Fuzzy Initial Value Problem and Improved Runge Kutta method with Butcher's Coefficients

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Abstract

This paper intends to find the solution to Fuzzy initial value problems. Here a fourth-order Improved Runge-Kutta method is proposed to find the solution to the problem. Along with the Butcher's table is also used. The find solutions are depicted in terms of fuzzy representation. The accuracy and efficiency of the proposed method are also checked by numerical examples. The proposed method is an extension of the existing methods.

Keywords: Fuzzy initial value problem, explicit improved Runge-Kutta method, Butcher's table, Trapezoidal fuzzy number.

1. Introduction

To increase the order of the Runge Kutta method has to increase the number of Taylor series terms used and thus the number of function evaluations. The Runge Kutta method of order p has a local error over the step size h of (h^{p+1}). Many authors have attempted to increase the efficiency of Runge Kutta methods by trying to minimize the number of function evaluations required. The proposed Improved Runge Kutta methods can be used for autonomous as well as non- autonomous systems and solving ordinary differential equations. The Improved Runge Kutta methods that arise from the classical Runge Kutta methods can also be considered as a special class of two-step methods. i.e., the approximate solution y_{n+1} is calculated using the values of y_n and y_{n-1}. The proposed method introduces the new terms of k_{-i}, which are calculated using k_i, from the previous step. The method proposed here has a minimum number of function evaluations than the Runge Kutta methods. The numerical method to solve the fuzzy initial value problems is introduced by various researchers like Ming Ma et al., [8] In this paper, the proposed method is used to solve the fuzzy initial value problem through explicit improved Runge Kutta method with Butcher's coefficients.

2. Preliminaries

2.1 Trapezoidal Fuzzy Number (TrFN)

A fuzzy number $\tilde{u} = \{u \mid u : R \to [0,1]\}$ and satisfies the following

1. $\tilde{u}$ is upper semi-continuous.

2. $\tilde{u}$ is fuzzy convex, if

$$u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\} \forall\, x, y \in R, 0 \leq \lambda \leq 1$$

3. $\tilde{u}$ is normal, $\exists\ x_0 \in\ R$ for which $u(x_0) = 1$

4. Closure of the set $\{x \in R,\ u(x) > 0\}$ is compact.

The parametric form of a fuzzy number $\tilde{u}$ is represented as a pair $(\underline{u},\ \overline{u})$ of maps $(\underline{u}(\delta),\ \overline{u}(\delta))$, $0 \le \delta \le 1$, such that

1.$\underline{u}(\delta)$ is a left continuous, bounded and monotonic increasing map.

2.$\overline{u}(\delta)$) is a left continuous, bounded and monotonic decreasing map

3.$\underline{u}(\delta) \le\ \overline{u}(\delta)$, for $\delta \in (0,1]$.

A trapezoidal fuzzy number is a four tuples $u = (a, b, c, d)$ such that $a < b < c < d$ with base is the interval $[a, d]$ and vertex $x = b, x = c,$ and its membership function is given by

$$u(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \dfrac{c-x}{c-b}, & c \le x \le d \end{cases}$$

And have, (1) $u > 0 \quad if\ \ a > 0$; (2) $u \ge 0 \quad if\ \ b > 0$;

(3) $u > 0 \quad if\ c > 0$; (4) $u > 0 \quad if\ d > 0$.

2.2 Fuzzy Arithmetic

Let $\tilde{u} = \left(\underline{u}(\delta),\ \overline{u}(\delta)\right)$, $\tilde{v} = \left(v(\delta),\ \overline{v}(\delta)\right)$, $0 \le \delta \le 1$ be arbitrary Fuzzy numbers andlet $k \in R$, the arithmetic operations on fuzzy numbers are defined by.

$\tilde{u} + \tilde{v} = (\underline{u}(\delta) + \underline{v}(\delta),\ \overline{u}(\delta) + \overline{v}(\delta))$

$\tilde{u} - \tilde{v} = (\underline{u}(\delta) + \overline{v}(\delta), \overline{u}(\delta) + \underline{v}(\delta))$

$$\tilde{u} \cdot \tilde{v} = \Big(min\{\underline{u}(\delta)\underline{v}(\delta), \underline{u}(\delta)\overline{v}(\delta), \overline{u}(\delta)\underline{v}(\delta), \overline{u}(\delta)\overline{v}(\delta)\},$$
$$max\,\{\underline{u}(\delta)\underline{v}(\delta), \underline{u}(\delta)\overline{v}(\delta), \overline{u}(\delta)\underline{v}(\delta), \overline{u}(\delta)\overline{v}(\delta)\}\Big)$$

$$C\tilde{u} = \begin{cases} \left(C\,\overline{u}(\delta), C\underline{u}(\delta)\right), & if\ C \ge 0 \\ \left(C\underline{u}(\delta), C\overline{u}(\delta)\right), & if\ C < 0 \end{cases}$$

