Computing the Number of Subgroups of Group $T_{4n}\times C_2$

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Abstract
The aim of this paper to compute the number of subgroups of the group $T_{4n}\times C_2$. S. R. Cavior in year 1975 presented the number of subgroups of the dihedral group computed it is equal to τ(n)+σ(n) and Shelash and Ashrafi computed the number of subgroups of the Dicyclic group T_4n, its equal to τ(2n)+σ(n). We in this project proved that the number of subgroups of direct product $T_{4n}\times C_2$ is equal to 2τ(2n)+τ(n)+3σ(n)+2σ(n/2)..

Keywords: Lattice Subgroups, Dicyclic Group, Direct Product.

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1. Introduction.

Let G is a finite group, Computing the number of subgroups of a finite group, it is one of the important problem and difficult problem in group theory. In the paper, we want compute the number of subgroups of the group $T_{4n}\times C_2$ and study the normality and cyclicity degree of this group. In [2] Cavior, 1975, [1] Calhoun,1987 and [6] Divid and authers in 1990 were computed the number of subgroups of the dicyclic group T_{4n}, it is equal to $\tau(2n)+\sigma(n)$, where $\tau(2n)$ is the number of divisors of n and $\sigma(n)$ is the sum of divisors of n [2] Mario and authers in 2012 computed the number of subgroups of group $C_m \times C_n$. In [6,7,8] computed the number of subgroups and the number of chains of subgroups of the finite of cyclic group. Marius Tărnăuceanu in [11] computed the number of subgroups for some of finite groups.
In [10,11], Shelash and Ashrifa could computed the number of subgroups of the Dicyclic group T_{4n} and found relation between the number of subgroups of the Dihedral group D_{2n} and Dicyclic group T_{4n}.

2. The Preliminaries.

In the section we refer [3,5,9-12]. we shall present of some of the definition and resulting, one of this important result it is Order of subgroup table of finite group this result is defined by Shelash and Ashrefi in [11,12]. Suppose $|G| = 2^r \prod_{i=1}^{s} p_i^{\alpha_i}$ is a prime factorization of $|G|, 2 < p_1 < p_2 < \ldots < p_s$ and δ is all odd divisors of $m = \prod_{i=1}^{s} p_i^{\alpha_i}$.

Table I. Order Subgroups Table when $|G| = n = 2^r\, m$

	1	2	2^2	...	2^r
δ	δ	2δ	$2^2\delta$	...	$2^r\delta$

For example, let order group is equal to 60, this mean the prime factorization of $|G| = 2^2.3.5$

	1	2	2^2
1	1	2	4
3	3	6	12
5	5	10	20
15	15	30	60

We can see that the order subgroup table has exactly $\tau(2^r)$ of rows and $\tau(m)$ of columns. Recall that $\tau(n)$ is the number of all divides of n and the $\sigma(n)$ is the summation of all divides of n. Frattine subgroup $F(G) = \bigcap_{M<G} M$, M is a maximal subgroup of G, it is one of important results in computational group theory, we know $Max(G) = Max(G/F(G))$. $R(G)$ is a intersection of all normal maximal subgroups and $L(G)$ is a intersection of all Self-Normalizing subgroups. it is clear that we can write $F(G) = R(G) \cap L(G)$

3. Main Result

The Dicyclic group is a define by $\langle a, b \mid a^{2n} = b^2 = e, bab = a^{-1} \rangle$ of order 4n. It is have two types of the subgroups $\langle a^d \rangle$ for all $d|2n$ and $\langle a^d, a^j b \rangle$ where $d \mid n$ and $1 \le j \le d$ computed in [9].
So the cyclic group C_2 is a define by $\langle c \mid c^2 = e \rangle$, we can definition the representation of direct product of Dicyclic group and Cyclic group $T_{4n} \times C_2 = \langle a, b, c \mid a^{2n} = b^4 = c^2 = e, a^n = b^2, bab^{-1} = a^{-1}, [a, c] = [b, c] = e \rangle$ and the center $Z(T_{4n} \times C_2) = \langle a^n, c \rangle$ for all n. The group $T_{4n} \times C_2$ have eight different types of the subgroups. We explain those in the following theorem.

