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Solving the Classical Problems in Field of Extremal Graph Theory: A Review

Mahaveer P. Gadiya
Faculty of Engineering,
MIT World Peace University,
Pune 38 (MS)India

Vinayak Kishan Nirmale
Faculty of Engineering,
MIT World Peace University,
Pune 38 (MS)India

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Abstract

In various field the field of mathematics plays vital role. For the structural modeling the graph theory is the best structure. This structural theory leads towards development and varies according to real world. The graph theory began in year1735 in presence of the problem of Koinsberg bridge .One most favorable graph theory is Extremal graph theory which is the branch of combinatorics (itself is area of mathematics generally it is an intersection of the extremal combinatorics and the graph theory. Extremal graph theory represents the global properties of the graph impacts local substructures. Extremal graph theory is dealing with quantitative connections in between global and local graph properties. This paper focused the basic concepts related to extremal graph theory along with Theorems .In short, this paper gave review of concept related to the extremal graph theory as a part of research in various field along with different application area. Simple classical problem is considered for the discussion.

Keywords*:* Classical graph theory, extremal, vertex, self-link. Turán’s,Mantel.

Introduction:
Section I

The study of graph in mathematics domain is termed as graph theory, is nothing but the mathematical structures allow the model which represents the relation between objects in pair. The important part of this structure is node called as vertices joined by the links i.e. edges. Graphs are an important principal and plays great role in discrete mathematics [1].

Mathematically, graphs are defined in terms of ordered pair. It is represented as $G(v,e)$.Here ‘ v’ represents the node means vertices while ‘e’ represents a set of links (edges) .Those are the unordered pairs. Mathematically this set is given as,

$$e \subseteq \{\{x, y\} | x, y \in v \text{ and } x \neq y\} \dots \dots \dots \dots \dots .. (1)$$

Such types of graphs are called as undirected simple graph. In x and y is called as endpoints. Simple graph structure is shown in fig 1.

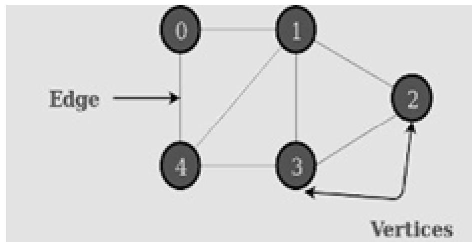

Fig.1 Graph structure

Graph theory is best classical toolbox for the engineers, Mathematicians and the scientist, even for any scientist having the different research area. Hence it becomes a powerful tool with complexity. As a graph theory supports to simplification, which allows for revisiting. Classical graph has edges and vertices .Edges or links generally represented by the straight line[2].

Fields of graph theory:

As, it is well-known to everyone that graph theory is an important field/area about the graphs. It is tool used to model the relationship. Basic structure of the graph is shown in fig.2 along with edge ,vertex (node) ,loop, self-arc, self link ,multiple arc etc. [3].

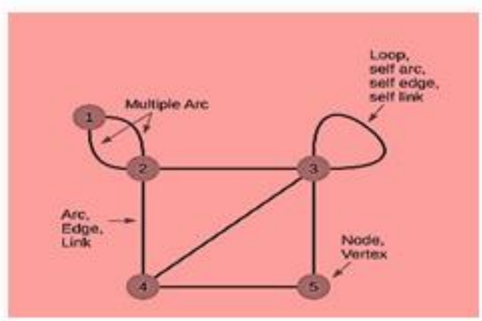

Fig.2 Different fields of graph

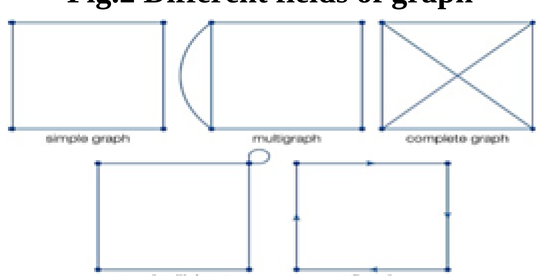

Fig.3 types of graph

An appropriate representation of graph is generally in the form of the matrices related to it. Very popular and relevant depiction of graph is adjacent matrix. Adjacent matrices is the metrics who has ij^{th} element is 1 where the existence of edge is in between ith and jth vertices of graph shown in fig.4 here the degree of vertex is defined in terms of adjacent matric $d(i)=\Sigma jA_{ij}$. It is powerful way