Let $D : \tilde{u} \times \tilde{u} \to R^+ \cup \{0\}$,

$D(u, v) = Sup_{\delta \in [0,1]} \max\{|\underline{u}(\delta) - \underline{v}(\delta)|, |\overline{u}(\delta) - \overline{v}(\delta)|\}$, be Hausdorff distance between fuzzy numbers, where $\tilde{u} = \left(\underline{u}(\delta),\ \overline{u}(\delta)\right), \tilde{v} = \left(v(\delta),\ \overline{v}(\delta)\right)$.

The following properties are well known:

- $D(u+w, v+w) = D(u, v),\ \forall\, u, v, w\ \in\ \tilde{u}$,
- $D(ku, kv) = |k| D(u, v), \qquad \forall\, k \in R,\ \ u, v\ \in\ \tilde{u}$,
- $D(u+v, w+e) = D(u, w) + D(v, e),\ \forall\, u, v, w, e\ \in\ \tilde{u}$.

2.3 $\boldsymbol{\delta}$-level set

Let F be the set of all fuzzy numbers, the δ-level set of fuzzy number $\tilde{u} \in F$, $0 \le \delta \le 1$, is defined by $[u]_\delta = \{x\ \in\ R\ /\ u(x)\ \ge\ \delta if\ 0\ \le\ \ \delta \le\ 1\}$.

The δ-level set $[u]_\delta = \left(\underline{u}(\delta),\ \ \overline{u}(\delta)\right)$is closed and bounded.

Lemma [8]

If the sequence of positive numbers $\{W_n\}_{n=0}^{N}$ satisfy

$$|W_{n+1}| \le A|W_n| + B, 0 \le n \le N-1,$$

for the given $A, B \in Z^+$, $|W_n| \le A^n |W_0| + B\frac{A^n - 1}{A-1}$,$0 \le n \le N-1$.

Lemma 2 [8]

If the sequence of positive numbers $\{W_n\}_{n=0}^{N}$, $\{V_n\}_{n=0}^{N}$ satisfy

$$|W_{n+1}| \le |W_n| + A \max\{|W_n|, |V_n|\} + B,$$

$$|V_{n+1}| \le |V_n| + A \max\{|W_n|, |V_n|\} + B,$$

for the given $A, B \in Z^+$, $U_n = |W_n| + |V_n|$,$0 \le n \le N$, then

$$U_n \le \overline{A^n} U_0 + \overline{B}\frac{\overline{A^n} - 1}{\overline{A} - 1}\ ,0 \le n \le N, \text{ where } \overline{A} = 1 + 2A \ \text{ and } \overline{B} = 2B.$$

Theorem 1[8]

Let $F(t\,, u\,, v\,)$ and $G(t\,, u\,, v\,)$be in $C^l(K)$ and its partial derivatives are bounded above K then, for random fixed δ, $0 \le \delta \le 1$, the approximate solutions $\underline{y}(t_{n+1}\,; \delta)$ and $\overline{y}(t_{n+1}\,; \delta)$meet the exact solutions $\underline{Y}(t\,; \delta)$ and $\overline{Y}(t\,; \delta)$ regularly in t.

Theorem 2 [8]

Let $F(t\,, u\,, v\,)$ and $G(t\,, u\,, v\,)$be in$C^l(K)$ and its partial derivatives are bounded above K, $2Lh < l$,then, for random fixed δ, $0 \le \delta \le 1$,the solutions $\underline{y}^i(t_n\,; \delta)$ and $\overline{y}^i(t_n\,; \delta)$, $i = 1, 2, \ldots$not diverge to the algebraic solutions $\underline{y}(t_n\,; \delta)$ and $\overline{y}(t_n\,; \delta)$ in $t_0 \le t_n \le t_N$, when $i \to \infty$.

