Numerical Solutions of Nonlinear Equations in Optimization

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Abstract

The goal of this paper is to find a better approximate value (whether it is maximize or minimize) for one-dimensional non-linear equations using one of the most famous algorithms in numerical optimization is the golden search method.

Keywords: Numerical Analysis, numerical optimization algorithm, the golden search method, nonlinear equations.

1. Introduction

The concept of optimization is now well established as a fundamental principle in the analysis of many complex decision or allocation problems. It has a certain philosophical elegance that is hard to argue with, and it frequently provides a level of operational simplicity that is indispensable. As a result of using this optimization methodology, one is able to focus on a single target meant to quantify performance and measure the quality of the decision when working with a complex problem. The selection of choice variable values may be limited by constraints that aim to maximize (or diminish, depending on the formulation) this one objective. Optimisation may be an appropriate framework for analysis.which be possible to isolate and characterize a problem's objective, be it profit or loss in a business environment, speed or distance in a physical challenge, projected return for hazardous investments or social welfare in government planning.

2. The one - dimensional search method [8]

We will discuss the problem of minimizing (or maximizing) an objective function, which is a problem with one dimension.Utilizing an iterative search technique, commonly referred to as a linesearch method, is the strategy that will be used. The following are some of the reasons why one-dimensional search algorithms are interesting to consider. First, these challenges are unique instances of search methods that are often applied to multivariable issues. Second, they are integrated into generic multivariable algorithms as components of those algorithms. iteration,. In an iterative algorithm, we start with an initial candidate solution $\zeta^{(0)}$ and generate a sequence of iterates $\zeta^{(1)}, \zeta^{(2)}, \ldots$ For each iteration

k = 0,1,2,..., the new point $\zeta^{(k+1)}$ depends on $\zeta^{(k)}$ and the objective function f. In other cases, the algorithm may just take use of the value of f at certain points, or its first derivative, or its second derivative.. In this search, we study salgorithm Golden section method.

3. Golden section method [3]

The optimal solution to a one-dimensional non-linear programming issue can be found using this method. The golden section approach is similar to other elimination procedures such as the Fibonacci method, binary search, and other search techniques in that we exclude the supplied region repeatedly, given the uncertainty period. However, there are a few things to note about the golden section approach, including the fact that it is wholly based on a single ratio known as the golden ratio.

Using the gold segment search method, you can locate the maximum or minimum of a single-modal function, respectively. A single minimum or maximum exists in the period for a monomodal function[a,b]).

in Figure 1, Selecting three points ζ_0 , ζ_1 and ζ_n $(\zeta_0 < \zeta_1 < \zeta_n)$ along the x-axis with the function's values $f(\zeta_0), f(\zeta_1)$ and $f(\zeta_n)$ respectively. Since $f(\zeta_1) > f(\zeta_0)$ and $f(\zeta_0) > f(\zeta_n)$, the maximum must be between ζ_0 and ζ_n .

the fourth point is represented by ζ_2 is chosen to be between the larger of the two intervals of $[\zeta_0, \zeta_1]$ and $[\zeta_1, \zeta_n]$.

Suppose the interval $[\zeta_0, \zeta_1]$ is lager than $[\zeta_1, \zeta_n]$, we would chose $[\zeta_0, \zeta_1]$ as the interval in which ζ_2 is chosen.

If $f(\zeta_2) > f(\zeta_1)$ then the new three points would be $\zeta_0 < \zeta_2 < \zeta_1$; else if $f(\zeta_2) < f(\zeta_1)$ then the new three points are $\zeta_2 < \zeta_1 < \zeta_n$ until the distance between the outer points is sufficiently small, this process repeats.

We selecte the first midpoint ζ_0 to equalize the ratio of the lengths as shown in Formula (1) where σ and τ are distance as shown in Figure (2). Note that $\sigma + \tau$ is equal to the distance between the lower and upper boundary points ζ_0 and ζ_n

$$\frac{\sigma}{\sigma + \tau} = \frac{\tau}{\sigma} \ldots(1)$$

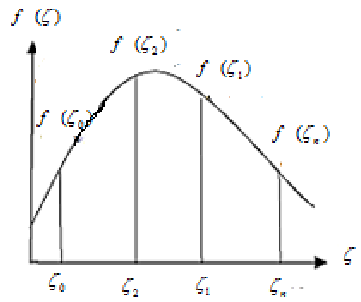

Figure (1) Gutter cross-section.

