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Comparison of Some Methods for Estimating the Reliability of Mixed Lindley Distribution Using Simulation

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Abstract:
In this paper, the reliability function of the mixed Lindley distribution was estimated using the standard Bayes method and the maximum Likelihood method and compared them using the simulation using Monte Carlo method using different sample sizes small, medium and large. The simulation results showed that the standard Bayes method is better than the maximum Likelihood method when the sample sizes are (n = 15,25), and the maximum Likelihood method is better when the sample sizes are large (n = 50,100).

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1- **Introduction:** The Lindley distribution is one of the continuous mixed distributions that have an important possibility to represent the various systems that belong to heterogeneous complex societies. Also, this distribution has a high flexibility in representing failure models and because most of the systems on the applied side are from complex societies. Heterogeneous this distribution was used to represent failure times in these heterogeneous systems.

2- **Objective**: This research aims to compare between the estimators of the maximum Likelihood and the standard Bayes method in estimating the reliability function of the mixed Lindley distribution and using simulation to determine the best method between them.

3- **Theoretical aspect:**

3-1 Lindley distribution:

It is one of the mixed distributions resulting from two variables, the first follows the exponential distribution with parameter (θ) and the second follows the gamma distribution with two parameters (θ) and (2) according to the following mixing formula [6] .

$$f(x;\theta) = \sigma f_1(x;\theta) + (1-\sigma) f_2(x;\theta) \quad \sigma,\theta > 0 \qquad \ldots\ldots\ldots\ldots (1)$$

whereas:

σ: represents the mixing ratio , $\sigma = \frac{\theta}{1+\theta}$

Therefore, the probability density function for a mixed Lindley distribution is:

$$f(x;\theta) = \frac{\theta^2}{(1+\theta)}(1+x)\, e^{-\theta x},\ x > 0\ ,\ \theta > 0 \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

And the cumulative distribution function (c.d.f) is:

$$F(x;\theta) = \int_0^x f(t)td = 1 - \left[1 + \frac{\theta}{(1+\theta)}x\right] e^{-\theta x}\ , \qquad x, \theta > 0 \quad \ldots\ldots\ldots(3)$$

3-2 Reliability function:

The reliability function of the mixed Lindley distribution is as follows:

$$R(x;\theta) = 1 - F(x;\theta) = \{1 + \frac{\theta}{(1+\theta)}x\}\ e^{-\theta x} = \frac{1+\theta+\theta x}{1+\theta}\ e^{-\theta x} \quad x, \theta > 0$$

3-3 Methods for estimating a parameter of the mixed Lindley distribution and the reliability function

Two methods of estimating the reliability of the mixed Lindley distribution will be discussed, which are the maximum potential (MLE) method and the standard Bayes method.

3-3-1 maximum Likelihood method:

The maximum Likelihood estimator (MLE) is the estimated value that makes the Likelihood function the greatest possible. The Likelihood function can be defined for the observations of the random variable that follows the mixed Lindley distribution as follows:

$$L(x1, x2, \ldots xn/\theta) = \prod_{i=1}^{n} f(xi, \theta) = \frac{\theta^{2n}}{(1+\theta)^n} \prod_{i=1}^{n}(1 + xi)\ e^{-\theta \sum_{i=1}^{n} x_i} \quad \ldots\ldots(5)$$

By taking the natural logarithm of equation (5), we get:

$$\ln L(x_1, x_2, \ldots, xn/\theta) = 2n \ln\theta - n\, ln(1+\theta) + \sum_{i=1}^{n}\ln(1+x_i) - \theta\sum_{i=1}^{n} x_i \ldots. (6)$$

By deriviting equation (6) above relative to the parameter θ, we get:

$$\frac{d\ln L(\theta; x1, x2, \ldots, xn/\theta)}{d\theta} = \frac{2n}{\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n} xi \qquad \ldots\ldots\ldots(7)$$

Equating the derivative to zero, we get the maximum Likelihood estimate for θ

$$2(1+\theta) - \theta - \theta(1+\theta)\bar{x} = 0$$

$$\bar{x}\theta^2 + (\bar{x} - 1)\theta - 2 = 0 \qquad \ldots\ldots\ldots(8)$$

And by solving equation . (8), we use the constitution method to solve equations of the second degree to get the maximum Likelihood value for the parameter (θ)

$$\hat{\theta}_{MLE} = \frac{-(\bar{x}-1) + \sqrt{(\bar{x}-1)^2 + 8\bar{x}}}{2\bar{x}} \qquad , \bar{x} > 0 \qquad \ldots\ldots\ldots(9)$$