3. Fuzzy Initial Value Problems (FIVP)

Consider the fuzzy initial value differential equation has the form:

$$\begin{cases} y'(t) = f\big(t, y(t)\big); \quad t \in [t_0,\ l] \\ y(t_0) = y_0, \end{cases} \qquad (1)$$

Here y is a fuzzy map in$t$, $f\ (t, y)$is a fuzzy map of $t$and fuzzy variable y, the derivative of y is denoted by y' and $y(t_0)\ =\ y_0$is a fuzzy number (in triangular shaped). The exact solution of the problem in (1) is

$[Y(t\,)]_\delta\ = \left[\underline{Y}\ (t\,;\delta), \overline{Y}(t\,;\delta)\ \right]$ be approximated by some

$[y(t\,)]_\delta\ = \left[\underline{y}\ (t\,;\delta), \overline{y}(t\,;\delta)\ \right]$.

$[y(t_0)]_\delta\ = \left[\underline{y}\ (t_0\,;\delta), \overline{y}(t_0\,;\delta)\ \right], \delta \in (0, 1]$

we write $f(t, y)\ =\ [\,\underline{f}(t, y), \overline{f}(t, y)]$ and $\underline{f}(t, y)\ =\ F[t, \underline{y}, \overline{y}\ ]$,

$\overline{f}(t, y)\ =\ G[t, \underline{y}, \overline{y}\ ]$. Because of $y'\ =\ f(t, y)$we have

$\underline{f}(t, y(t); \delta)\ =\ F[t, \underline{y}\ (t\,;\delta), \overline{y}(t\,;\delta)\ ]$

$\overline{f}(t, y(t); \delta)\ =\ G[t, \underline{y}\ (t\,;\delta), \overline{y}(t\,;\delta)]$

The extension principle gives the membership map as

$f(t, y(t))(s)\ =\ sup\{y(t)(\tau\,)/\ s\ =\ f\ (t, \tau\,)\}, s \in R$

so fuzzy number $f(t, y(t))$. From this it follows that

$\left[f\big(t, y(t)\big)\right]_r\ =\ \left[\underline{f}(t, y(t); \delta), \overline{f}(t, y(t); \delta)\right], \delta \in (0,1]$,

where

$\underline{f}(t, y(t); \delta)\ =\ min\{f(t, u) | u \in [y(t)]_\delta\}$

$\overline{f}(t, y(t); \delta)\ =\ max\{f(t, u) | u \in [y(t)]_\delta\}$.

Theorem 3 [8] If a function f satisfy the following

$|f(t, u) - f(t, u')| \leq g(t, |u - u'|), \quad t \geq 0, u, u' \in R$, where $g: R^+ \to R^+$ is a continuous function and $\delta \to g(t, \delta)$ is increasing, the initial value problem $u'(t) = g\big(t, u(t)\big),\ u(0) = u_0$, has a solution on R^+ for $u_0 = 0$, then the Fuzzy initial value problem (1) has a unique fuzzy solution.

4. Explicit Improved Runge Kutta Method

The family of explicit Improved Runge Kutta methods is a generalization of the Runge Kutta method. It is given by

$$y_{n+1} = y_n + h\left(b_1 k_1 - b_{-1}k_{-1} + \sum_{i=2}^{s} b_i(k_i - k_{-i})\right) \qquad (2)$$

where,

$k_1 = f(t_n, y_n),\ k_{-1} = f(t_{n-1}, y_{n-1}),$

$$k_i = f\left(t_n + hc_i, y_n + h\sum_{j=1}^{i-1} a_{ij}k_j\right),$$

$$k_{-i} = f\left(t_{n-1} + hc_i, y_{n-1} + h\sum_{j=1}^{i-1} a_{ij}k_{-j}\right), 2 \le i \le s.$$

Taylor Series expansion for y_{n+1} around y_n is given by

$$y_{n+1} = y_n + hf + \frac{h^2}{2!}\left(f_x + ff_y\right) + \frac{h^3}{3!}\left(f_y f_x + f_y^2 f + f_{xx} + 2f_{xx}f + f_{yy}f^2\right) + O(h^4),$$

where $y' = f(t, y)$ and by expanding (1) in the Taylor series expansion and comparing both the series in terms of h, we have the order conditions of the method.

The Butcher's (1987) equations of conditions for order 5 are given below

which are used for the elementary differentials up to order 5.