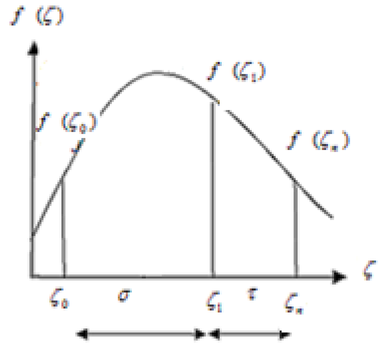

Figure (2) Determine the first midpoint

The second midpoint ζ_2 is chosen similarly in the interval σ to satisfy the following ratio in (2) where the distances of σ and τ are shown in Figure 3.

$$\frac{\tau}{\sigma} = \frac{\tau - \sigma}{\tau} \quad ...(2)$$

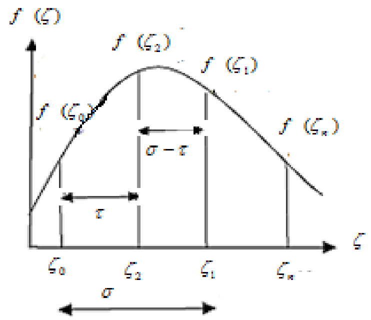

Figure (3) Determine the second midpoint

The ratio in Formulas (1) and (2) are equal and have a special value known as the Golden Ratio. The Golden Ratio has been used since ancient times in various fields such as architecture, design, art and engineering. to ascertain the value of the Golden Ratio let $T = \sigma/\tau$ then Formula (1) can be written as

$1+T=\frac{1}{T}$ or

$T^2+T-1=0$ ...(3)

The positive root of the formula can be obtained by using the quadratic formula (3) :-

$$T=\frac{-1+\sqrt{1-4(-1)}}{2}\Rightarrow T=0.61803$$

To put it another way, the points ζ_1 and ζ_2 are picked so that the distance between them and the search region's limits is equal to the golden Ratio, as shown in Figure (4).

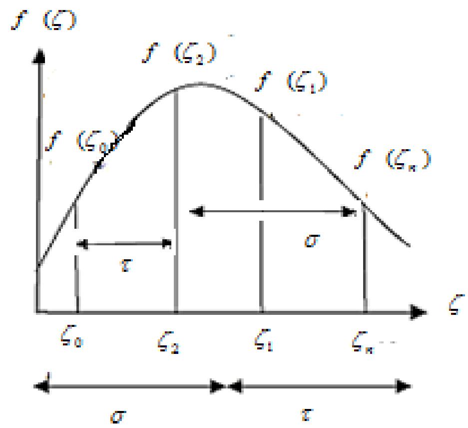

Figure (4) The relationship between the points ζ_1 and ζ_2 and boundary points

Then we find a new, smaller interval in which the function's maximum value is obtained. We know that the new interval is either $[\zeta_0, \zeta_2, \zeta_1]$ or. $[\zeta_2, \zeta_1, \zeta_n]$

The function is assessed at the intermediate points ζ_2 andto determine which of these intervals ζ_1 will be taken into consideration in the following iteration,If, then the new region of $f(\zeta_2) > f(\zeta_1)$ interest will be $[\zeta_0, \zeta_2, \zeta_1]$; else if $f(\zeta_2) < f(\zeta_1)$ then the new region of interest will be $[\zeta_2, \zeta_1, \zeta_n]$.

In Figure (4), we see that $f(\zeta_2) > f(\zeta_1)$, therefore our new region of interest is $[\zeta_0, \zeta_2, \zeta_1]$. It's worth mentioning that the limits of the new, smaller territory ζ_0 and ζ_1 have been determined. We already have one of the midpoints namely ζ_2, which is conveniently positioned at a point where the distance between the boundaries is the Golden Ratio. Then find the midpoint once more, The process of determining a new smaller region of interest and a new intermediate point will be continued until the distance between the boundary points is small enough.

