Antimagic Labeling of Complete Graphs and its Application in Chessboard

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Abstract
The Purpose in this work is to extend antimagic labeling in complete graph K_n and defined special type of labeling such as Maximum Arithmetic Antimagic Labeling (MAAL), Minimum Arithmetic Antimagic Labeling (MIAAL), Maximum Product Antimagic Labeling (MPAL), Minimum ProductAntimagic Labeling (MIPAL). In this research we will check whether while usingthese labelings on Knit satisfies or not (ie. K_n is MAAL or not and similarly to other 3 as we mentioned above). Also discussed some applicationsof antimagic labeling in Chessboard.

Keywords: Antimagic Labeling, Maximum Arithmetic Antimagic Labeling (MAAL), Minimum Arithmetic Antimagic Labeling (MIAAL), Maximum Product Antimagic Labeling (MPAL), Minimum Product Antimagic Labeling(MIPAL).

1 Introduction

Hartsfield and Ringel introduced the concept of antimagic labeling which is assigning distinct vertices with distinct values in graph such that sum of edge labels incident to each vertex, the sum will be different. In order to extend antimagic labeling in complete graph K_n, We introduce a new labeling definition MAAL, MIAAL, MPAL, MIPAL in this work. We will show that complete graph is MAAL, MIAAL, MIPAL, MPAL. Also additionally We will show some graphs on chess also satisfy antimagic labeling, if so then MAAL, MIPAL, MIAAL, MPAL also exist.

Definition 1.1

A graph G is said to **antimagic** if the addition of incident edge labels in each vertex is distinct[1].

Example 1.2

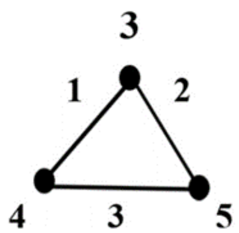


Fig 1: $K3$ with Antimagic Labeling

Definition 1.3

A graph G=(V, E) is said to be **Arithmetic Labeling** if its vertices $u \in V$ are labeled with natural numbers from 1 to n and it is denoted by η and each edge u, $V \in E$ are labeled with the condition $f(uv) = \eta(u) + \eta(v)$.

Example 1.4

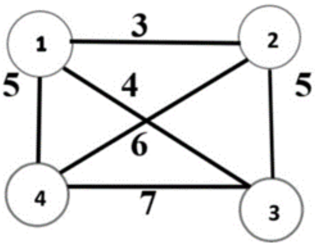


Fig 2: K4 with Arithmetic Labeling

Definition 1.5

A Arithmetic Labeling Graph G= (V, E) is said to be **Minimum Arithmetic Labeling** if each edge $e = uv \in E$ Satisfies the condition $f(uv) = \eta(u) + \eta(v) + \min(\eta(u), \eta(v))$.

Example 1.6

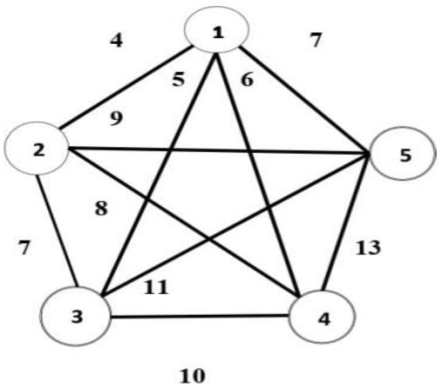


Fig 3: *K*5 with Minimum Arithmetic Labeling

Definition 1.7

A graph G=(V, E) is said to be **Minimum Arithmetic Antimagic Labeling** (MIAAL) if it is both Minimum Arithmetic Labeling and Antimagic Labeling.

Example 1.8

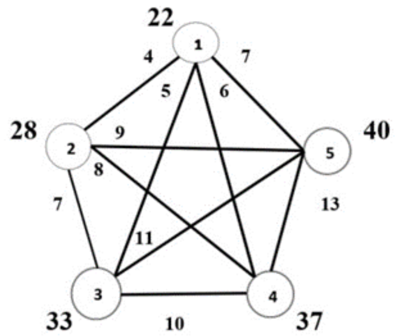


Fig4: $K5$ with Minimum Arithmetic Antimagic Labeling (MIAAL)

Theorem 1.9 A complete graph K_n , n>2 is MIAAL.