1st Order	$b_1 - b_{-1} = 1$
2nd Order	$b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2}$
3rd Order	$\sum_i b_i c_i = \frac{5}{12}$
4th Order	$\sum_i b_i {c_i}^2 = \frac{5}{12}, \sum_{ij} b_i c_j^2 a_{ij} = \frac{1}{6}$
5th Order	$\sum_i b_i c_i^3 = \frac{31}{120}, \sum_{ij} b_i {c_j}^2 a_{ij} c_i = \frac{31}{240},$ $\sum_{ij} b_i {c_j}^2 a_{ij} = \frac{31}{360}, \sum_{ijk} b_i a_{ij} a_{jk} c_k = \frac{31}{720}$

By the order conditions in above Table, we derived the Improved Runge Kutta methods of orders 4. To determine the free parameters of the third and fourth order methods we minimized the error norm for the methods of order 4 and 5, respectively. Hence, by satisfying as many equations as possible, for the fifth order method, we obtained the optimized fourth order method with 4-stages $p = 4$,

$s = 4$. The coefficients of the improved fourth order Runge Kutta methods are in following Table. The integer s (the number of stages), and the coefficients a_{ij} , b_i , c_i , $i = 2, 3, ..., s$.

The matrix $[a_{ij}]$ is called the Runge Kutta matrix, while the b_i and c_i are known as the weights and the nodes. These data are usually arranged in a mnemonic device, known as a Butcher tableau [2]:

$$
\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & & & & \\
\vdots & \vdots & & & & \\
c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & \\
\hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}
$$

The Runge–Kutta method is consistent if $\sum_{j=1}^{i-1} a_{ij} = c_i \, , i = 2,3, \dots s$

The Coefficients of 4th Order improved Runge Kutta method

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{5} & \frac{1}{5} & & & \\
\frac{3}{5} & 0 & \frac{3}{5} & & \\
\frac{4}{5} & \frac{2}{15} & \frac{4}{25} & \frac{38}{75} & \\
\hline
 & \frac{307}{288} & \frac{-25}{144} & \frac{25}{144} & \frac{125}{288}
\end{array}
$$

5. Explicit Improved Runge-Kutta Fourth Order

The explicit improved Runge Kutta fourth order methods is to prompt the variance among the standards of y at t_{n+1} and t_n as

$$y_{n+1} = y_n + h\left(b_1k_1 - b_{-1}k_{-1} + \sum_{i=2}^{s} b_i(k_i - k_{-i})\right)$$

where,

$k_1 = f(t_n, y_n)$

$k_2 = f\big(t_n + hc_2, y_n + h(a_{21}k_1)\big)$

$k_3 = f\big(t_n + hc_3, y_n + h(a_{31}k_1 + a_{32}k_2)\big)$

$k_4 = f\big(t_n + hc_4, y_n + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\big)$

$k_{-1} = f(t_{n-1}, y_{n-1})$

$k_{-2} = f\big(t_{n-1} + hc_2, y_{n-1} + h(a_{21}k_{-1})\big)$

$k_{-3} = f\big(t_{n-1} + hc_3, y_{n-1} + h(a_{31}k_{-1} + a_{32}k_{-2})\big)$

$k_{-4} = f\big(t_{n-1} + hc_4, y_{n-1} + h(a_{41}k_{-1} + a_{42}k_{-2} + a_{43}k_{-3})\big)$

From Butcher table,

$c_1 = 0$	$c_2 = \frac{1}{5}$	$c_3 = \frac{3}{5}$	$c_4 = \frac{4}{5}$	$a_{11} = 0$
$b_1 = \frac{307}{288}$	$b_2 = \frac{-25}{144}$	$b_3 = \frac{25}{144}$	$b_4 = \frac{125}{288}$	$a_{43} = \frac{38}{75}$
$a_{21} = \frac{1}{5}$	$a_{31} = 0$	$a_{32} = \frac{3}{5}$	$a_{41} = \frac{2}{15}$	$a_{42} = \frac{4}{25}$
$b_{-1} = \frac{19}{288}$				

Hence,

$$y_{n+1} = y_n + h\left(\frac{307}{288}k_1 - \frac{19}{288}k_{-1} - \frac{25}{144}(k_2 - k_{-2}) + \frac{25}{144}(k_3 - k_{-3}) + \frac{125}{288}(k_4 - k_{-4})\right)$$

where,

$$k_1 = f(t_n, y_n),\; k_{-1} = f(t_{n-1}, y_{n-1})$$

$$k_2 = f\left(t_n + \frac{1}{5}h, y_n + h\left(\frac{1}{5}k_1\right)\right)$$

$$k_3 = f\left(t_n + \frac{3}{5}h, y_n + h\left(\frac{3}{5}k_2\right)\right)$$

$$k_4 = f\left(t_n + \frac{4}{5}h, y_n + h\left(\frac{2}{15}k_1 + \frac{4}{25}k_2 + \frac{38}{75}k_3\right)\right)$$

$$k_{-2} = f\left(t_{n-1} + \frac{1}{5}h, y_{n-1} + h\left(\frac{1}{5}k_{-1}\right)\right)$$

$$k_{-3} = f\left(t_{n-1} + \frac{3}{5}h, y_{n-1} + h\left(\frac{3}{5}k_{-2}\right)\right)$$

$$k_{-4} = f\left(t_{n-1} + \frac{4}{5}h, y_{n-1} + h\left(\frac{2}{15}k_{-1} + \frac{4}{25}k_{-2} + \frac{38}{75}k_{-3}\right)\right)$$