4. The Golden Section Search Algorithm

To find the maximum of a function $f\ (\zeta)$, use the following algorithm:

Initialization: Determine ζ_l and ζ_n which is known to contain the maximum of the function $f(\zeta)$.
The first step: find two points ζ_1 and ζ_2 such that
$\zeta_1 = \zeta_0 + \omega$, $\zeta_2 = \zeta_n - \omega$

Where $\omega = \frac{\sqrt{5}-1}{2}(\zeta_n - \zeta_0)$

The second step: We calculate $f\ (\zeta_1)$ and $\zeta_0, \zeta_1, \zeta_2$ then determine new , $f\ (\zeta_1) > f\ (\zeta_2)$ .If $f\ (\zeta_2)$ andas shown in Formulas set (5). Note that the only new calculation is done to determine the ζ_n new ζ_1

$$\begin{aligned}
\zeta_0 &= \zeta_2 \\
\zeta_2 &= \zeta_1 \\
\zeta_n &= \zeta_n \\
\zeta_1 &= \zeta_0 + \omega \qquad \text{...(5)}
\end{aligned}$$

If $f\ (\zeta_1) < f\ (\zeta_2)$,then determine new $\zeta_0, \zeta_1, \zeta_2$ and ζ_n as shown in Formulas set (6). Note that the only new calculation is done to determine the new ζ_2

$$\begin{aligned}
\zeta_0 &= \zeta_0 \\
\zeta_n &= \zeta_1 \\
\zeta_1 &= \zeta_2 \\
\zeta_2 &= \zeta_n - \omega \qquad \text{...(6)}
\end{aligned}$$

The third step: If $\frac{\zeta_n + \zeta_0}{2}$ (a sufficiently small number), then the maximum occurs at $\zeta_n - \zeta_0 < \in$ and stop iterating, else go to The second step.

5. Examples

Example(1):- apply the golden section search method to maximize the function $f\ (\zeta) = \ln \zeta + \cos(\zeta)$ on the interval $[1, 2]$.

Solution : $\omega = \frac{\sqrt{5}-1}{2}(\tau - \sigma) \Rightarrow \omega = 0.618034$

$\zeta_1 = \sigma + \omega \Rightarrow \zeta_1 = 1.618034$, $f\left(\zeta_1\right) = 0.433992$

$\zeta_2 = \tau - \omega \Rightarrow \zeta_2 = 1.381966$, $f\left(\zeta_2\right) = 0.511217$

$\sigma = 1$ is the upper term of interval

$\tau = 2$ is the lower term of interval

$\omega = \frac{\sqrt{5}-1}{2}(\tau - \sigma) \Rightarrow \omega = 0.618034$ is golden number

Since $f\left(\zeta_2\right) > f\left(\zeta_1\right)$ Then Delete the interval $\left[\sigma, \zeta_1\right]$

We tested the golden search method to find the best approximate solution based on the table (1), which includes k means the number of iterations and r means length of interval.

This test was on the non-linear optimization problem, and the results showed the good approximate solution appears after fourteen iterations. As we can see 12-14 the results.

table (1)

K	r	ζ_1	$f\left(\zeta_1\right)$	ζ_2	$f\left(\zeta_2\right)$
1	1	1.618034	0.433992	1.381966	0.511217
2	0.381966	1.236068	0.433992	1.618034	0.540448
3	0.145898	1.618034	0.511217	1.291796	0.531428
4	0.055728	1.270510	0.433992	1.618034	0.535212
5	0.021286	1.618034	0.531428	1.278640	0.533815
6	0.008131	1.275535	0.433992	1.618034	0.534356
7	0.003106	1.618034	0.533815	1.276721	0.534150
8	0.001186	1.276268	0.433992	1.618034	0.534229
9	0.000453	1.618034	0.534150	1.276441	0.534199
10	0.000173	1.276375	0.433992	1.618034	0.534210
11	0.000066	1.618034	0.534199	1.276400	0.534206
12	0.000025	1.276391	0.433992	1.618034	0.534207
13	0.000010	1.618034	0.534206	1.276394	0.534207
14	0.000004	1.276393	0.433992	1.618034	0.534207

the maximize e function $f(\zeta) = \ln\zeta + \cos(\zeta)$ on the interval $[1,2]$ is 1.276393