Proof:

Let G=(V, E) be a graph having vertices $u \in V$ and E be its edges.

We shall prove this theorem by induction hypothesis.

For n=3 (while labeling the vertices with natural numbers 1 to 3)

Let $u_1=1$, $u_2=2$ and $u_3=3$

$$f(u_1u_2) = \eta(u_1) + \eta(u_2) + \min\{\eta(u_1),\ \eta(u_2)\} = 4$$

$$f(u_1u_3) = \eta(u_1) + \eta(u_3) + \min\{\eta(u_1),\ \eta(u_3)\} = 5$$

$$f(u_2u_3) = \eta(u_2) + \eta(u_3) + \min\{\eta(u_2),\ \eta(u_3)\} = 7$$

Now $f(u_1) = f(u_1u_2) + f(u_1u_3) = 4 + 5 = 9$

$$f(u_2) = f(u_1u_2) + f(u_2u_3) = 4 + 7 = 11$$

$$f(u_3) = f(u_1u_3) + f(u_2u_3) = 5 + 7 = 12$$

Here $f(u_1) \neq f(u_2) \neq f(u_3)$.

Thus the theorem holds for n=3.

Assume that the theorem holds for all K_n where n=k-1.

Then we have to prove that the theorem is true for n=k.

Consider $u_s=s$, $u_{s+1}=s+1$, $u_{s+2}=s+2$, …, $u_{s+t}=s+t(=k)$ be the vertices of graph G.

$$f(u_s\, u_{s+1}) = \mathrm{s} + 1 + 2\{\min\{u_{s,}\ u_{s+1}\} = s + 1 + 2s = 3s + 1$$

$$f(u_s\, u_{s+2}) = \mathrm{s} + 2\{\min\{u_{s,}\ u_{s+2}\} = s + 2 + 2s = 3s + 2$$

In general, $f(u_s\, u_{s+t=k}) = \mathrm{s} + \mathrm{t} + 2\min\{u_{s,}\ u_{s+t}\} = s + t + 2s = 3s + t$

Now $f(u_s) = f(u_s\, u_{s+t=k}) = 3\text{s}+1+3\text{s}+2+3\text{s}+3+....+3\text{s}+\text{t}$

$f(u_{s+1}) = f(u_s\, u_{s+1})\sum_{s=1}^{n} f(u_s\, u_{s+t=k}) = 3\text{s}+1+3\text{s}+4+3\text{s}+5+3s+6....+3\text{s}+2+\text{t}$

In general, $f(u_{s+t}) = f(u_s\, u_{s+t}) + f(u_{s+1}\, u_{s+t}) + f(u_{s+2}\, u_{s+t}) + ... + f(u_n\, u_{s+t})$

(Or) $f(u_s\; u_k) + f(u_{s+1}\; u_k) + f(u_{s+2}\; u_k) + ... + f(u_n\; u_k)$

$f(u_{s+t}) = 3s+t+3s+2+t+3s+4+t+...+2n+s+t\,(or)\,2s+k+2s+2+k+...+2n+k$

Thus $f(u_s) \neq f(u_{s+1}) \neq ... \neq f(u_{s+t=k})$

Thus the theorem is true for n=k.

Therefore the complete graph K_n is MIAAL , except K_2.

Hence proved.

Definition 1.10

A Arithmetic Labeling graph G=(V, E) is said to be **Maximum Arithmetic Labeling** if each edge satisfies the condition $g(uv) = \eta(u) + \eta(v) + \max\{\eta(u), \eta(v)\}$.

