Some Unicyclic Graphs, Bistar Graph and their Super Domination Number

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Abstract

Consider simple connected undirected graph having V as the vertex set and E as the edge set. A set $S \subseteq V$ is called the super dominating set of G if $\forall v \in \bar{S}, \exists u \in S : N(u) \cap \bar{S} = \{v\}$, where $\bar{S} = V \setminus S$. The least cardinality of the super dominating sets in G is the super domination number, $\gamma_{sp}(G)$. Some graphs were selected and analysed to obtain the super domination number.

Keywords: Super dominating set; Super dominating number; Helm Graph
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1 Introduction

Let us consider the graphs which are simple, connected, undirected and finite. Let V and E represent the set of all vertices and edges of G. The open neighborhood of a vertex $u \in V$ of a graph G, N (u) is the collection of all vertices adjacent to the vertex $u \in V$. For a set $S \subseteq V$ and we define the complement of S as $\bar{S} = V(G) - S$. The private neighbor of the vertex v with respect to the set $\bar{S}$ is the vertex u ,if $N(u) \cap \bar{S} = \{v\}$.

A set $S \subseteq V, S$ is said to be dominating set of G if $N(v) \cap S \neq \emptyset, \forall v \in \bar{S}$. Consider all the dominating sets in G, the set with least cardinality among these dominating sets is the domination number of G, $\gamma(G)$.

The theoretical background about domination and the preliminary theorems were obtained from the book [6]. The concepts, definition and basic results of super domination number in several types of graphs are discussed in [7] by Lemanska et al.

A set $S \subseteq V$ is said to be a super dominating set of G, if $\forall v \in \bar{S}, \exists u \in S$ such that $N(u) \cap \bar{S} = \{v\}$. The minimum cardinality of a super dominating set in G called Super dominating number, $\gamma_{sp}(G)$.

In this paper we analysed super domination number of some graphs. When a new concept is introduced, the required definitions will be introduced.

Some basic results in super domination which can be used in the current research are given below [7],

2 Preliminary Results

1. For a Path Graph P_n, $n \geq 3$, $\gamma_{sp}(P_n) = \left\lceil \frac{n}{2} \right\rceil$.
2. For a Complete Graph K_n, $n \geq 2$, $\gamma_{sp}(K_n) = n - 1$.
3. For a Cycle Graph C_n , $n \geq 3$,

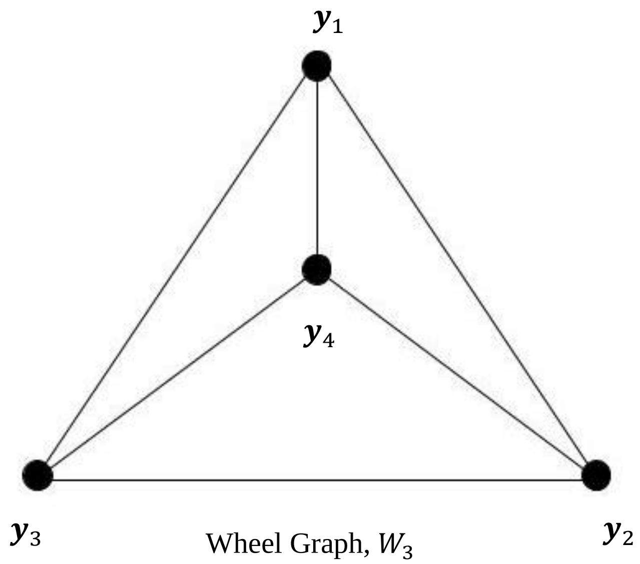
$$\gamma_{sp}(C_n) = \begin{cases} \left\lceil \frac{n+1}{2} \right\rceil, & n \equiv 1,2 (\text{mod } 4) \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise} \end{cases}$$

3 Super Domination Number of some particular Graphs

We analysed the super domination number of Wheel graph, Helm graph, Fan graph, and Bistar graph to obtain their super domination number.

3.1 Wheel Graph

The Wheel Graph is created by attaching an extra vertex to the cycle graph C_n for $n \geq 3$, then joining the additional vertex to each of the n vertices in C_n with new edges. It is denoted by W_n.



Wheel Graph, W_3

Theorem 3.1.1:- Consider a wheel graph W_n with $n \geq 3$. The super domination number of W_n is given by $\gamma_{sp}(W_n) = n$.

Proof: Consider $S = \{y_1, y_2, \dots, y_n\} \subseteq V(W_n)$ and $\bar{S} = \{y_{n+1}\}$. Clearly, S is a dominating set of W_n as $N(y_{n+1}) \cap S \neq \emptyset$. Also we have,

$N(y_i) \cap \bar{S} = \{y_{n+1}\}$ for $i = 1,2, \dots, n$ and $\forall\, y_i \in S$.