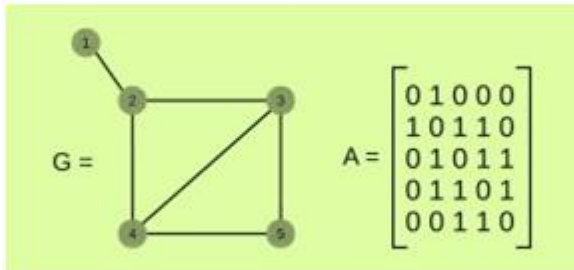

Fig. 4 Matric representation of the graph.

Graph theory is the relation and connection between nodes/vertices with line. This theory plays an important role in field of computer science. It is helpful for demonstration of concept of scientific discipline. This theory used for modeling and analysis of the network. It is generally topological, approbation qualitative as well as quantitative. Applications of graph theory are to find shortest path in network ,to analyze geographical approaches. In mathematics domain , network analysis is applied in various graph concepts for modeling of the functional connectivity [4].The overall substructure within a network summarizes at specific node. Also this theory is used to judge the eigenvalue a specific node. Many researchers were focused on the nodel properties. Rest paper is organized as: Section II literature survey, section III includes mathematical concepts and theorems related to extremal graph theory. Section IV and section V summarizes the conclusion andreferences.

Section II Literature survey:

Abello, J. et al. (1998). In this study, they described new method for designing extremal graph algorithms. It is used to create simple extremal algorithms for minimum spanning trees, computing connected components, maximal matching in undirected graphs and multi-graphs and bottleneck minimum spanning trees. Their I/O bounds take part with those of older methods. Theiralgorithms are helpful to standardcheck pointing and programming language optimization tools. The data-structural approach in this study will develop better graph algorithms that utilize parallel disks also remains open [5].

Balaji N et al. (2021). The main objective of this study is to describe the importance of graph theory ideas in different areas of computer science and its applications for developer which they can use graph theory methods for study. In this study, they described a survey for project ideas of graph theory [6].

Gowda, D.V et al. (2021), In this paper, they stated that for designers and programmers, graph theory is an extraordinarily rich field. Basically, graph is used to solve some very complicated issues like visualization, lower costs, program analysis, etc. To calculate network devices , an optimum traffic routing like switches and routers use graphics. This study highlights the main advanced developments in the area of graph theory and different applications in area of engineering[7].

Aydın B. (2015) In this study, they have presented a review of the works carted out in the sector of social networks (SN) and computer science (CS) that employs the concepts of graph theory (GT). They analyzed graph properties and good combination of graphs are chooses for their problems. They also provides practical examples and existing use in most of diverse applications of CN and CS domains. Moreover, they provided practical examples and explanations of use of graph theory to maintain the importance of graphs in modern research. In the future, they will work on new graphical tools and applications for the vehicular network management because of the emergence of smart cities [8].

Section III Discussions

Definition 3.1: The simple graph 'g' consists with nonempty set ' v' , called as node or vertices of 'g', and a set 'e' is set having two element which is subset of ' v' .Then members of 'e' is known as edges of 'g', and it can be written as g d.v; e/.

Definition 3.2: The two nodes in simple graph are adjacent if nodes joined with link (edge).The total number of link or edges incident to node(vertex) is known as degree of node (vertex).It is represented by deg.v/.It means degree of node is equal to total number of adjacent vertices.

Definition 3.3: In n node graph g=(v,e) where v is $\{v_1,v_2,\ldots,v_n\}$ then adjacent metrics for g is nxn size matrix

$$A_g = \{a_{ij}\} \ldots \ldots \ldots \ldots \ldots . . (2)$$

Where ,

$$a_{ij}=\begin{cases} 1 \; if \; set \; of \; v \in e \\ 0 \qquad otherwise \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

If the graph is weighted graph then instead of 1,weights of respective are present in the matrix. Adjacency matrix is an important part of th graph theory,having weights shown in blows fig.5.