Example 1.11

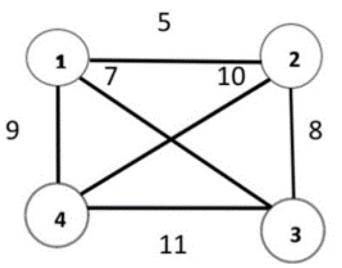


Fig 5: K4 with Maximum Arithmetic Labeling

Definition 1.12

A graph G=(V, E) is said to be **Maximum Arithmetic Antimagic Labeling** (MAAL) if it is both Maximum Arithmetic Labeling and Antimagic Labeling.

Example 1.13

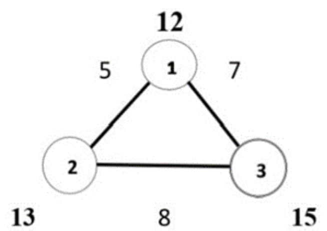


Fig 6: K_3 with Maximum Arithmetic Antimagic Labeling (MAAL)

Theorem 1.14 For n>2, a complete graph K_n is MAAL.

Let G=(V, E) be a graph having vertices $p \in V$ and E be its edges.

We shall prove this theorem by induction hypothesis.

For n=3 (while labeling the vertices with natural numbers 1 to 3)

Let $p_1 = 1, p_2 = 2$ *and* $p_3 = 3$

$$g(p_1\ p_2) = \eta(p_1) + \eta(p_2) + \max\{\eta(p_1), \eta(p_2)\} = 5$$

$$g(p_1\ p_3) = \eta(p_1) + \eta(p_3) + \max\{\eta(p_1), \eta(p_3)\} = 7$$

$$g(p_2\ p_3) = \eta(p_2) + \eta(p_3) + \max\{\eta(p_2), \eta(p_3)\} = 8$$

Now $g(p_1) = g(p_1\ p_2) + g(p_1\ p_3) = 5 + 7 = 12$

$$g(p_2) = g(p_1\ p_2) + g(p_2\ p_3) = 5 + 8 = 13$$

$$g(p_3) = g(p_1\ p_3) + g(p_2\ p_3) = 7 + 8 = 15$$

Here $g(p_1) \neq g(p_2) \neq g(p_3)$.

Thus the theorem holds for n=3.

Assume that the theorem holds for all K_n where n=k-1.

Then we have to prove that the theorem is true for n=k.

Also $p_y = y, p_{y+1} = y+1, p_{y+2} = y+2, \ldots, p_{y+m} = y+m(=k)$ be the vertices of graph G.

$$\text{g}(p_y\ p_{y+1}) = y + 1 + 2\{\max\{p_y, p_{y+1}\} = y + 2\ \{\max y, y+1\} = y + 2(y+1) = y + 2y + 2 = 3y + 2$$
$$\text{g}(p_y\ p_{y+2}) = y + \{\max\{y, y+2\} = y + 2(y+2) = 3y + 4$$

In general, $\text{g}(p_y\ p_{y+m=k}) = y + 2\max\{y, y+m\} = y + 2(y+m) = 3y + 2m$

Now $\text{g}(p_y) = \sum_{y=1}^{n} g(p_y\ p_{y+m=k}) = 3y + 2 + 3y + 4 + \ldots + 3y + 2m$

$$\text{g}(p_{y+1}) = \text{g}(p_y\ p_{y+1}) + \sum_{y=1}^{n} g(p_{y+1}\ p_{y+m=k}) = 3y + 2 + 3y + 5 + \ldots + 3y + 1 + 2m$$

In general, $g(p_{y+m}) = g(p_y\ p_{y+m}) + g(p_{y+1}\ p_{y+m}) + g(p_{y+2}\ p_{y+m}) + \ldots + g(p_n\ p_{y+m})$

(Or) $g(p_y\ p_k) + g(p_{y+1}\ p_k) + g(p_{y+2}\ p_k) + \ldots + g(p_n\ p_k)$

$$g(p_{y+m}) = 3y + 2m + 3y + 1 + 2m + \ldots + n + 2y + 2m\,(or)\ y + 2k + y + 1 + 2k + \ldots + n + 2k$$

Thus $g(p_y) \neq g(p_{y+1}) \neq \ldots \neq g(p_{y+m=k})$

Thus the theorem is true for n=k.