Since the estimator of the greatest possibility is an estimator characterized as an estimator (one to one), the estimator of the maximum Likelihood of the Reliability function is:

$$\hat{R}_{MLE}(xi) = \frac{1-\hat{\theta}_{mle}+\ \hat{\theta}_{mle}xi}{1+\hat{\theta}_{mle}} e^{-\hat{\theta}_{mle}x_i} \qquad \ldots\ldots\ldots\ldots(10)$$

3-3-2 Standard Bayes Method:

The standard Bayes method depends on the posterior probability density function and the squared loss function, and the posterior probability density function can be obtained from the application of the inverse Bayes formula.

$$(\theta/x) = \frac{L(x/\theta)g(\theta)}{\int_{\forall\theta} L(x/\theta)g(\theta)d\theta} f$$

Where: L (x/θ) is the Likelihood probability function for a sample of size n observations

g (θ) : represents the initial probability density function for the parameter and the initial probability density function for the parameter can be obtained by relying on the formula (θ), which states that the parameter to be estimated if it is within the range

(0, ∞), the initial probability function follows a regular logarithmic distribution, meaning that:

$$(\theta) \approx \frac{1}{\theta} = \frac{c}{\theta} \qquad , \theta > 0 \qquad , \theta > 0g$$

Since c is the constant of proportionality

Using the quadratic loss function

$$\text{Loss} = (\hat{\theta} - \theta)^2$$

The piez estimator for the parameter θ is the Expected of density function for the posterior distribution:

$$\hat{\theta}_{Bayes} = E(\theta/x) = \int_{\forall\theta} \theta f(\theta/x) d\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

And to find the formula for the Bayes estimator for parameter θ by employing the squared loss function and the posterior probability density function which are:

$$f(\theta/x) = \frac{\frac{1}{\theta(1+\theta)^n} \prod_{i=1}^{n}(1+xi)\, e^{-\theta} \sum_{i=1}^{n} xi}{\prod_{i=1}^{n}(1+xi) \int_0^\infty \frac{1}{\theta(1+\theta)^n e^{-\theta \Sigma x_i}}} \qquad \ldots\ldots\ldots.. (13)$$

From Equation No. (12), since the Bayesian estimator is the expectation for the posterior distribution, then:

$$\hat{\theta}_{Bayes} = E(\theta/x) = \frac{\int_0^\infty \theta \frac{1}{\theta(1+\theta)^n} e^{-\theta \Sigma x_i} d\theta}{\int_0^\infty \theta \frac{1}{\theta(1+\theta)n} e^{-\theta \Sigma x_i} d\theta} \qquad \ldots\ldots\ldots(14)$$

The integral can be found in equation . (14) using the Lindley approximation, as follows:

$$I(x) = E[h(\theta)] = \frac{\int_{\forall\theta} h(\theta) \exp[L(\theta,x)+g(\theta)]d\theta}{\int_{\forall\theta} \exp[L(\theta,x)+g(\theta)]d\theta}$$

where:

I(x) approximate output:

$h(\theta)$: parameter function θ

L (θ, x): the logarithm of the maximum possibility function

$g(\theta)$: the logarithm of the initial probability density function

The product of the approximate integration is as follows:

$$I(x) = h(\hat{\theta}) + 0.5[(\hat{h}_{\theta\theta} + 2\hat{h}_\theta \hat{p}_\theta)\hat{\sigma}_{\theta\theta}] + 0.5[(\hat{h}_\theta . \hat{\sigma}_{\theta\theta})(\hat{L}_{\theta\theta} . \hat{\sigma}_{\theta\theta})] \quad \text{...........(15)}$$

where:

$\hat{h}_\theta = \frac{\partial h(\hat{\theta})}{\partial \hat{\theta}}$, $\hat{h}_{\theta\theta} = \frac{\partial^2 h(\hat{})}{\partial \hat{\theta}^2}$, $P_\theta = \frac{\partial g(\hat{\theta})}{\partial \hat{\theta}}$