6. Explicit Runge-Kutta method for solving fuzzy initial value problem

The exact solution of the problem in (2)

$[Y(t\,)]_\delta \; = \left[\underline{Y}\,(t\,;\delta), \overline{Y}(t\,;\delta)\,\right]$ be estimated by some

$[y(t\,)]_\delta \; = \left[\underline{y}\,(t\,;\delta), \overline{y}(t\,;\delta)\,\right]$.

The grating points are$h = \frac{T-t_0}{N}$, $t_1 = t_0 + i\,h; 0 \le i \le N$.

Now, we define

$y_{n+1} = y_n + h\left(\frac{307}{288}k_1 - \frac{19}{288}k_{-1} - \frac{25}{144}(k_2 - k_{-2}) + \frac{25}{144}(k_3 - k_{-3}) + \frac{125}{288}(k_4 - k_{-4})\right)$ as

$$\begin{aligned}\underline{Y}(t_{n+1};\delta) = {} & \underline{Y}(t_n;\delta) \\ & + h\left(\frac{307}{288}\underline{k_1}\big(t, y(t;\delta)\big) - \frac{19}{288}\underline{k_{-1}}\big(t, y(t;\delta)\big)\right. \\ & - \frac{25}{144}\Big(\underline{k_2}\big(t, y(t;\delta)\big) - \underline{k_{-2}}\big(t, y(t;\delta)\big)\Big) \\ & + \frac{25}{144}\Big(\underline{k_3}\big(t, y(t;\delta)\big) - \underline{k_{-3}}\big(t, y(t;\delta)\big)\Big) \\ & \left. + \frac{125}{288}\Big(\underline{k_4}\big(t, y(t;\delta)\big) - \underline{k_{-4}}\big(t, y(t;\delta)\big)\Big)\right)\end{aligned}$$

Where,

$$k_1 = F\left(t_n, \underline{Y}(t;\delta), \overline{Y}(t;\delta)\right);$$

$$k_2 = F\left(t_n + \frac{h}{5}, \underline{Y}(t;\delta) + \frac{hk_1}{5}, \overline{Y}(t;\delta) + \frac{hk_1}{5}\right)$$

$$k_3 = F\left(t_n + \frac{3h}{5}, \underline{Y}(t;\delta) + \frac{3hk_2}{5}, \overline{Y}(t;\delta) + \frac{3hk_2}{5}\right)$$

$$k_4 = F\left(t_n + \frac{4h}{5}, \underline{Y}(t;\delta) + h\left(\frac{2}{15}k_1 + \frac{4}{25}k_2 + \frac{38}{75}k_3\right), \overline{Y}(t;\delta) + h\left(\frac{2}{15}k_1 + \frac{4}{25}k_2 + \frac{38}{75}k_3\right)\right)$$

$$k_{-1} = F\left(t_{n-1}, \underline{Y}(t;\delta), \overline{Y}(t;\delta)\right)$$

$$k_{-2} = F\left(t_{n-1} + \frac{h}{5}, \underline{Y}(t;\delta) + \frac{hk_{-1}}{5}, \overline{Y}(t;\delta) + \frac{hk_{-1}}{5}\right)$$

$$k_{-3} = F\left(t_{n-1} + \frac{3h}{5}, \underline{Y}(t;\delta) + \frac{3hk_{-2}}{5}, \overline{Y}(t;\delta) + \frac{3hk_{-2}}{5}\right)$$

$$k_{-4} = F\left(t_{n-1} + \frac{4h}{5}, \underline{Y}(t;\delta) + h\left(\frac{2}{15}k_{-1} + \frac{4}{25}k_{-2} + \frac{38}{75}k_{-3}\right), \overline{Y}(t;\delta) + h\left(\frac{2}{15}k_{-1} + \frac{4}{25}k_{-2} + \frac{38}{75}k_{-3}\right)\right)$$

