P}_{\frac{1}{h}}(x)$, of degree $\leq m, 0 < m \leq \frac{1}{h}\ (\frac{1}{h} \in R^+)$, have the form

$$\mathcal{P}_m(x) = \mathcal{P}_{0,\frac{1}{h}}(x) + \sum_{j=1}^{N}\mathcal{P}_{x_j}(x)\mathcal{P}_{j,\frac{1}{h}}(x)$$

$\mathcal{P}_{j,\frac{1}{h}}(x)$, ia a polynomial of degree $\leq m$, $0 < m \leq \frac{1}{h}$, such that

$$\mathcal{P}_{j,\frac{1}{h}}(x) = \frac{1}{h_j}\int\limits_{x_{j-1}}^{x_{j+1}} F(t)dt, h_j = x_{j+1}-x_j\ , j = 1, \dots, N$$

$\forall\ F \in W^{(v)}_{\omega,p}(I), F_{\frac{1}{h}}(F,x) = F_{\frac{\alpha^k}{h}}(F,x)$, and $F_{\frac{\alpha^k}{h}}(F,x) = \frac{\alpha^k}{h}\int_{kbh}^{kbh+bh} F(t)dt$

Now $\forall\ x \in [kbh, kbh+bh), k = \frac{1}{h}, \dots, \frac{1}{h} - 1$, we have when $x_1 - x_0 = h_0 = \frac{hb}{\alpha^k}, K_i = 1$, that

$F_{\frac{1}{h}}(F,x) = \frac{1}{h_0}\int_{-b}^{-b+h} F(t)dt = \frac{1}{h_j}\int_{x_j}^{x_{j+1}} F(t)dt$

Then $F_{\frac{1}{h}}(F,x) = \mathcal{P}_{0,\frac{1}{h}}(x)$, when $j = 0$. $\mathcal{P}_{x_j}$ is a polynomial of degree $\le m, 0 < m \le \frac{1}{h}$ ($\frac{1}{h} \in R^+$)(m which is depended on (a_1, a_2, v, p), such that $\mathcal{P}_{x_j}(x) = \aleph_j(x) - \mathcal{P}_j(x), \forall \in I$, we have $\sqrt{1 - x_i^2} < \sqrt{1 - x^2}$, and $h\sqrt{1 - x_i^2} + h^2 < h\sqrt{1 - x^2} + h^2$ when $\varphi(x) = \sqrt{1 - x^2}$ and since $sin\gamma_{i,\frac{1}{h}} = \sqrt{1 - x_i^2}$ then we get

$\frac{1}{(h\varphi(x)+h^2)^v} \le \frac{1}{\left(h\sqrt{1-x_i^2}+h^2\right)^v} < \frac{1}{\left(hsin\gamma_{i,\frac{1}{h}}\right)^v}$ (12)

Now, to find the estimation (11), let

$$T^p = \left\|\frac{F_{\frac{1}{h}}(F) - \mathcal{P}_m}{\Delta^v_{\frac{1}{h}}}\right\|^p_{L_{\omega,p}(I)} = \int_I \left|\frac{F_{\frac{1}{h}}(F,x) - \mathcal{P}_m(x)}{\Delta^v_{\frac{1}{h}}(x)}\right|^p \omega(x)dx$$

$$\le \sum_i \int_{x_i}^{x_{i+1}} \left|\frac{\frac{1}{h_0}\int_{x_0}^{x_1} F(t)dt - \frac{1}{h_0}\int_{x_0}^{x_1} F(t)dt - \sum_{j=1}^N (\aleph_j(x) - \mathcal{P}_j(x))\, \mathcal{P}_{j,\frac{1}{h}}(x)}{\Delta^v_{\frac{1}{h}}(x)}\right|^p \omega(x)dx$$

See that $\aleph_j(x) = \mathcal{P}_j(x) = 1$, when $x > x_j$, $(x > x_0)$ when $j = 0$, hence

$$T^p \le \sum_i \int_{x_i}^{x_{i+1}} \left|\frac{\sum_{j=1}^N (\aleph_j(x) - \mathcal{P}_j(x))\, \mathcal{P}_{j,\frac{1}{h}}(x)}{\Delta^v_{\frac{1}{h}}(x)}\right|^p \omega(x)dx$$

$$\le \mathcal{C}\sum_i \left(\Delta^v_{\frac{1}{h}}(x)\right)^{-vp} \left(\sum_{j=1}^N \left|\mathcal{P}_{j,\frac{1}{h}}(x)\right| \left|\aleph_j(x) - \mathcal{P}_j(x)\right|\right)^p \int_{x_i}^{x_{i+1}} \omega(x)dx$$

$$\le \mathcal{C}\sum_i \left(\Delta^v_{\frac{1}{h}}(x)\right)^{-vp} \left(\sum_{j=1}^N \left|\mathcal{P}_{j,\frac{1}{h}}(x)\right| sup_{x_i \le x \le x_{i+1}}\left|\aleph_j(x) - \mathcal{P}_j(x)\right|\right)^p \int_{x_i}^{x_{i+1}} \omega(x)dx$$