Therefore, y_i is the private neighbourhood of y_{n+1} w.r.t $\bar{S}$.

Hence by definition of super dominating set, S is a super dominating set of W_n. Thus $|S| \leq n$. To check if $|S| < n$.

Let us assume that $|S| < n$. Choose $S = \{y_j : 1 \leq j \leq n-1\}$ then the vertices in $\bar{S}$ doesn't have a private neighbour as

$$N(y_i) \cap \bar{S} = \{y_{n+1}\}, i = 1,2,\cdots,n-1$$

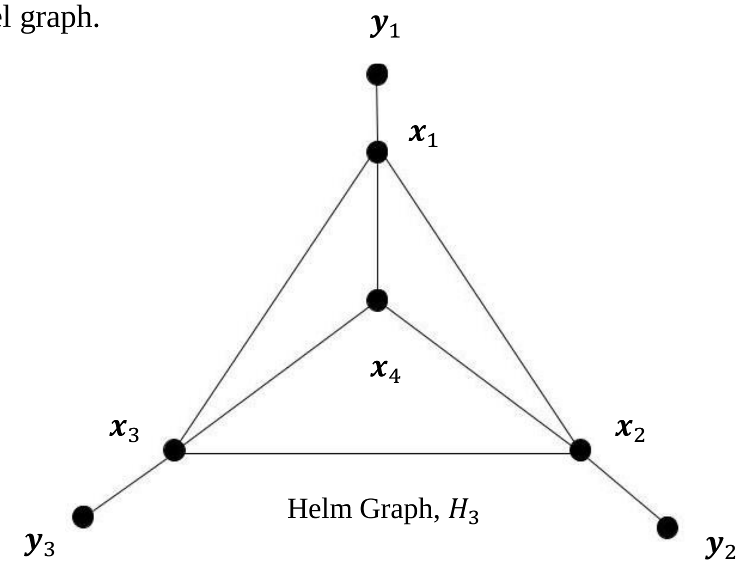
$$N(y_{n-1}) \cap \bar{S} = \{y_n, y_{n+1}\},$$

which is a contradiction to the definition. Thus, the minimum number of elements in the super dominating set is,

$$\gamma_{sp}(W_n) = n.$$

3.2 Helm Graph

Helm Graph, H_n, is a graph constructed by attaching a pendant edge to each node of the cycle C_n to an $n-$wheel graph.



Helm Graph, H_3

Theorem 3.2.1:- The super domination number of a Helm graph H_n, $n \geq 3$, is given by $\gamma_{sp}(H_n) = n + 1$.

Proof: Let H_n be a helm graph with $V(H_n) = \{x_i: 1 \leq i \leq n+1\} \cup \{y_j: 1 \leq j \leq n\}$ and$E(H_n) = \{x_i y_j; 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{x_i x_{i+1}: 1 \leq i \leq n\} \cup \{x_i x_{n+1}: 1 \leq i \leq n\} \cup \{x_1 x_n\}$where $\{y_i : 1 \leq i \leq n\}$ are the pendant vertices attached to the edges at $x_1, \ldots, x_n$. Consider $S = \{y_1, \ldots, y_n, x_{n+1}\} \subseteq V(H_n)$. S is definitely a dominating set of H_n, since $N(y_j) \cap S \neq \emptyset$ for all $y_j \in S$, where $\bar{S} = \{x_1, \ldots, x_n\}$.The set S is a super dominating set of $H_n$because

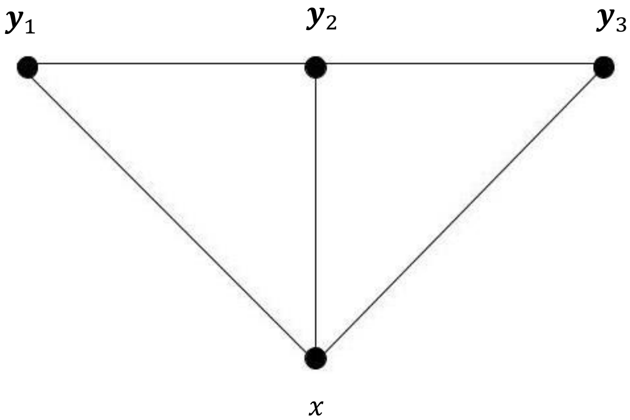
$N(y_j) \cap \bar{S} = \{x_i\}$ for $i = 1,2, \ldots, n$.