And,

$$\begin{aligned}\overline{Y}(t_{n+1};\delta) = \overline{Y}(t_n;\delta) &+ h\left(\frac{307}{288}\overline{k_1}(t, y(t;\delta)) - \frac{19}{288}\overline{k_{-1}}(t, y(t;\delta))\right. \\ &- \frac{25}{144}\left(\overline{k_2}(t, y(t;\delta)) - \overline{k_{-2}}(t, y(t;\delta))\right) \\ &+ \frac{25}{144}\left(\overline{k_3}(t, y(t;\delta)) - \overline{k_{-3}}(t, y(t;\delta))\right) \\ &\left. + \frac{125}{288}\left(\overline{k_4}(t, y(t;\delta)) - \overline{k_{-4}}(t, y(t;\delta))\right)\right)\end{aligned}$$

Where,

$$k_1 = G\left(t_n, \underline{Y}(t;\delta), \overline{Y}(t;\delta)\right);$$

$$k_2 = G\left(t_n + \frac{h}{5}, \underline{Y}(t;\delta) + \frac{hk_1}{5}, \overline{Y}(t;\delta) + \frac{hk_1}{5}\right)$$

$$k_3 = G\left(t_n + \frac{3h}{5}, \underline{Y}(t\,;\delta) + \frac{3hk_2}{5}, \overline{Y}(t\,;\delta) + \frac{3hk_2}{5}\right)$$

$$k_4 = G\left(t_n + \frac{4h}{5}, \underline{Y}(t\,;\delta) + h\left(\frac{2}{15}k_1 + \frac{4}{25}k_2 + \frac{38}{75}k_3\right), \overline{Y}(t\,;\delta) + h\left(\frac{2}{15}k_1 + \frac{4}{25}k_2 + \frac{38}{75}k_3\right)\right)$$

$$k_{-1} = G\left(t_{n-1}, \underline{Y}\,(t\,;\delta), \overline{Y}(t\,;\delta)\right)$$

$$k_{-2} = G\left(t_{n-1} + \frac{h}{5}, \underline{Y}(t\,;\delta) + \frac{hk_{-1}}{5}, \overline{Y}(t\,;\delta) + \frac{hk_{-1}}{5}\right)$$

$$k_{-3} = G\left(t_{n-1} + \frac{3h}{5}, \underline{Y}(t\,;\delta) + \frac{3hk_{-2}}{5}, \overline{Y}(t\,;\delta) + \frac{3hk_{-2}}{5}\right)$$

$$k_{-4} = G\left(t_{n-1} + \frac{4h}{5}, \underline{Y}(t\,;\delta) + h\left(\frac{2}{15}k_{-1} + \frac{4}{25}k_{-2} + \frac{38}{75}k_{-3}\right), \overline{Y}(t\,;\delta) + h\left(\frac{2}{15}k_{-1} + \frac{4}{25}k_{-2} + \frac{38}{75}k_{-3}\right)\right)$$

Also we have

$$\underline{y}(t_{n+1};\delta) = \underline{y}(t_n;\delta) + h\left(\frac{307}{288}\underline{k_1}(t, y(t;\delta)) - \frac{19}{288}\underline{k_{-1}}(t, y(t;\delta)) - \frac{25}{144}\left(\underline{k_2}(t, y(t;\delta)) - \underline{k_{-2}}(t, y(t;\delta))\right) + \frac{25}{144}\left(\underline{k_3}(t, y(t;\delta)) - \underline{k_{-3}}(t, y(t;\delta))\right) + \frac{125}{288}\left(\underline{k_4}(t, y(t;\delta)) - \underline{k_{-4}}(t, y(t;\delta))\right)\right)$$

Where,

$$k_1 = F\left(t_n, \underline{y}\,(t\,;\delta), \overline{y}(t\,;\delta)\right);$$

$$k_2 = F\left(t_n + \frac{h}{5}, \underline{y}(t\,;\delta) + \frac{hk_1}{5}, \overline{y}(t\,;\delta) + \frac{hk_1}{5}\right)$$

$$k_3 = F\left(t_n + \frac{3h}{5}, \underline{y}(t\,;\delta) + \frac{3hk_2}{5}, \overline{y}(t\,;\delta) + \frac{3hk_2}{5}\right)$$

$$k_4 = F\left(t_n + \frac{4h}{5}, \underline{y}(t\,;\delta) + h\left(\frac{2}{15}k_1 + \frac{4}{25}k_2 + \frac{38}{75}k_3\right), \overline{y}(t\,;\delta) + h\left(\frac{2}{15}k_1 + \frac{4}{25}k_2 + \frac{38}{75}k_3\right)\right)$$

$$k_{-1} = F\left(t_{n-1}, \underline{y}\,(t\,;\delta), \overline{y}(t\,;\delta)\right)$$