Now by using the inequality (12) and Lemma ((3.7),(3.8) and (3.9)) and by setting the smallest value to a_1, m we get

$$T^p \leq \mathcal{C}(a_2,p,v,b)\alpha^{k(1-vp)}h^{vp-1} = \mathcal{C}\left(\frac{h}{\alpha^k}\right)^{vp-1}$$

$$\leq \mathcal{C}(a_2,p,v,b)h_i, (h_i = x_{i+1} - x_i)$$

$= \mathcal{C}(a_2,p,v,b)(x_{i+1} - x_i)$, by using property (3) we get

$= \mathcal{C}\ \frac{hb\,K_i}{\alpha^i}$, for $i = Log_\alpha \frac{1}{h}$ then

$T^p \leq \mathcal{C}\frac{h\,Log_\alpha \frac{1}{h}}{\alpha^{Log_\alpha \frac{1}{h}}} = \mathcal{C}\ h^2 Ln\ \frac{1}{h}$, hence

$$\left\|\frac{F_{\frac{1}{h}}(F) - \mathcal{P}_m}{\Delta^v_{\frac{1}{h}}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^{\frac{2}{p}}\left(Ln\frac{1}{h}\right)^{\frac{1}{p}}, \mathcal{C} = \mathcal{C}(\alpha,\alpha_1,a_2,v,p,\mathfrak{b})$$

Such that $a_2 > -b, \alpha \in (0,1), \alpha_1 = Ln\alpha \in R^+, v \in (0,b], h \in (0,\mathfrak{b}), 1 \leq p < \infty, b$ =positive integer.

Theorem 3.11. $\forall\ F \in W^{(v)}_{\omega,p}(I)$, there exist a polynomial $\mathcal{P}_m$, of degree $\leq m, 0 < m \leq \frac{1}{h}$ ($\frac{1}{h} \in R^+$)(m which is depends on a_1, a_2, v, p) such that

$$\left\|\frac{F - \mathcal{P}_m}{(h\varphi + h^2)^v}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^{\frac{2}{p}}\left(Ln\frac{1}{h}\right)^{\frac{1}{p}}, \mathcal{C} = \mathcal{C}(\alpha,\alpha_1,a_1,a_2,v,p,\mathfrak{b})$$

Such that $a_1, a_2 > -b, \alpha \in (0,1), \alpha_1 = Ln\alpha \in R^+, v \in (0,b], h \in (0,\mathfrak{b}), 1 \leq p < \infty, b$ =positive integer.

Proof: $\forall\ F \in W^{(v)}_{\omega,p}(I)$, then by theorem (3.6) there exist a polynomial $\mathcal{P}_m$, of degree $\leq m, 0 < m \leq \frac{1}{h}$ ($\frac{1}{h} \in R^+$)(m which is depends on a_1, a_2, v, p),there exist $F_{\frac{1}{h}}(F,x) = F_{\frac{h}{\alpha^k}}(F,x), \forall\ x \in A_k$ then

$$\left\|\frac{F - \mathcal{P}_m}{(h\varphi + h^2)^v}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}(p)\left\|\frac{F - F_{\frac{1}{h}}(F)}{(h\varphi + h^2)^v}\right\|_{L_{\omega,p}(I)} + \mathcal{C}(p)\left\|\frac{F_{\frac{1}{h}}(F) - \mathcal{P}_m}{(h\varphi + h^2)^v}\right\|_{L_{\omega,p}(I)}$$

Then by theorem (3.6) and (3.10) we get

$$\left\|\frac{F - \mathcal{P}_m}{(h\varphi + h^2)^v}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^{\frac{2}{p}}\left(Ln\frac{1}{h}\right)^{\frac{1}{p}}, \mathcal{C} = \mathcal{C}(\alpha,\alpha_1,a_1,a_2,v,p,\mathfrak{b})$$

Such that $a_1, a_2 > -b, \alpha \in (0,1), \alpha_1 = Ln\alpha \in R^+, v \in (0,b], h \in (0,\mathfrak{b}), 1 \leq p < \infty, b$ =positive integer.

Theorem 3.12. $\forall\ F \in W_{\omega,p}^{(v)}(I)$, there exist a polynomial $\mathcal{P}_{m}$, of degree $\leq m, 0 < m \leq \frac{1}{h}\, (\frac{1}{h} \in R^{+})$($m$ which is depends on a_1, a_2, v, p) such that

$$\left\|\frac{F-\mathcal{P}_m}{(h\varphi+h^2)^v}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^{\frac{2}{p}} \left(Ln\frac{1}{h}\right)^{\frac{1}{p}} \text{ If and only if } \left\|\frac{F-\mathcal{P}_m}{(\varphi+h)^v}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^{\frac{2}{p}+v} \left(Ln\frac{1}{h}\right)^{\frac{1}{p}}$$

$$,\mathcal{C} = \mathcal{C}(\alpha, \alpha_1, a_1, a_2, v, p, \mathfrak{b})$$

Such that $a_1, a_2 > -b, \alpha \in (0,1), \alpha_1 = Ln\alpha \in R^{+}, v \in (0, b], h \in (0, \mathfrak{b}), 1 \leq p < \infty, b =$positive integer.