$\hat{L}_{\theta\theta} = \frac{\partial^2 L(\hat{\theta})}{\partial \hat{\theta}^2}$, $\hat{L}_{\theta\theta\theta} = \frac{\partial^3 L(\hat{\theta})}{\partial \hat{\theta}^3}$, $\hat{\sigma}_{\theta\theta} = \frac{1}{\hat{L}_{\theta\theta}}$

$h(\theta) = \frac{1}{\theta}$, $\therefore \hat{h}_\theta = -\frac{1}{\theta^2}$, $\hat{h}_{\theta\theta} = \frac{1}{\theta^3}$

$L(\theta, x) = 2n \ln\theta - n\ln(1+\theta) - \theta\sum_{i=1}^{n} xi + \sum_{i=1}^{n}\ln(1 + x_i)$

$g(\theta) = - n \ln\theta$

$\therefore \hat{p}_\theta = -\frac{n}{\theta}$, $\hat{L}_{\theta\theta} = \frac{-2n}{\hat{\theta^2}} + \frac{n}{(1+\hat{\theta})^2}$

$\hat{L}_{\theta\theta\theta} = \frac{4n}{\hat{\theta}^3} - \frac{2n}{(1+\hat{\theta})^3}$, $\hat{\sigma}_{\theta\theta} = \frac{\hat{\theta}^2(1+\hat{\theta})^2}{2n(1+\hat{\theta})^2 - n\hat{\theta}^2}$

And by substituting in equation No. (15), we get a Bayes estimator

$$\hat{\theta}_{Bayes} = I(x) = -\frac{1}{\hat{\theta}^2} + 0.5\left[\left(\frac{2}{\hat{\theta}^3} + 2 - \frac{1}{\hat{\theta}^2} - \frac{1}{\hat{\theta}}\right)\left(\frac{\hat{\theta}^2(\mathbf{1}+\hat{\theta})^2}{\mathbf{2n}(\mathbf{1}-\hat{\theta})^2 - n\hat{\theta}^2}\right)\right] + \mathbf{0.5}\left[\left(-\frac{1}{\hat{\theta}^2}\frac{\theta^2(1+\theta^2)}{2h(1+\theta)^2 - n\theta^2}\right)\left(\left(\frac{4n}{\theta^3} - \frac{2n}{(1+\theta)^2}\right)\frac{\theta^2(\mathbf{1}+\theta)^2}{\mathbf{2n}(\mathbf{1}+\theta^2) - n\theta^2}\right)\right]$$

$$\hat{\theta}_{Bayes} = -\frac{1}{\theta^2} + \mathbf{0.5}\left[\frac{4}{\theta^3}\frac{\theta^2(\mathbf{1}+\theta)^2}{\mathbf{2n}(\mathbf{1}+\theta^2) - n\theta^2}\right] + 0.5\left[\frac{-(\mathbf{1}+\theta)^2}{\mathbf{2n}(\mathbf{1}+\theta^2) - n\theta^2}\left(\frac{4n(\mathbf{1}+\theta)^3 - 2n\theta^3}{2n\theta(1+\theta)^3 - n\theta^3}\right)\right] \quad \text{.......(16)}$$

The estimation of the reliability function using the Bayes method is:

$$\hat{\theta}_{Bayes} = \frac{1+\hat{\theta}\,Bayes + \hat{\theta}\,Bayes\,xi}{1+\hat{\theta}\,Bayes} e^{-\hat{\theta}_{Bayes} x_i} \quad xi > 0 \quad ..(17)$$

4- Experimental aspect:

1-4 Introduction:

The simulation was used the Monte Carlo method to compare the estimation of the reliability functions of the Lindley distribution, where the initial default values for the Lindley distribution parameter θ were chosen, which are (0.5,1,1.5,2,2.5,3), and small, medium and large sizes of samples were selected (15,25,50,100). The data of the random variable that follows the mixed Lindley distribution was generated depending on the selected sample sizes and the default parameter values θ using the (accept-reject) method to generate the data according to the following steps:

1- Generate the random variable: u that follows the standard uniform distribution $u_i \sim U_c(0,1)$

2- Generating random variables y_i & w_i

where :

$$y_i \sim \exp(\theta)$$

$$w_i \sim \text{gamma}(2, \theta)$$

فأذا كانت $\frac{\theta}{\theta+1} \leq u_i$ فان $x_i = y_i$

3- If it was $u_i \leq \frac{\theta}{\theta+1}$ so $x_i = y_i$

Otherwise, the $x_i = w_i$

4- After obtaining the data, the parameter θ was estimated using two methods of maximum Likelihood $(\hat{\theta}_{mle})$ and Bayes method $(\hat{\theta}_{Bayes})$ And then find the estimator of the reliability function $\hat{R}(xi)$ And both the method of the maximum Likelihood and the method for Bayes .