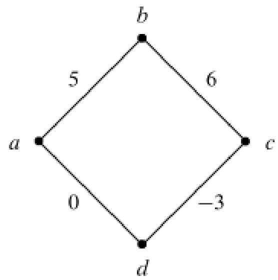

Fig:5 Graph with weights

Fig.5 is indicate four node structure with weights, as per eq.,the general matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \text{ and for above fig the weighted matrix is } \begin{bmatrix} 0 & 5 & 0 & 1 \\ 5 & 0 & 6 & 0 \\ 0 & 6 & 0 & -3 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$

a. Extremal graphtheory:

Extremal graph is largest graph having order n .It doesn't have sub graph. Definition and mathematical concepts for extremal graph theory varies according to researchers.in 1959 Goodman defines the extremal theory as

$$N(n) = \begin{cases} \frac{1}{3} m(m-1)(m-2) & for\ n = 2m \\ \frac{1}{3} 2m(m-1)(4m+1) & for\ n = 4m+1 \\ \frac{1}{3} 2m(m+1)(4m-1) & for\ n = 4m+3 \end{cases} \quad \text{........(4)}$$

It is also known as Goodman's formula whereas Schwenk defined the extremal graph theory as follows

$$N(n) = \binom{n}{3} - \left\lfloor \frac{1}{2} n \left\lfloor \frac{1}{4} (n-1)^2 \right\rfloor \right\rfloor \quad \text{................(5)}$$

Where, floor function is represented by$\lfloor x \rfloor$.

b. Theorems of extremal graphtheory:

Generally, extremal result in the graph theory is all about to minimize or to maximize quantity between all the graphs belongs to few class. Graph H and the n belongs to Z^{+} (positive integer) ,defines an extremal number $ex\ (n,\ H)$ is the maximum of n edges on a graph with n vertices with absence of sub-graph. The extremal number in an extremal graph theory can be rounded by function which is depends on the value of n [9].

Theorem 3.1: Mantel Theorem: Assume $n \geq 2$ and G is n vertex triangle-free graph. Then

$$ex(n, K_3) = \lfloor\frac{n^2}{4}\rfloor \ldots\ldots\ldots\ldots\ldots(6)$$

Assume the complete bipartite $K_{l,n-l}$ where $l = 1, 2, \ldots\ldots\ldots, n-1$.Basically this graph is triangle free. It has $l(n-l)$ edges.it is maximum at $l = \lfloor\frac{n}{2}\rfloor$ or $\lceil\frac{n}{2}\rceil$.Both the things are equivalent .for example $K_{3,4}$ and $K_{4,3}$ known as isomorphic.

Maximum value is $k(n-k) = \lfloor\frac{n^2}{4}\rfloor$ $where$ $k = \lfloor\frac{n}{2}\rfloor$

When n is even, then $k(n-k) = \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4}$ $is\ the\ integer.$

When n is odd, $k(n-k) = \frac{n-1}{2} \cdot \frac{n+1}{2} = \frac{n^2-1}{4} = \lfloor\frac{n^2}{4}\rfloor$

Finally resulted into

$$ex(n, K_3) \geq \lfloor\frac{n^2}{4}\rfloor \ldots\ldots\ldots\ldots(7)$$

Consider H be the sub graph of G having n vertices and $\lfloor\frac{n^2}{4}\rfloor$ edges. v be the vertex in H with minimum degree : $d_H(v)$.The $H' = H - \{v\}$.it is a H's sub-graph on n-1 vertices. It has an edges : $|E(H')| = |E(H)| - \delta_H(v) \geq \lfloor\frac{n^2}{4}\rfloor - \ldots \lfloor\frac{n^2}{4}\rfloor \ldots \lfloor\frac{n}{4}\rfloor \ldots \lfloor\frac{(n-1)^2}{4}\rfloor$

For odd and even value of n we can verify the mathematical expressions/statement given in belows table.