Proof: Let $\left\|\frac{F-\mathcal{P}_m}{(h\varphi+h^2)^v}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^{\frac{2}{p}} \left(Ln\frac{1}{h}\right)^{\frac{1}{p}}$ that is $\left\|\frac{F-\mathcal{P}_m}{(h\varphi+h^2)^v}\right\|^p_{L_{\omega,p}(I)} \leq \mathcal{C}h^2\ Ln\frac{1}{h}$ then

$\left\|\frac{F-\mathcal{P}_m}{(h\varphi+h^2)^v}\right\|^p_{L_{\omega,p}(I)} \leq \mathcal{C}h^2\ Ln\frac{1}{h}$ if and only if $\frac{1}{h^{vp}}\ \left\|\frac{F-\mathcal{P}_m}{(\varphi+h)^v}\right\|^p_{L_{\omega,p}(I)} \leq \mathcal{C}h^2\ Ln\frac{1}{h}$

And $\frac{1}{h^{vp}}\ \left\|\frac{F-\mathcal{P}_m}{(\varphi+h)^v}\right\|^p_{L_{\omega,p}(I)} \leq \mathcal{C}h^2\ Ln\frac{1}{h}$ if and only if $\left\|\frac{F-\mathcal{P}_m}{(\varphi+h)^v}\right\|^p_{L_{\omega,p}(I)} \leq \mathcal{C}h^{2+vp}\ Ln\frac{1}{h}$

Hence $\left\|\frac{F-\mathcal{P}_m}{(\varphi+h)^v}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^{\frac{2}{p}+v} \left(Ln\frac{1}{h}\right)^{\frac{1}{p}}, \mathcal{C} = \mathcal{C}(\alpha, \alpha_1, a_1, a_2, v, p, \mathfrak{b})$

Such that $a_1, a_2 > -b, \alpha \in (0,1), \alpha_1 = Ln\alpha \in R^{+}, v \in (0, b], h \in (0, \mathfrak{b}), 1 \leq p < \infty, b =$positive integer.

Lemma 3.13. Let $F \in W_{\omega,p}^{(v)}(I)$,have the derivative such that $\left(\int_{-\mathfrak{b}}^{\mathfrak{b}} \left|\frac{\acute{F}(x)}{(\varphi(x)+h)^s}\right|^p\ \omega(x)dx\right)^{\frac{1}{p}} < M$, then there exist a polynomial q of a degree $\leq\ m$

$0 < m \leq \frac{1}{h}\, (\frac{1}{h} \in R^{+})$, such that $\left\|\frac{F-q}{(\varphi+h)^{s+1}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}hM, \mathcal{C} = \mathcal{C}(a_1, a_2, p, b, s), \frac{1}{h} \geq [s] + 2, m([s]+2) > \frac{1}{h}, a_1, a_2 > -b, 1 \leq p < \infty, b =$positive integer, s is the integer function of the greatest integer.

Proof: Let $m \in N\ \ni\ (s+2)(m-1) \leq \frac{1}{h}$, let $x = \mathfrak{b}\, cos\, \theta$, and q^{*} be a polynomial of degree $\leq\ m\ \ni q(x) = q^{*}(cos\, \theta\,)$, using trigonometric identities we get

$\varphi(x) = sin\theta, \omega(x) = 2^{a_1+a_2}\ \left|sin\frac{\theta}{2}\right|^{2a_1} \left|cos\frac{\theta}{2}\right|^{2a_2}, a_1, a_2 \geq \frac{-1}{2}$ and $a_1, a_2 \leq \frac{p-1}{2}, 1 \leq p < \infty, dx = -\mathfrak{b}sin\theta d\theta$, hence we get from the above relation.

Let $B = \left\|\frac{F-q}{(\varphi+h)^{s+1}}\right\|_{L_{\omega,p}(I)} = \left(\int_{-\mathfrak{b}}^{\mathfrak{b}} \left|\frac{F(x)-q(x)}{(\varphi(x)+h)^{s+1}}\right|^p\ \omega(x)dx\right)^{\frac{1}{p}}$

$$=\left(\int_{\theta=0}^{\pi}\left|\frac{F(\cos\theta)-q^{*}(\cos\theta)}{(|\sin\theta|+h)^{s+1}}\right|^{p} \mathfrak{b}2^{a_1+a_2}\left|\sin\frac{\theta}{2}\right|^{2a_1}\left|\cos\frac{\theta}{2}\right|^{2a_2}|\sin\theta|d\theta\right)^{\frac{1}{p}}$$

Let $T(t)$ be a trigonometric polynomial which is satisfy (12).Since $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, then we get

$$B=\left(\mathcal{C}\int_{-\pi}^{\pi}\left|\frac{F(\cos\theta)-\int_{-\pi}^{\pi}F(\cos(\theta+t))\,T(t)dt}{(|\sin\theta|+h)^{s+1}}\right|^{p} \mathfrak{b}2^{a_1+a_2}\left|\sin\frac{\theta}{2}\right|^{2a_1+1}\left|\cos\frac{\theta}{2}\right|^{2a_2+1}d\theta\right)^{\frac{1}{p}}$$

$$\leq \mathcal{C}(a_1,a_2,b,p)\int_{-\pi}^{\pi}T(t)dt\int_{u=0}^{t}\left(\int_{-\pi}^{\pi}\left|\frac{\acute{F}(\cos(\theta+u))\sin(\theta+u)}{(|\sin\theta|+h)^{s+1}}\right|^{p}\left|\sin\frac{\theta}{2}\right|^{2a_1+1}\left|\cos\frac{\theta}{2}\right|^{2a_2+1}d\theta\right)^{\frac{1}{p}}du$$