Therefore the complete graph K_n is MAAL , except K_2.

Hence proved.

Definition 1.15

A graph G=(V, E) is said to be **product labeling** if its vertices $u \in V$ are labeled with natural numbers from 1 to n and it is denoted by η and each edge $uv \in E$ are labeled with the condition $\eta(uv) = n(u)n(v)$.

Example 1.16

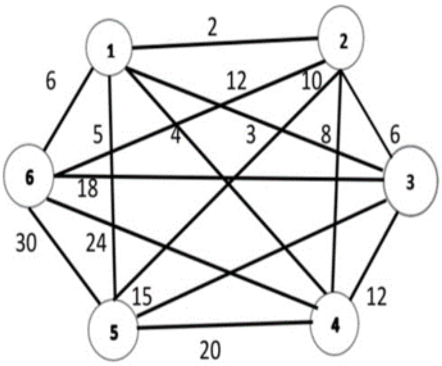


Fig 7: K_6 with Product Labeling

Definition 1.17

A graph G=(V, E) is said to **Minimum Product Labeling** if each edge $e = uv \in E$ satisfies the condition $\eta(uv) = n(u)n(v) + \min\{n(u), n(v)\}$.

Definition 1.18

A graph G=(V, E) is said to be **Minimum Product Antimagic Labeling** (MIPAL) if it is both Minimum Product Labeling and Antimagic Labeling.

Example 1.19

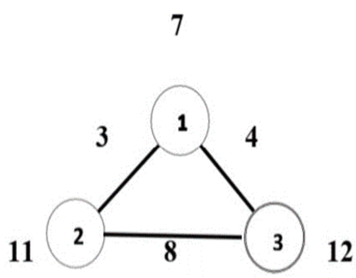


Fig 8: K_3 with Minimum Product Antimagic Labeling

Theorem 1.20 A complete graph K_n , $n \geq 3$ is MIPAL.

Proof:

Let G=(V, E) be a graph having vertices $h \in V$ *and* E be its edges.

We shall prove this theorem by induction hypothesis.

For n=3 (while labeling the vertices with natural numbers 1 to 3 we obtain **Fig 1.9**)

Let $h_1=1$, $h_2=2$ and $h_3=3$

$n(h_1h_2) = \eta(h_1)\ \eta(h_2) + \min\{\eta(h_1), \eta(h_2)\} = 2+1 = 3$

$n(h_1h_3) = \eta(h_1)\ \eta(h_3) + \min\{\eta(h_1), \eta(h_3)\} = 3+1 = 4$

$n(h_2h_3) = \eta(h_2)\ \eta(h_3) + \min\{\eta(h_2), \eta(h_3)\} = 6+2 = 8$

Now $n(h_1) = \eta(h_1h_2) + \eta(h_1h_3) = 3+4 = 7$

$n(h_2) = \eta(h_1h_2) + \eta(h_2h_3) = 3+8 = 11$

$n(h_3) = \eta(h_1h_3) + \eta(h_2h_3) = 4+8 = 12$

Here $n(h_1) \neq \eta(h_2) \neq \eta(h_3)$.

Thus the theorem holds for n=3.

Assume that the theorem holds for all K_n where n=k-1.

Then we have to prove that the theorem is true for n=k.

Let $h_r = r,\ h_{r+1} = r+1,\ h_{r+2} = r+2, \ldots,\ h_{r+q} = r+q(=k)$ be the vertices of graph G.