Table1: expression for n(even/odd)

Quantity	n even	n odd
$k = \lfloor n/2 \rfloor$	$\frac{n}{2}$	$\frac{n-1}{2}$
$\left\lfloor \frac{2\lfloor n^2/4 \rfloor}{n} \right\rfloor$	$\frac{n}{2} = k$	$\frac{n-1}{2} = k$
$\ell = \left\lfloor \frac{n-1}{2} \right\rfloor$	$\frac{n}{2} - 1 = k - 1$	$\frac{n-1}{2} = k$
$n - 1 - \ell$	$\frac{n}{2} = k$	$\frac{n-1}{2} = k$
$d_H(v) = \left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$	$\frac{n}{2} = k$	$\frac{n-1}{2} = k$

Centralized types of the scenarios (problems) in extremal theory , which has the maximum edges in graph with finite number of nodes (vertices) which doesn't have sub- graphs. Mantel's theorem is one of the most popular theorems in extremal graph theory [10].

Theorem 3.2: Turán's Theorem: Consider Let $n \geq 1$ and G is the n-vertex graph having K_{m+1}.Then$\left|E(G)\right|\leq t_{m}(n)$,havinganequalityifandonlyif$G=T_{m}(n)$.Thistheorem is used to Avoid certain size cliques .Mantel theorem is the case of Turán's Theorem with m=2.Turán's Theorem is the base of the start of extremal graph theory field[11].

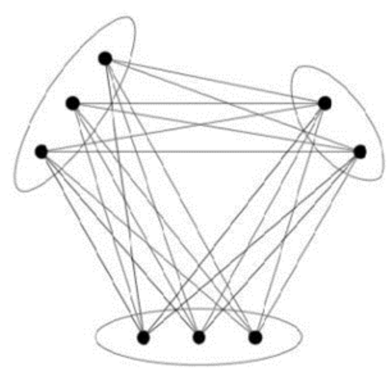

Fig.6 Turán's graph T3

Theorem 3.3: (van der Waerden, 1927). Consider r, k $\in$ Z^+. it is existing least

N = W(r, k) $\in$ Z^+ such that, if an elements of [N] are r-coloured, then it represents the monochromatic arithmetic progression of length k [12].

Theorem3.4(Szemer´edi,1969,finiteversion).Letk∈Z^+,δ∈(0,1].Itexiststheleast N = N (k, δ) $\in$ Z^+ like each subset of [N] has at least δN element having an arithmetic progression of length k [13].

Theorem 3.5 : (Simonovits, 1968). Consider graph H with chromatic number r having colour-critical edge. Then n0 exists for n ≥ n0, ex(n; H) = t(n, r − 1). Furthermore, EX (n, H) = {T(n, r − 1)}[14] and[15].

Theorem 3.6 : Ramsey'sTheorem

Assume number R such that R = R(m1 , . . . , mc ; r) for n > R, then for such all colorings, I is the colour and an m_i -element set S ⊆ { 1, . . . , n } ,where all the r- elements are subsets of set S having color I[16].

Theorem 3.7 : Ramsey's Theorem (1930) — Version for graphs

Let consider R such that R = R(m1 , . . . , m_c) (the Ramsey Number) for n > R ,then all the edge colorings are of K_n with c colors ,must have a monochromatic clique Kmi
Of few color i[17].

c. Example: To find number of edges (How to solve simple classicalproblem)

For n vertices with m parts, the total edges are maximized, all the parts are much more closure .Hence all parts are [n/m] or [n/m].When the graph consist of those parameters termed as Turán graph. This graph is denoted by T_m (n) whereas number of

edges are represented as t_m (n). Mathematically it is represented as $\frac{1}{2}(1-\frac{1}{2})n^2$.

Let see eg. For calculating number of edges. Numbers of vertices are 7and number of parts are 3. then Turán graph is represented as$T_3(7)=K_{2,2,3}$.

Then total number of edges= t_3 (7) = 2 · 2 + 2 · 3 + 2 · 3=4+6+6

= 16 edges

d. Applications:

The dominant application areas for the graph theory are electrical engineering, information science, linguistics, computer network science, physics, biotechnology, chemistry including theoretical applications. Many researchers were dealing with graph theory to solve the problems related to weak vertices, valance, edge detection issues [9] and [18].

Section IV

Conclusion:

Here, we have done the complete review study of extremal graph theory. We studied the basic terms related to graph theory in detail. As a part of discussion, we overviewed the theorems which explain the mathematical study of the extremal graph theory. Some application of extremal graph theory is addedlisted.

SectionV

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