Theorem 3.1. Suppose that the $n = 2^r m$ where m is odd number, the number of all subgroups of the group $T_{4n} \times C_2$ is equal to following:

$$Sub(T_{4n} \times C_2) = 2\tau(2n) + \tau(n) + 3\sigma(n) + 2\sigma(\frac{n}{2})$$

Proof: From the following diagram, we can find the all of probability of types of subgroup in group $T_{4n} \times C_2$

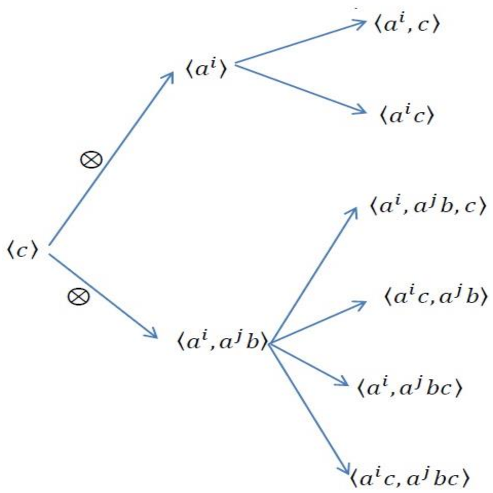

Figure 1. Direct product of subgroups.

From above lattice product subgroups, we can obtain on all of subgroups. Those define by the following:

[1] A subgroups $G_1 = \langle a\char`^ i \rangle \; for \; all \; i|2n$;

[2] A subgroups $G_2 = \langle a^i, c\rangle \; for \; all \; i|2n$;

[3] A subgroups $G_3 = \langle a^i, a^j b\rangle \; for \; all \; i|n$ and $1 \leq j \leq i$;

[4] A subgroups $G_4 = \langle a^i, a^j bc\rangle \; for \; all \; i|n$ and $1 \leq j \leq i$;

[5] A subgroups $G_5 = \langle a^i, a^j b, c \rangle \; for \; all \; i|n$ and $1 \leq j \leq i$;

[6] A subgroups $G_6 = \langle a^i c, a^j b \rangle \; for \; all \; i|\frac{n}{2}$ and $1 \leq j \leq i$;

[7] A subgroups $G_7 = \langle a^i c, a^j bc \rangle \; for \; all \; i|\frac{n}{2}$ and $1 \leq j \leq i$;

[8] A subgroups $G_8 = \langle a^i c \rangle$ for all $i|n$.

The first type of subgroups has the subgroup be equal to $Sub(G_1) = \tau(n)$ where the function $\tau(n)$ is the number of all divisors of n there exists exactly one subgroup of this type and so we obtain $\tau(n)$ cyclic subgroups contained in G_1. The second type of subgroups $Sub(G_2) = \tau(n)$, a similar argument as Part(1). The number subgroups of the third type $Sub(G_3) = \sigma(n)$, in this case, it is easy to see that $\langle a^i, a^j b\rangle = \langle a^u, a^v b\rangle$if and only if $i = v$ and $j = u$. Since $i|n$, all divisors of n are $n, \frac{n}{2}, \cdots, 1$ are $\sum_{i|n} i = \sigma(n)$. The forth type and fifth type of the subgroups is $Sub(G_4) = Sub(G_5) = \sigma(n)$ for any type of those, a similar argument as Part (3). The number subgroups of the sixth type be $Sub(G_6) = Sub(G_7) = \sigma(\frac{n}{2})$ And the $Sub(G_8) = \tau(n)$. In the final the number of subgroups of the group $T_{4n} \times C_2 = 2\tau(2n) + \tau(n) + 3\sigma(n) + 2\sigma(\frac{n}{2})$. The end proof.