We have $\int_{-\pi}^{\pi}T(t)dt=1$,then

$$B \leq \mathcal{C}\int_{u=0}^{t}\left(\int_{-\pi}^{\pi}\left|\frac{\acute{F}(\cos(\theta+u))}{(|\sin\theta|+h)^{s}}\right|^{p}\left(\frac{|\sin(\theta+u)+h|}{|\sin\theta|+h}\right)^{sp}\left(\frac{|\sin(\theta+u)|^{p}\left|\sin\frac{\theta}{2}\right|^{2a_1+1}\left|\cos\frac{\theta}{2}\right|^{2a_2+1}}{(|\sin\theta|+h)^{p}}\right)d\theta\right)^{\frac{1}{p}}du$$

Let $B_1=\left(\frac{|\sin(\theta+u)+h|}{|\sin\theta|+h}\right)^{sp}$, since

$\sin(\theta+u)=2\sin\left(\frac{\theta+u}{2}\right)\cos\left(\frac{\theta-u}{2}\right)$, then $|\sin(\theta+u)|\leq|\sin\theta|+|\sin u|$ (13)

$$B_1\leq\left(\frac{|\sin\theta+h|+|\sin u|}{|\sin\theta|+h}\right)^{sp}=\left(1+\frac{|\sin u|}{h+|\sin\theta|}\right)^{sp}$$

$=\left(1+\frac{\frac{1}{h}|\sin u|}{1+|\sin\theta|}\right)^{sp}$, then $\frac{|\sin u|}{1+\frac{|\sin\theta|}{h}}=u$, at $\theta=[-\pi,\pi]$

$B_1\leq\mathcal{C}\left(1+\frac{u}{h}\right)^{sp}, \frac{1}{h}<m$ then $B_1\leq\mathcal{C}(1+mu(s+2))^{sp}$

Now let $B_2=\frac{|\sin(\theta+u)|^{p}\left|\sin\frac{\theta}{2}\right|^{2a_1+1}\left|\cos\frac{\theta}{2}\right|^{2a_2+1}}{(|\sin\theta|+h)^{p}}$

Since $|\sin(\theta+u)|^{p}\leq 2^{p}\left|\sin\left(\frac{\theta+u}{2}\right)\right|^{p}\left|\cos\left(\frac{\theta+u}{2}\right)\right|^{p}$

$$=2^{p}\left|\sin\left(\frac{\theta+u}{2}\right)\right|^{p-(2a_1+1)}\left|\cos\left(\frac{\theta+u}{2}\right)\right|^{p-(2a_2+1)}\left|\sin\left(\frac{\theta+u}{2}\right)\right|^{(2a_1+1)}\left|\cos\left(\frac{\theta+u}{2}\right)\right|^{(2a_2+1)}$$

Hence

$$B_2 \le C\frac{\left|sin\frac{\theta+u}{2}\right|^{p-(2a_1+1)}\left|cos\frac{\theta+u}{2}\right|^{p-(2a_2+1)}\left|sin\left(\frac{\theta+u}{2}\right)\right|^{(2a_1+1)}\left|cos\left(\frac{\theta+u}{2}\right)\right|^{(2a_2+1)}\left|sin\left(\frac{\theta}{2}\right)\right|^{(2a_1+1)}\left|cos\left(\frac{\theta}{2}\right)\right|^{(2a_2+1)}}{(|sin\theta|+h)^p}$$

Since $\frac{1}{3}\left(\left|sin\frac{\theta}{2}\right|+h\right)\left(\left|cos\frac{\theta}{2}\right|+h\right) \le (|sin\theta|+h)$

Hence $\frac{1}{(|sin\theta|+h)^p} < \frac{1}{\left(\left|sin\frac{\theta}{2}\right|+h\right)^p\left(\left|cos\frac{\theta}{2}\right|+h\right)^p}$.Also $\left|sin\frac{\theta}{2}\right|^{2a_1+1} < \left(\left|sin\frac{\theta}{2}\right|+h\right)^{2a_1+1}$

$\left|cos\frac{\theta}{2}\right|^{2a_2+1} < \left(\left|cos\frac{\theta}{2}\right|+h\right)^{2a_2+1}$, and

$$\left|sin\frac{\theta+u}{2}\right|^{p-(2a_1+1)} < \left(\left|sin\frac{\theta+u}{2}\right|+h\right)^{p-(2a_1+1)}$$

$$\left|cos\frac{\theta+u}{2}\right|^{p-(2a_2+1)} < \left(\left|cos\frac{\theta+u}{2}\right|+h\right)^{p-(2a_2+1)}$$