$$k_{-2} = F\left(t_{n-1} + \frac{h}{5}, \underline{y}(t\,;\delta) + \frac{hk_{-1}}{5}, \overline{y}(t\,;\delta) + \frac{hk_{-1}}{5}\right)$$

$$k_{-3} = F\left(t_{n-1} + \frac{3h}{5}, \underline{y}(t\,;\delta) + \frac{3hk_{-2}}{5}, \overline{y}(t\,;\delta) + \frac{3hk_{-2}}{5}\right)$$

$$k_{-4} = F\left(t_{n-1} + \frac{4h}{5}, \underline{y}(t\,;\delta) + h\left(\frac{2}{15}k_{-1} + \frac{4}{25}k_{-2} + \frac{38}{75}k_{-3}\right), \overline{y}(t\,;\delta) + h\left(\frac{2}{15}k_{-1} + \frac{4}{25}k_{-2} + \frac{38}{75}k_{-3}\right)\right)$$

And,

$$\bar{y}(t_{n+1};\delta) = \bar{y}(t_n;\delta) + h\left(\frac{307}{288}\overline{k_1}\big(t, y(t;\delta)\big) - \frac{19}{288}\overline{k_{-1}}\big(t, y(t;\delta)\big) - \frac{25}{144}\left(\overline{k_2}\big(t, y(t;\delta)\big) - \overline{k_{-2}}\big(t, y(t;\delta)\big)\right) + \frac{25}{144}\left(\overline{k_3}\big(t, y(t;\delta)\big) - \overline{k_{-3}}\big(t, y(t;\delta)\big)\right) + \frac{125}{288}\left(\overline{k_4}\big(t, y(t;\delta)\big) - \overline{k_{-4}}\big(t, y(t;\delta)\big)\right)\right)$$

Where,

$$k_1 = G\left(t_n, \underline{y}\,(t\,;\delta), \overline{y}(t\,;\delta)\right);$$

$$k_2 = G\left(t_n + \frac{h}{5}, \underline{y}(t\,;\delta) + \frac{hk_1}{5}, \overline{y}(t\,;\delta) + \frac{hk_1}{5}\right)$$

$$k_3 = G\left(t_n + \frac{3h}{5}, \underline{y}(t\,;\delta) + \frac{3hk_2}{5}, \overline{y}(t\,;\delta) + \frac{3hk_2}{5}\right)$$

$$k_4 = G\left(t_n + \frac{4h}{5}, \underline{y}(t\,;\delta) + h\left(\frac{2}{15}k_1 + \frac{4}{25}k_2 + \frac{38}{75}k_3\right), \overline{y}(t\,;\delta) + h\left(\frac{2}{15}k_1 + \frac{4}{25}k_2 + \frac{38}{75}k_3\right)\right)$$

$$k_{-1} = G\left(t_{n-1}, \underline{y}\,(t\,;\delta), \overline{y}(t\,;\delta)\,\right)$$

$$k_{-2} = G\left(t_{n-1} + \frac{h}{5}, \underline{y}(t\,;\delta) + \frac{hk_{-1}}{5}, \overline{y}(t\,;\delta) + \frac{hk_{-1}}{5}\right)$$

$$k_{-3} = G\left(t_{n-1} + \frac{3h}{5}, \underline{y}(t\,;\delta) + \frac{3hk_{-2}}{5}, \overline{y}(t\,;\delta) + \frac{3hk_{-2}}{5}\right)$$

$$k_{-4} = G\left(t_{n-1} + \frac{4h}{5}, \underline{y}(t\,;\delta) + h\left(\frac{2}{15}k_{-1} + \frac{4}{25}k_{-2} + \frac{38}{75}k_{-3}\right), \overline{y}(t\,;\delta) + h\left(\frac{2}{15}k_{-1} + \frac{4}{25}k_{-2} + \frac{38}{75}k_{-3}\right)\right)$$