Hence, $|S| \leq n+1$. Consider the case when $|S| < n+1$, say n. Let $S = \{y_1, \ldots, y_n\}$ then S will no longer be a dominating set which violates the condition of super domination.

Hence, $|S| = n + 1 \Longrightarrow \gamma_{sp}(H_n) = n + 1$.

3.3 Fan Graph

The Fan graph, F_n, is the graph constructed by attaching all of the vertices in the path P_n to a single vertex.



Fan Graph, F_3

Theorem 3.3.1:- The super domination number of a Fan Graph F_n , $n \geq 3$ is given by $\gamma_{sp}(F_n) = n$.

Proof: Consider a Fan Graph with $V(F_n) = \{y_j : 1 \leq j \leq n\} \cup \{x\}$and $E(F_n) = \{y_j y_{j+1} : 1 \leq j \leq n-1\} \cup \{y_j x: 1 \leq j \leq n\}$. Let the vertices $\{y_j : 1 \leq j \leq n\}$lies on the path graph P_n which are joined to the central vertex x in F_n.

Choose $S = \{y_1, y_2, \ldots, y_n\} \subseteq V(F_n)$ and $\bar{S} = \{x\}$. Obviously S is a dominating set of F_n as $N(y_n) \cap S \neq \emptyset$. Also, S is a super dominating set since, $N(y_j) \cap \bar{S} = N(y_j) \cap \{x\} = \{x\}\ \forall y_j \in S$, $j = 1, 2, \ldots, n$. Therefore, y_n is the private neighbourhood of x w.r.t $\bar{S}$.

Hence by definition of super dominating set S is a super dominating set of F_n. Thus $|S| \leq n$. To check if $|S| < n$.

Let us assume that $|S| < n$. Choose $S = \{y_j : 1 \leq j \leq n-1\}$ then the vertices in $\bar{S}$ does not have a private neighbour as

$$N(y_i) \cap \bar{S} = \{y_n\}, i = 1,2, \ldots \ldots, n-1$$

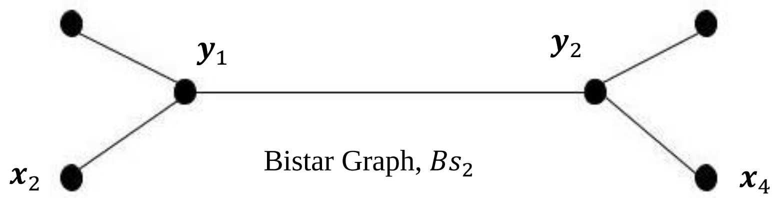
$$N(y_{n-1}) \cap \bar{S} = \{y_n, x\},$$

which is a contradiction to the definition. Thus, the minimum number of elements in the super dominating set is n. Hence, $\gamma_{sp}(F_n) = n$.

3.4 Bistar Graph

The central vertices of two star graphs are connected by an edge to form a Bistar Graph, Bs_n.

x_1 x_3



Bistar Graph, Bs_2

Theorem 3.4.1:- Consider a bistar graph Bs_n with $n \geq 3$. The super domination number is given by $\gamma_{sp}(Bs_n) = 2n$.
Proof: Let $\{x_j y_1 : 1 \leq j \leq n\}$, $\{x_j y_2 : 1 \leq j \leq n\}$ be the vertices in the star graph's $K_{1,n}$. Choose $S = \{x_1, x_2, \dots, x_{2n}\} \subseteq V(Bs_n)$ and $\bar{S} = \{y_1, y_2\}$.Obviously, S is a Dominating set of Bs_n.

We know that, $\forall\, u \in S$,

$$N(u) \cap \bar{S} = \{x_j\} \cap \{y_i\} = \{y_i\}, y_i \in \bar{S}, i = 1,2 .$$

S is a Super Dominating Set and $|S| = |\{x_1, x_2, \dots, x_{2n}\}| = 2n$. To check if S is minimum super dominating set
If $|S| > 2n$, S contains at least a vertex more than $|S|$, say $2n + 1$. Thus, $|\bar{S}| = 1$.Next let us assume that $|S| < 2n$. Then the vertices x_j , y_i in $\bar{S}$ is adjacent to some vertices say x_j: $1 \leq j \leq 2n - 1$ i.e,

$$\bar{S} = \{y_1, y_2, x_{n+1}\}.$$

So we have, $|S| \geq 2n$. The least cardinality of a super dominating set is the super domination number, so we get,

$$\gamma_{sp}(Bs_n) = 2n.$$

4 Conclusion

The exact value of super domination number of certain graphs are obtained in this paper. We further extended our study to general graphs and obtained the relationship of super domination number and domination number. We have initiated our study in comb product of graphs to obtain their super domination number.

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