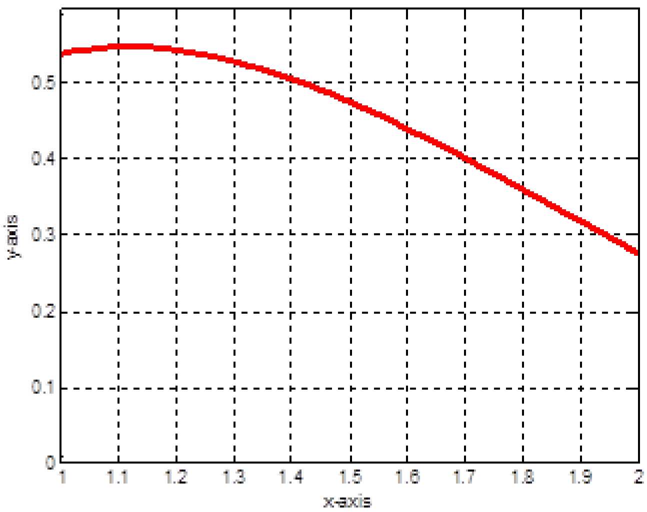

Figure (5) graph of Function $\left(f(\zeta)=\ln\zeta+\cos(\zeta)\right)$

Example (2):- apply the golden section search method to maximize the function $f(\zeta)=2e^{\zeta}-\dfrac{\zeta^2}{10}$ on the interval $[0,1]$.

Solution :-

$\sigma=0$ is the upper term of interval

$\tau=1$ is the lower term of interval

$\omega=\dfrac{\sqrt{5}-1}{2}(\tau-\sigma)\Rightarrow\omega=0.618034$ is golden number

$\zeta_1=\sigma+\omega\Rightarrow\zeta_1=0.618034$, $f(\zeta_1)=3.672357$

$\zeta_2=\tau-\omega\Rightarrow\zeta_2=0.381966$, $f(\zeta_2)=2.915735$

We tested the golden search method to find the best approximate solution based on the table (2), which includes k means the number of iterations and r means length of interval.

This test was on the non-linear optimization problem, and the results showed the good approximate solution appears after fourteen iterations.

table (2)

k	r	ζ_1	$f(\zeta_1)$	ζ_2	$f(\zeta_2)$
1	1	0.618034	3.672357	0.381966	2.915735
2	0.381966	0.381966	2.915735	0.763932	4.235042
3	0.145898	0.708204	2.915735	0.381966	4.010527
4	-0.236068	0.472136	4.010527	0.708204	3.184539

5	0.090170	0.673762	3.184539	0.472136	3.877811
6	-0.145898	0.527864	3.877811	0.673762	3.362751
7	0.055728	0.652476	3.362751	0.527864	3.798006
8	-0.090170	0.562306	3.798006	0.652476	3.477809
9	0.034442	0.639320	3.477809	0.562306	3.749511
10	-0.055728	0.583592	3.749511	0.639320	3.550873
11	0.021286	0.631190	3.550873	0.583592	3.719851
12	-0.034442	0.596748	3.719851	0.631190	3.596794
13	0.013156	0.626165	3.596794	0.596748	3.701638
14	-0.021286	0.604878	3.701638	0.626165	3.625471

the minimize e function $f(\zeta) = 2e^{\zeta} - \frac{\zeta^2}{10}$ on the interval $[0,1]$ is 0.604878**.**