Hence $B_2 \le C\left(\frac{\left|sin\frac{\theta+u}{2}\right|+h}{\left|sin\frac{\theta}{2}\right|+h}\right)^{p-(2a_1+1)}\left(\frac{\left|cos\frac{\theta+u}{2}\right|+h}{\left|cos\frac{\theta}{2}\right|+h}\right)^{p-(2a_2+1)}\left|sin\left(\frac{\theta+u}{2}\right)\right|^{(2a_1+1)}\left|cos\left(\frac{\theta+u}{2}\right)\right|^{(2a_2+1)}$

Let $D = D_1D_2$ and $D_1 = \left(\frac{\left|sin\frac{\theta+u}{2}\right|+h}{\left|sin\frac{\theta}{2}\right|+h}\right)^{p-(2a_1+1)}$, $D_2 = \left(\frac{\left|cos\frac{\theta+u}{2}\right|+h}{\left|cos\frac{\theta}{2}\right|+h}\right)^{p-(2a_2+1)}$

Now to find D_1 and D_2 at $\theta = [-\pi,\pi]$ then

$$D_1 = \left(\frac{\left|sin\frac{\theta+u}{2}\right|+h}{\left|sin\frac{\theta}{2}\right|+h}\right)^{p-(2a_1+1)}$$

u Offset by π, hence $sin\frac{\theta+u}{2} = sin\left(\frac{\pi}{2}+\frac{u}{2}\right) = sin\frac{u}{2} - 1$

$$D_1 \le \left(\frac{1+\frac{1}{h}\left|sin\frac{u}{2}\right|}{1+\frac{1}{h}}\right)^{p-(2a_1+1)}, \frac{1}{h} < m(s+2)$$

$$D_1 < (1+m(s+2))^{(2a_1+1)-p}\,(1+mu(s+2))^{p-(2a_1+1)}, a_1 \ge \frac{-1}{2}$$

Hence $D_1 < C(m,p,s)(1+um(s+2))^{p-(2a_1+1)}$

By inequality (13) and since $cos\left(\frac{\theta+u}{2}\right) = -\,sin\frac{u}{2}$ then we get

$D_2 = \left(\frac{\left|sin\frac{u}{2}\right|+h}{h}\right)^{p-(2a_2+1)} = \left(1+\frac{\left|sin\frac{u}{2}\right|}{h}\right)^{p-(2a_2+1)}$ then

$$D_2 < \left(1+\frac{u}{h}\right)^{p-(2a_2+1)}, \frac{1}{h} < m(s+2)$$

Hence $D_2 < (1+um(s+2))^{p-(2a_2+1)}$

Then we get $D \le \mathcal{C}(m,p,s)\big(1+um(s+2)\big)^{p-(2a_1+1)}\,(1+um(s+2))^{p-(2a_2+1)}$

$$B_2 \le \mathcal{C}(m,\mathrm{a}_1,a_2,p)\big(1+um(s+2)\big)^p\left(sin\frac{\theta+u}{2}\right)^{2a_1+1}\left(cos\frac{\theta+u}{2}\right)^{2a_2+1}$$

$$B \le \mathcal{C}\int_0^t\left(\int_{-2\pi}^{2\pi}\left|\frac{\acute{F}(cos(\theta+u))}{(|sin(\theta+u)+h|)^s}\right|^p\big(1+um(s+2)\big)^{sp}\big(1 + um(s+2)\big)^p\left(sin\frac{\theta+u}{2}\right)^{2a_1+1}\left(cos\frac{\theta+u}{2}\right)^{2a_2+1}d(\theta+u)\right)^{\frac{1}{p}}du$$

$$= \mathcal{C}\left(\int_{-2\pi}^{2\pi}\left|\frac{\acute{F}(cos(\theta+u))}{(|sin(\theta+u)+h|)^s}\right|^p\left(sin\frac{\theta+u}{2}\right)^{2a_1+1}\left(cos\frac{\theta+u}{2}\right)^{2a_2+1}d(\theta + u)\right)\left(\int_0^t(1+um(s+2))^{s+1}\,du\right)$$

$$\le \mathcal{C}M\int_0^t\big(1+mu(s+2)\big)^{s+1}du$$

$\le \frac{\mathcal{C}M}{m(s+2)^2}\big(1+tm(s+2)\big)^{s+1}$, since $m(s+2) > \frac{1}{h}$ and $\forall h\in(0,1), t\ge 0$ then

$B \le \frac{\mathcal{C}Mt}{(s+2)}\,(1+tm(s+2))^{s+1}$, we have $\int_{-\pi}^{\pi}T(t)dt = 1$

$B \le \frac{\mathcal{C}M}{(s+2)}\int_{-\pi}^{\pi}|t|\big(1+|t|m(s+2)\big)^{s+1}\,T(t)dt$

$$\le \frac{\mathcal{C}M}{(s+2)}\int_{-\pi}^{\pi}|t|T(t)dt + m^{s+1}|t|^{s+2}(s+2)^{s+1}T(t)dt$$

$$\le \frac{\mathcal{C}M}{(s+2)}\left(\frac{\mathcal{C}_1}{m}+\frac{\mathcal{C}_2 m^{s+1}}{m^{s+2}}\right)$$

$\leq \frac{\mathcal{C}Mm^{-1}}{s+2}$, since $\frac{1}{h} \geq s+2, m(s+2) > \frac{1}{h}$ then $\frac{1}{m(s+2)} < h$. Hence

$\left\|\frac{F-q}{(\varphi+h)^{s+1}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}hM, \mathcal{C} = \mathcal{C}(a_1, a_2, p, b, s), \frac{1}{h} \geq [s]+2, m([s]+2) > \frac{1}{h}, a_1, a_2 > -b, 1 \leq p < \infty, b =$Positive integer, and s is the integer function of the greatest integer.