5- Calculate the mean square error (MSE) for the reliability function estimator as follows:

$$MSE\ \hat{R}(xi) = \frac{\sum_{j=1}^{k}\left(\hat{R}(xi) - R(xi)\right)^2}{k} \qquad \ldots\ldots\ldots(18)$$

where :

k: the number of iterations of the simulation experiment (k=1000)

R(xi): represents the real value of the reliability function

$\hat{R}(xi)$: represents the estimated value of the reliability function

4-2 Presentation of the simulation results:

The simulation results are shown in Table (1).

Table No. (1) shows the mean values of the error squares for the reliability function estimates by the two methods of greatest possibility and Bayes method, for all sample sizes and default values

n	θ	$\hat{R}(t)_{mle}$	$\hat{R}(t)$ Bayes	The best
15	0.5	1.38607E-02	1.26839E-02	BAYES
	1	5.44424E-03	6.79277E-03	MLE
	1.5	3.03814E-03	3.94516E-03	BAYES
	2	4.12221E-03	5.13846E-04	BAYES
	2.5	4.84925E-03	3.65925E-03	BAYES
	3	1.04777E-05	2.02394E-05	MLE
25	0.5	4.78595E-03	4.22233E-03	BAYES
	1	2.21201E-03	1.97551E-03	BAYES
	1.5	1.21532E-03	1.10286E-03	BAYES
	2	3.76095E-04	1.69870E-03	MLE
	2.5	2.88364E-03	1.37611E-03	BAYES
	3	4.98665E-04	1.11208E-05	BAYES
50	0.5	3.16781E-03	3.09610E-03	BAYES
	1	1.23573E-03	1.35837E-03	MLE
	1.5	6.97865E-04	7.30854E-04	MLE
	2	1.52634E-04	1.62981E-04	MLE
	2.5	1.54921E-02	1.84903E-03	BAYES
	3	3.23456E-03	3.42963E-03	MLE
100	0.5	9.84348E-04	9.78758E-04	BAYES
	1	4.03512E-05	4.09653E-04	MLE
	1.5	2.11404E-04	2.16294E-04	MLE
	2	2.74596E-05	2.80958E-05	MLE
	2.5	5.72171E-05	5.71856E-05	BAYES
	3	3.36951E-08	1.23275E-06	MLE

4.3 Analyzing the results:

By observing the results in Table No. (1), the following was reached:

First: The Bayes method is better than the maximum Likelihood method in estimating the reliability function when the sample size is 15 and for all default values with a priority ratio of 2/3.

Second: The Bayes method is better than the maximum Likelihood method when the sample size is 25 and for all default values with a priority ratio of 5/6

Third: The maximum Likelihood method is better than the Bayes method in estimating the reliability function at the sample size of 50 and for all default values with a priority ratio of 4/6

Fourth: The maximum Likelihood method is better than the Bayes method in estimating the reliability function at a sample size of 100 with a priority ratio of 4/6

Fifth: We note that the MSE values approach zero as the sample size increases for the two methods.

Sixth: We note that the Bayes method is better than the maximum Likelihood method for all sample sizes, with an advantage ratio of 13/24.

The preference ratio can be summarized as in the following table (2):

Table. (2) : the percentage of preference for the two estimation methods for the reliability function

Sample volume N	MLE		BAYES	
	the number	The ratio	the number	The ratio
15	2	0.333	4	0.667
25	1	0.167	5	0.833
50	4	0.667	2	0.333
100	4	0667	2	0.333
TOTAL	11	0.458	13	0.542

5- Conclusions and recommendations

5-1 Conclusions

1- The maximum Likelihood method is better than the Bayes method when the sample size is large.

2- The Bayes method is better than the maximum Likelihood method when the sample size is small.

3- The estimated values get closer to the real values, When the sample size is large .

5.2 Recommendations:

1- We recommend using the maximum Likelihood method in estimating the reliability function of the Lindley distribution when the sample size is large.

2- We recommend using the Bayes method in estimating the reliability function of the Lindley distribution when the sample size is small.

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