Define,

$$F(t, y(t,\delta)) = h\left(\frac{307}{288}\underline{k_1}(t, y(t;\delta)) - \frac{19}{288}\underline{k_{-1}}(t, y(t;\delta)) - \frac{25}{144}\left(\underline{k_2}(t, y(t;\delta)) - \underline{k_{-2}}(t, y(t;\delta))\right) + \frac{25}{144}\left(\underline{k_3}(t, y(t;\delta)) - \underline{k_{-3}}(t, y(t;\delta))\right) + \frac{125}{288}\left(\underline{k_4}(t, y(t;\delta)) - \underline{k_{-4}}(t, y(t;\delta))\right)\right)$$

$$G(t, y(t,\delta)) = h\left(\frac{307}{288}\overline{k_1}(t, y(t;\delta)) - \frac{19}{288}\overline{k_{-1}}(t, y(t;\delta)) - \frac{25}{144}\left(\overline{k_2}(t, y(t;\delta)) - \overline{k_{-2}}(t, y(t;\delta))\right) + \frac{25}{144}\left(\overline{k_3}(t, y(t;\delta)) - \overline{k_{-3}}(t, y(t;\delta))\right) + \frac{125}{288}\left(\overline{k_4}(t, y(t;\delta)) - \overline{k_{-4}}(t, y(t;\delta))\right)\right)$$

Thus,

$[Y(t_n)]_\delta = \left[\underline{Y}\,(t_n\,;\delta), \overline{Y}(t_n\,;\delta)\,\right]$ and $[y(t_n)]_\delta = \left[\underline{y}\,(t_n\,;\delta), \overline{y}(t_n\,;\delta)\,\right]$ are the exact and approximate solutions at t_n, $0 \le n \le N$.

The solution at grid points, $l = t_0 \le t_1 \le t_2 \le \cdots \le t_N = m$ and

$h = \frac{m-l}{N} = t_{n+1} - t_n$.

By above Equations,

let

$$\underline{Y}(t_{n+1};\delta) = \underline{Y}(t_n;\delta) + F[t_n, Y(t_n;\delta)]$$

$$\overline{Y}(t_{n+1};\delta) = \overline{Y}(t_n;\delta) + G[t_n, Y(t_n;\delta)]$$

and

$$\underline{y}(t_{n+1};\delta) = \underline{y}(t_n;\delta) + F[t_n, y(t_n;\delta)]$$

$$\overline{y}(t_{n+1};\delta) = \overline{y}(t_n;\delta) + G[t_n, y(t_n;\delta)]$$

By Lemma 1 and Lemma 2,

$lim_{h\to 0}\underline{y}(t,\delta) = \underline{Y}(t,\delta)$ *and* $lim_{h\to 0}\overline{y}(t,\delta) = \overline{Y}(t,\delta)$.

Let $F(t,u,v)$ and $G(t,u,v)$ be found by replacing $[y(t)]_\delta = [u,v]$

$$F(t,u,v) = h\Bigg(\frac{307}{288}\underline{k_1}(t,u,v) - \frac{19}{288}\underline{k_{-1}}(t,u,v) - \frac{25}{144}\Big(\underline{k_2}(t,u,v) - \underline{k_{-2}}(t,u,v)\Big) + \frac{25}{144}\Big(\underline{k_3}(t,u,v) - \underline{k_{-3}}(t,u,v)\Big) + \frac{125}{288}\Big(\underline{k_4}(t,u,v) - \underline{k_{-4}}(t,u,v)\Big)\Bigg)$$

$$G(t,u,v) = h\Bigg(\frac{307}{288}\overline{k_1}(t,u,v) - \frac{19}{288}\overline{k_{-1}}(t,u,v) - \frac{25}{144}\Big(\overline{k_2}(t,u,v) - \overline{k_{-2}}(t,u,v)\Big) + \frac{25}{144}\Big(\overline{k_3}(t,u,v) - \overline{k_{-3}}(t,u,v)\Big) + \frac{125}{288}\Big(\overline{k_4}(t,u,v) - \overline{k_{-4}}(t,u,v)\Big)\Bigg)$$

The territory where F and G are well-defined, therefore

$K = \{(t,u,v) \mid 0 \le t \le T, -\infty < v < \infty, -\infty < u \le v\}$.

By Theorem 2, the approximate solutions $\underline{y}(t;r)$ and $\overline{y}(t;r)$ converges to the precise solution $\underline{Y}(t;r)$ and $\overline{Y}(t;r)$ consistently in t.

7. **Numerical Example:**

The following problems are solved for $t \in [0,10]$

Problem 1: Consider the IVP, $y' = \frac{-ty}{(1+t^2)}$, $y(0) = 1$;

The exact solutions $y(t) = \frac{-1}{\sqrt{(1+t^2)}}$

Problem 2: Consider the IVP, $y' = ycos\ t,\ y(0) = 1$;

The exact solutions $y(t) = e^{\sin t}$

8. Conclusion

In this paper, an Explicit Improved Runge-Kutta method is introduced and the fuzzy initial valued problem with Butcher coefficient is solved by using the proposed method, tested some initial value problems to show the efficiency and accuracy of the proposed methods.

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