Lemma 3.14. Let $F \in W_{\omega,p}^{(v)}(I)$, have the derivative such that $\left(\int_{-\mathfrak{b}}^{\mathfrak{b}} \left|\frac{\acute{F}(x)-\frac{d}{dx}H(x)}{(\varphi(x)+h)^s}\right|^p \omega(x)dx\right)^{\frac{1}{p}} < M$, then there exist a polynomial P_m of a degree $\leq m$, $0 < m \leq \frac{1}{h}$ $(\frac{1}{h} \in R^+)$, satisfy

$\left\|\frac{F-P_m}{(\varphi+h)^{s+1}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}hM, M > 0, h \in (0,\mathfrak{b}), \mathcal{C} = \mathcal{C}(a_1, a_2, p, b, s), a_1, a_2 > -b, 1 \leq p < \infty, b =$Positive integer, s is the integer function of the greatest integer.

Proof: By Lemma 3.13, let $F^* \in W_{\psi,p}^{v}(I)$, have the derivative such that

$F^*(x) = F(x) - H(x)$, and $H(x)$ the Heaviside function, then $\acute{F}^*(x) = \acute{F}(x) - \frac{d}{dx}H(x)$ if $H(x)$ is a piecewise function then $\frac{d}{dx}H(x) = \delta_a(x)$, Where $\delta_a(x)$ Dirac function.And if $H(x)$ is an indicator function then $\frac{d}{dx}H(x) = 0$, as $x > 0$, then

$\left\|\frac{\acute{F}^*(x)}{(\varphi+h)^s}\right\|_{L_{\omega,p}(I)} \leq M$, also by lemma (3.13) then there exist a polynomial P_m of degree $\leq m$, such that $P_m(x) = Q_m(x) + H(x), Q_m$ is a polynomial of degree $\leq m$, then by lemma (3.13)

$$\left\|\frac{F-P_m}{(\varphi+h)^{s+1}}\right\|_{L_{\omega,p}(I)} = \left\|\frac{F^*+H-Q_m-H}{(\varphi+h)^{s+1}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}hM$$

$\mathcal{C} = \mathcal{C}(a_1, a_2, s, p, b), a_1, a_2 \geq \frac{-1}{2}, M > 0, h \in (0,\mathfrak{b}), 1 \leq p < \infty, b =$Positive integer, s is the integer function of the greatest integer.

Theorem 3.15. $\forall\, F \in W_{\omega,p}^{(v)}(I), 1 \leq p < \infty, 0 \leq \mu < v$,then there exist a polynomial P_m of a degree $\leq m$, $0 < m \leq \frac{1}{h}$ $(\frac{1}{h} \in R^+)$,such that

$\left\|\frac{F-P_m}{(\varphi+h)^{\mu}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^v, h \in (0,\mathfrak{b}), \mathcal{C} = \mathcal{C}(a_1, a_2, p, b, v, \mu), v \in (0,\mathfrak{b}), a_1, a_2 \geq -\frac{1}{2}, 1 \leq p < \infty, b =$Positive integer.

Proof: $\forall\, F \in W_{\omega,p}^{(v)}(I)$, let $F_{\frac{1}{h}}(F,x) = F_{\frac{h}{\alpha^k}}(F,x)$, hence

$$\left\|\frac{F-P_m}{(\varphi+h)^{\mu}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}\left\|\frac{F-F_{\frac{1}{h}}(F)}{(\varphi+h)^{\mu}}\right\|_{L_{\omega,p}(I)} + \mathcal{C}\left\|\frac{F_{\frac{1}{h}}(F)-P_m}{(\varphi+h)^{\mu}}\right\|_{L_{\omega,p}(I)}$$

$$= \mathbb{L}_1 + \mathbb{L}_2$$

Let $\mathbb{L}_1^p = \left\|\frac{F-F_{\frac{1}{h}}(F)}{(\varphi+h)^\mu}\right\|_{L_{\omega,p}(I)}^p = \int_I \left|\frac{F(x)-F_{\frac{1}{h}}(F,x)}{(\varphi(x)+h)^\mu}\right|^p \omega(x)dx$

We have from theorem (3.6)

$$\int_{A_k} \left|F(x) - F_{\frac{h}{\alpha^k}}(F,x)\right|^p \omega(x)dx \leq \mathcal{C}\left(\frac{h}{\alpha^k}\right)^{-vp}$$

Now by property (2) we get $\alpha^{-k} < \sqrt{1-t_k^2} < \sqrt{1-x^2} = \varphi(x)$

$\frac{1}{(\varphi(x)+h)^\mu} \leq \left(\frac{1}{\alpha^{-k}+h}\right)^\mu$ we have $h < \alpha^{-k} + h$ then $\left(\frac{1}{\alpha^{-k}+h}\right)^\mu \leq \left(\frac{1}{h}\right)^\mu < \left(\frac{1}{h}\right)^v, (\mu < v)$

then $\frac{1}{(\varphi(x)+h)^\mu} \leq \left(\frac{1}{h}\right)^v$.