Corollary 3.2. Let n is odd number. Then

$$Sub(T_{4n} \times C_2) = 2\tau(2n) + \tau(n) + 3\sigma(n).$$

Proof: Clear that in the number theory, the $\sigma(\frac{n}{m}) = 0$ when $m \nmid n$ since n is an odd number then $Sub(T_{4n} \times C_2) = 2\tau(2n) + \tau(n) + 3\sigma(n)$.

In [9] Shelash and Ashrafi presented a new method to find the number subgroups for each order in group. We will in this project using This method to find the number of subgroups for each order in group $T_{4n}\times C_2$.
In the first we must computing when n is odd number and when n is even number.

Theorem 3.4. Let $n = 2^r m$ and $m = \prod_{i=1}^{s} p_i^{\alpha_i}$ be a positive integer. The number of subgroups table of the group $T_{4n}\times C_2$ is given by the following table:

a) If n is an odd number this mean $r = 0$, then

j	1	2	3	4
	1	2	4	8
δ_i	1	3	$\frac{2n}{\delta_i}+1$	$\frac{n}{\delta_i}$

b) If n is an even number this mean $r \geq 1$, then

j	1	2	3	4	5	⋯	r+2	r+3	r+4
	1	2	4	8	16	⋯	2^{r+1}	2^{r+2}	2^{r+3}
δ_i	1	3	$\frac{2n}{\delta_i}+3$	$\frac{3n}{\delta_i}+3$	$\frac{3n}{2\delta_i}+3$	⋯	$\frac{3n}{2^{(r-2)}\delta_i}+3$	$\frac{3n}{2^{(r-1)}\delta_i}+1$	$\frac{n}{2^r\delta_i}$

Hence, δ_i be divisors of m
Now, we mention some examples that verify the Theorem 3.4.

Example 3.5. Taken $n = 15$, thus the number of subgroups table is given by the following.

*	1	2	2^2	2^3
1	1	3	31	15
3	1	3	11	5
5	1	3	7	3
15	1	3	3	1

From Theorem 3.1. we can see the order subgroup satisfy the above table by the following table.

*	1	2	2^2	2^3

1	$\langle a^{30}\rangle$	$\langle a^{15}\rangle$ $\langle a^{30},c\rangle$ $\langle a^{15}c\rangle$	$\langle a^{15},c\rangle$ $\langle a^{15},a^{j}b\rangle_{1\leq j\leq 15}$ $\langle a^{15},a^{j}bc\rangle_{1\leq j\leq 15}$	$\langle a^{15},a^{j}b,c\rangle_{1\leq j\leq 15}$
3	$\langle a^{10}\rangle$	$\langle a^{5}\rangle$ $\langle a^{10},c\rangle$ $\langle a^{5}c\rangle$	$\langle a^{5},c\rangle$ $\langle a^{5},a^{j}b\rangle_{1\leq j\leq 5}$ $\langle a^{5},a^{j}bc\rangle_{1\leq j\leq 5}$	$\langle a^{5},a^{j}b,c\rangle_{1\leq j\leq 5}$
5	$\langle a^{6}\rangle$	$\langle a^{3}\rangle$ $\langle a^{6},c\rangle$ $\langle a^{3}c\rangle$	$\langle a^{3},c\rangle$ $\langle a^{3},a^{j}b\rangle_{1\leq j\leq 3}$ $\langle a^{3},a^{j}bc\rangle_{1\leq j\leq 3}$	$\langle a^{3},a^{j}b,c\rangle_{1\leq j\leq 3}$
15	$\langle a^{2}\rangle$	$\langle a\rangle$ $\langle a^{2},c\rangle$ $\langle ac\rangle$	$\langle a,c\rangle$ $\langle a,ab\rangle$ $\langle a,abc\rangle$	$\langle a,ab,c\rangle$

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