$n(h_r\, h_{r+1}) = r\,(r+1) + \min\{r, r+1\} = r^2 + r + r = r^2 + 2r$

$n(h_r\, h_{r+2}) = r\,(r+2) + \min\{r, r+2\} = r^2 + 2r + r = r^2 + 3r$

In general, $n(h_r\, h_{r+q=k}) = r\,(r+q) + \min\{r, r+q\} = r^2 + rq + r$

Then $n(h_r) = \sum_{r=1}^{n} n\,(h_r\, h_{r+q=k}) = r^2 + 2r + r^2 + 3r + \ldots + r^2 + rq + r$

$n(h_{r+1}) = n(h_r\, h_{r+1}) + \sum_{r=1}^{n} n\,(h_{r+1}\ \ h_{r+q=k}) = r^2 + 2r + r^2 + 4r + 3 + \ldots + r^2 + 2r + 1 + rq + r$

In general, $n(h_{r+q}) = n(h_r\, h_{r+q}) + n(h_{r+1}\, h_{r+q}) + n(h_{r+2}\, h_{r+q}) + \ldots + n(h_n\, h_{r+q})$

(Or) $n(h_r\, h_k) + n(h_{r+1}\, h_k) + n(h_{r+2}\, h_k) + \ldots + n(h_n\, h_k)$

$n(h_{r+q}) = r^2 + rq + r + r^2 + 2r + 1 + rq + q + \ldots + nr + nq + n$

Thus $n(h_r) \neq n(h_{r+1}) \neq ... \neq n(h_{r+q=k})$

Thus the theorem is true for n=k.

Therefore the complete graph K_n is MIPAL , except K_2.

Hence proved.

Definition 1.21

A graph G=(V, E) is said to **Maximum Product Labeling** if each edge $e = uv \in E$ satisfies the condition $f(uv) = \eta(u)\eta(v) + \max\{\eta(u), \eta(v)\}$.

Example 1.22

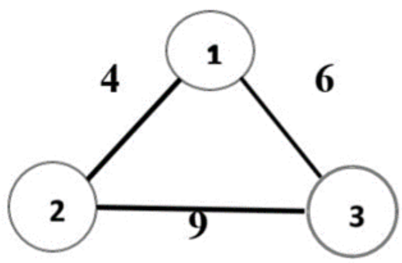


Fig 9: K_3 with Maximum Product Labeling

Definition 1.23

A graph G=(V, E) is said to be **Maximum Product Antimagic Labeling** (mapal) if it is both Maximum Product Labeling and Antimagic Labeling.

Example 1.24

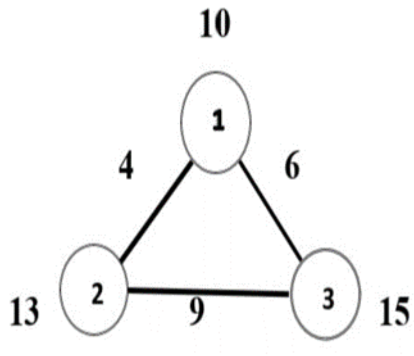


Fig 10: K_3 with Maximum Product Antimagic Labeling

Theorem 1.25 For $n \geq 3$, a complete graph K_n is MPAL.

Proof:

The proof of this theorem is similar as that of theorem 1.20. replace the minimum by maximum value and add it to the product labeling values.

2 Applications of Antimagic labeling on Chess-board

Consider a 2×2 chessboard how it normally looks like (see Fig 2.1)