$\mathbb{L}_1^p \leq \mathcal{C}\sum_{k=0}^{Log_\alpha \frac{1}{h}} \left(\frac{\alpha^{-k}}{h}\right)^{-vp} \left(\frac{1}{h}\right)^{vp}$

$\mathbb{L}_1^p \leq \mathcal{C}\left(\alpha^{Log_\alpha \frac{1}{h}}\right)^{-vp} = \mathcal{C}h^{vp}$

Now, let $P_m(x) = P_{j,\frac{1}{h}}(x) = \int_{x_j}^{x_{j+1}} F(t)dt$ and $F_{\frac{1}{h}}(F,x) = F_{\frac{h}{\alpha^k}}(F,\mathrm{x}) = \frac{\alpha^k}{h}\int_{bkh}^{bkh+bh} F(t)dt$, see that $F_{\frac{1}{h}}(F,x) = P_{j,\frac{1}{h}}(x)$, when $j = 0$, hence $\mathbb{L}_2 = 0$.

$$\left\|\frac{F-P_m}{(\varphi+h)^\mu}\right\|_{L_{\omega,p}(I)}^p \leq \mathcal{C}\,\mathbb{L}_1^p \leq \mathcal{C}\,h^{vp}$$

$\left\|\frac{F-P_m}{(\varphi+h)^\mu}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^v, h \in (0,\mathfrak{b}), \mathcal{C} = \mathcal{C}(a_1,a_2,p,b,v,\mu), v \in (0,\mathfrak{b}), a_1,a_2 \leq -\frac{1}{2}, 1 \leq p < \infty, b =$Positive integer.

Lemma 3.16. $\forall\, F \in W_{\omega,p}^{(v)}(I), 1 \leq p < \infty$ we have

$\left\|\frac{F-\mathcal{P}_m}{(h\varphi+h^2)^\mu}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}\left(\frac{1}{h}\right)^{(\mu-\mathrm{v})}$ *if* and only if $\left\|\frac{F-\mathcal{P}_m}{(\varphi+h)^\mu}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^v, \mathcal{C} = \mathcal{C}(a_1,a_2,v,\mu,p,\mathfrak{b})$. Such that $a_1,a_2 \geq -\frac{1}{2}, v \in (0,b], 0 \leq \mu < v, h \in (0,\mathfrak{b}), 1 \leq p < \infty, b =$positive integer.

Proof: Let $\left\|\frac{F-\mathcal{P}_m}{(\varphi+h)^\mu}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^v$,since

$\left\|\frac{h^\mu(F-\mathcal{P}_m)}{(h\varphi+h^2)^\mu}\right\|_{L_{\omega,p}(I)} = \left\|\frac{F-\mathcal{P}_m}{(\varphi+h)^\mu}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^v$ if and only if

$h^{\mu}\left\|\frac{F-\mathcal{P}_m}{(h\varphi+h^2)^{\mu}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^{v}$, hence $\left\|\frac{F-\mathcal{P}_m}{(h\varphi+h^2)^{\mu}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}h^{v-\mu} = \mathcal{C}\left(\frac{1}{h}\right)^{\mu-v}$

$\mathcal{C}=\mathcal{C}(a_1,a_2,v,\mu,p,\mathfrak{b})$.Such that $a_1,a_2 \geq -\frac{1}{2}, v \in (0,b], 0 \leq \mu < v, h \in (0,\mathfrak{b}), 1 \leq p < \infty, b =$positive integer.

4. Proof of Theorem A

1) By using Theorem (3.12) when $F \in W_{\omega,p}^{(v)}(I)$, then $F^{(k)} \in W_{\omega,p}^{(v)}(I)$, and there exist a polynomial Q_n of degree $\leq k, 0 < k \leq n$. Let $\frac{1}{h_1} \in N \ni \frac{1}{h} \leq \frac{1}{h_1}$, which is satisfy Theorem (3.12)

$\left\|\frac{F-\mathcal{P}_m}{(\varphi+h_1)^{v}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}\frac{h^{\frac{2}{p}}\left(Ln\frac{1}{h_1}\right)^{\frac{1}{p}}}{\left(\frac{1}{h_1}\right)^{v}}$. Since $\frac{1}{h} \leq \frac{1}{h_1}$ then $(\varphi(x)+h)^{-v} \leq (\varphi(x)+h_1)^{-v}$, hence

$$\left\|\frac{F^{(r)}-Q_n}{(\varphi+h)^{v}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}\left\|\frac{F^{(r)}-Q_n}{(\varphi+h_1)^{v}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}\frac{h^{\frac{2}{p}}\left(Ln\frac{1}{h_1}\right)^{\frac{1}{p}}}{\left(\frac{1}{h_1}\right)^{v}}$$

$$\leq \mathcal{C}\frac{h^{\frac{2}{p}}\left(\frac{1}{h}\right)^{v}\left(Ln\frac{1}{h}\right)^{\frac{1}{p}}}{\left(\frac{\frac{1}{h}}{h_1}\right)^{v}} = \mathcal{C}\frac{h^{\frac{2}{p}}\left(Ln\frac{1}{h}\right)^{\frac{1}{p}}}{\left(\frac{1}{h}\right)^{v}}\left(\frac{\frac{1}{h}}{\frac{1}{h_1}}\right)^{v} \quad (14)$$