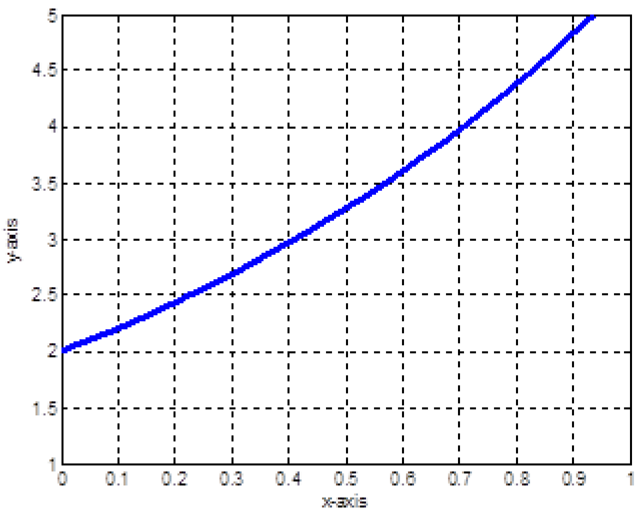

Figure (6) graph of Function $\left(f(\zeta) = 2e^{\zeta} - \frac{\zeta^2}{10}\right)$

Example(3):- use the golden section search method to find maximize the function $f(\zeta) = \zeta^3 + 2\zeta^2 + 5\zeta + 8$ on the interval $[-1,1]$.

Solution :-

$\sigma = 1$ is the upper term of interval

$\tau = -1$ is the lower term of interval

$\omega = \frac{\sqrt{5}-1}{2}(\tau - \sigma) \Rightarrow \omega = 0.618034$ is golden number

$\zeta_1 = \sigma + \omega \Rightarrow \zeta_1 = 0.236068$, $f(\zeta_1) = 9.236068$

$\zeta_2 = \sigma - \omega \Rightarrow \zeta_2 = \text{-}0.236068$, $f(\zeta_2) = 7.321213$

We tested the golden search method to find the best approximate solution based on the table (3), which includes k means the number of iterations and r means length of interval.
This test was on the non-linear optimization problem, and the results showed the good approximate solution appears after nine iterations.

table (3)

K	r	ζ_1	$f(\zeta_1)$	ζ_2	$f(\zeta_2)$
1	2.000000	0.236068	9.236068	-0.236068	7.321213
2	0.763932	-0.236068	7.321213	0.527864	11.651743
3	0.291796	0.416408	7.321213	-0.236068	10.604812
4	-0.472136	-0.055728	10.604812	0.416408	7.792443
5	0.180340	0.347524	7.792443	-0.055728	10.035933
6	-0.291796	0.055728	10.035933	0.347524	8.238614
7	0.111456	0.304952	8.238614	0.055728	9.713144
8	-0.180340	0.124612	9.713144	0.304952	8.578023
9	0.068884	0.278640	8.578023	0.124612	9.524398

the minimize e function $f(\zeta) = \zeta^3 + 2\zeta^2 + 5\zeta + 8$ on the interval $[-1,1]$ is 0.124612**.**

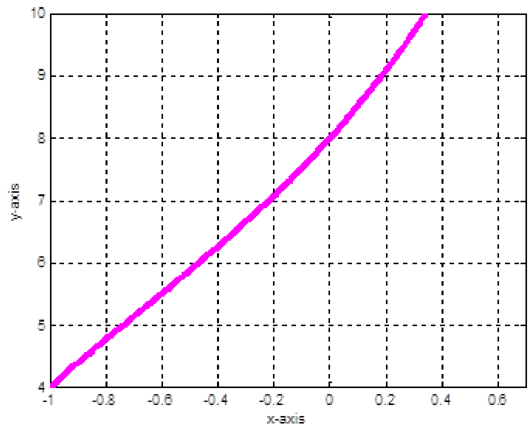

Figure (7) graph of Function $f(\zeta) = \zeta^3 + 2\zeta^2 + 5\zeta + 8$

6.Applications of the golden section method

The golden section method is an old mathematical technique that has been utilized in a number of disciplines, including painting, drawing, and design. It has also been employed in blasting engineering, where it was used to blow through the hole and destroy the engineering of tall structures.

7. Conclusion

In this search,We talked about the golden section strategy which refines the search by excluding particular regions based only on function evaluations. In the golden section approach no gradient computation is necessary. The number 0.61803, known as the golden number in aesthetics, which has significance in aesthetics.We used various unrestricted optimization problems and got the results shown in the search.

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