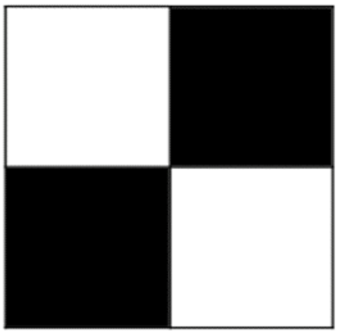


Fig 2.1: 2×2 chessboard

Convert 2×2 chessboard as graph as mentioned below

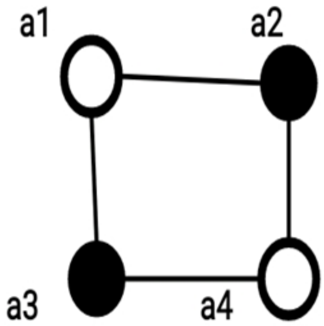
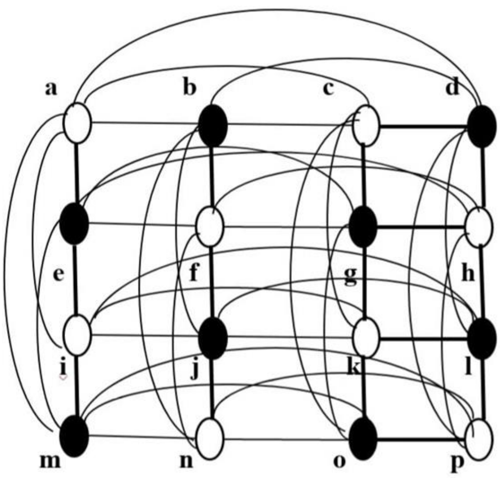


Fig 2.2: Graphical Representation of 2×2 chessboard

Result:

R_4 [4] is Antimagic.

Proof:



R_4 :Rook Graph with 4 vertices

Here a is adjacent with the vertices b, c, d, e, i, m.

thus we are having 6 edges ab, ac, ad, ae, ai, am .

now label the edges with natural numbers from 1 to 6 as there are 6 edges.

So $f(a) = 1+2+3+4+5+6 = 21$

The edges connected to b are ab, bc, bf, bn, bd, bj .

Now label the edges connected to b

from 7 to 11 as per antimagic labeling distinct labels only used.

Hence $f(b)$ =1+7+9+11+8+10=**46**.

Now label the edges of c ,bc=7, ac=2, cd=12, cg=13, ck=14, co=15

$f(c)$ =7+2+12+13+14+15=**63**.

For d , ad=3, bd=8, cd=12, dh=16, dl=17, dp=18.

$f(d)$ =3+8+12+16+17+18=**74**.

For e, ae=4, ef=19, ei=20, em=21, eg=22, eh=23

$f(e)$ =4+19+20+21+22+23 =**109**

For f,

ef=19, bf=9, fh=24, fg=25, fj=26, fh=27

$f(f)$=19+9+24+25+26+27=130

also for g, fg=25, og=30, gk=29, gh=28, gc=13, eg=22.

$f(g)$=**147**

for h, gh=28, hd=16, fh=27, eh=23, ph=32, lh=31.

$f(h)$=**157**

next for i,

ai=5, ie=20, im=33, ij=34, ik=35, il=36

$f(i)$=**163**

for j,

ij=34, jk=37, jl=38, jn=39, jf=26, bj=10.

$f(j)$=**184**

for *k*,

ok=41, *kl*=40, *kj*=37, *ki*=35, *kg*=29, *kc*=14

$f(k)$=**196**

for *l*,

kl=40, *lp*=42, *ld*=17, *lh*=31, *il*=36, *jl*=38

$f(l)$=**204**

for *m*,

im=33, *em*=21, *am*=6, *mn*=43, *mo*=44, *mp*=45

$f(m)$=**192**

for *n*,

mn=43, *on*=47, *np*=46, *nj*=39, *nf*=27, *bn*=11

$f(n)$=**213**

for *o*,

no=47, *mo*=44, *ok*=41, *og*=30, *oc*=15, *op*=48

$f(o)$=**177**

for *p*,

op=48, *np*=46, *mp*=45, *pl*=42, *ph*=32, *dp*=18

$f(p)$=**231**

Therefore, $f(a)\neq f(b)\neq f(c)\neq f(d)\neq f(e)\neq f(f)\neq f(g)\neq f(h)\neq f(i)\neq f(j)\neq f(k)\neq$ $f(l)\neq f(m)\neq f(n)\neq f(o)\neq f(p)$

The given Rooks graph R_4 is antimagic. R_4 is also MAAL, MIAAL, MPAL, MIPAL.

Remark:

N_4 [4] is also MAAL, MIAAL, MPAL, MIPAL.

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