Let $Q_n(x) = \int_0^x \dots \int_0^x Q_m(u)du + H(x)$ (15)

Where $\int_0^x \dots \int_0^x integral$ for (k-th) times, $H(x)$ Heaviside function, and $Q_m(u)$ is a polynomial of degree $\leq \frac{1}{h}$, and $Q_n^{(k)}(x) = Q_{\mu}(x)$. Now by using Lemma (3.14) and application the inequality (14) for one time we get

$$\left\|\frac{F^{(k-1)}-\acute{Q}_n}{(\varphi+h)^{v+1}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}\frac{h^{\frac{2}{p}}\left(\mathrm{Ln}\frac{1}{h}\right)^{\frac{1}{p}}}{\left(\frac{1}{h}\right)^{v+1}}\left(\frac{\frac{1}{h}}{\frac{1}{h_1}}\right)^{v+1}$$

We repeat this application for k many times, and so we get

$$\left\|\frac{F^{(k-k)}-Q_n^{(k)}}{(\varphi+h)^{v+1}}\right\|_{L_{\omega,p}(I)} \leq \mathcal{C}\frac{h^{\frac{2}{p}}\left(Ln\frac{1}{h}\right)^{\frac{1}{p}}}{\left(\frac{1}{h}\right)^{v+k}}\left(\frac{\frac{1}{h}}{\frac{1}{h_1}}\right)^{v+k}$$

Since $\frac{1}{h} \leq \frac{1}{h_1}$ then $\left(\frac{\frac{1}{h}}{\frac{1}{h_1}}\right)^{v+k} \leq 1$, hence

$$\left\|\frac{F-Q_m}{(\varphi+h)^{v+k}}\right\|_{L_{\omega,p}(I)} \le \mathcal{C}\frac{h^{\frac{2}{p}}\left(Ln\frac{1}{h}\right)^{\frac{1}{p}}}{\left(\frac{1}{h}\right)^{v+k}} = \mathcal{C}h^{\frac{2}{p}+v+k}\left(Ln\frac{1}{h}\right)^{\frac{1}{p}}$$

$\mathcal{C} = \mathcal{C}(a_1, a_2, p, b, v, k, \alpha, \alpha_1)$,Such that $a_1, a_2 > -b, 1 \le p < \infty, v \in (0, \mathfrak{b}], k > 0, \alpha \in (0,1), \alpha_1 = Ln\alpha \in R^+$ and $\mathfrak{b}$ positive integer.

2) This case can be proof by using the same method in (1) and by using theorem (3.15) for $0 \le \mu < v$, we have for $\frac{1}{h} \le \frac{1}{h_1}$ that $(\varphi(x)+h)^{-\mu} \le (\varphi(x)+h)^{-\mu}$ then

$\left\|\frac{F^{(k)}-Q_n}{(\varphi+h)^{\mu}}\right\|_{L_{\omega,p}(I)} \le \mathcal{C}\left\|\frac{F^{(k)}-Q_n}{(\varphi+h_1)^{\mu}}\right\|_{L_{\omega,p}(I)} \le \frac{\mathcal{C}}{\left(\frac{1}{h_1}\right)^{v}}$, since $\frac{1}{h} \le \frac{1}{h_1}$ then $\frac{1}{\left(\frac{1}{h_1}\right)^{v}} < \frac{1}{\left(\frac{1}{h}\right)^{v}}$

Hence $\left\|\frac{F^{(k)}-Q_n}{(\varphi+h)^{\mu}}\right\|_{L_{\omega,p}(I)} \le \frac{\mathcal{C}}{\left(\frac{1}{h}\right)^{v}}$ (16)

Now by using lemma (3.14) and application the inequality (16) for one time then we get

$\left\|\frac{F^{(k-1)}-\acute{Q}_n}{(\varphi+h)^{\mu+1}}\right\|_{L_{\omega,p}(I)} \le \frac{\mathcal{C}}{\left(\frac{1}{h}\right)^{v+1}}$. And by repeat this application for k many times, and so we get

$\left\|\frac{F^{(k-k)}-Q_n^{(k)}}{(\varphi+h)^{\mu+k}}\right\|_{L_{\omega,p}(I)} \le \frac{\mathcal{C}}{\left(\frac{1}{h}\right)^{v+k}}$, using the inequality (15) then we get

$\left\|\frac{F-Q_m}{(\varphi+h)^{\mu+k}}\right\|_{L_{\omega,p}(I)} \le \frac{\mathcal{C}}{\left(\frac{1}{h}\right)^{v+k}} = \mathcal{C}h^{v+k}$,

$\mathcal{C} = \mathcal{C}(a_1, a_2, p, b, v, k, \mu, s)$,Such that $a_1, a_2 > -b, 1 \le p < \infty, v \in (0, \mathfrak{b}], k > 0, \mathfrak{b}$ positive integer and s is the integer function of the greatest integer.

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