Some Common Fixed Point Results in 2-Fuzzy 2-Rectangular Metric Spaces

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Abstract
In this paper, 2-fuzzy 2-rectangular fuzzy metric space is defined and a sufficient condition for the existence of unique common fixed point of 2-fuzzy 2-contractive mapping on complete 2-fuzzy 2-rectangular metric space is established.

Keywords: 2-fuzzy 2-rectangular metric space, 2-fuzzy 2-weakly commuting, 2-fuzzy 2-weakly compatible, 2-fuzzy 2- contractive.

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1.Introduction

The concept of fuzzy set was introduced by L.A.Zadeh(1965) [10] in his seminal paper. Shen et al.(2012)[8] introduced the notion of control function of fuzzy metric spaces. Nagoor Gani and Mohamed Althaf (2018)[5] established some results on fixed point theorem by using control function.

Bose and Sahani (1987)[1] have studied the fixed point theorem for fuzzy mapping coincidence points. Fang (1992) [2] developed fixed point theorems for different types of mapping. Soni (2018)[7] came with a new class of implict function to prove the concept of fixed point theorems in fuzzy metric space and its applications to the fuzzy system of functional equations. Pavan kumar et al.(2018)[6] introduced the notion of compatible mapping known as variants of compatible maps and proved fixed point theorem for these mappings. Singh and Chouhan(2000)[9] introduced the concepts of compatible maps in fuzzy metric space and proved fixed point theorems in fuzzy metric space. Muhammed Arshad et al.(2013)[4] introduced some common fixed point results in rectangular metric spaces.

In this paper, 2-fuzzy 2-rectangular metric space is defined and a sufficient condition for the existence of unique common fixed point of 2-fuzzy 2-contractive mapping on complete 2-fuzzy 2-rectangular metric space is established.

2. Preliminaries

Definition 2.1

Let X be a universe of discourse, a fuzzy set is defined as $\tilde{A} = \{(x, \mu_A(x)): x \in X\}$ which is characterized by a membership function $\mu_A(x): X \to [0,1]$where $\mu_A(x)$ denotes the degree of membership of the element x to the set A.

Definition 2.2

Let X be a non empty set and $F(X)$ be the set of all fuzzy sets in X. If $f \in F(X)$ then $f = \{\frac{x,\mu}{x} \in X$and $\mu \in (0,1]\}$. Clearly f is bounded function for $|f(x)| \leq 1$. Let K be the space of real numbers then $F(X)$ is a linear space over the field K where the addition and scalar multiplication are defined by

$$f + g = \{(x,\mu) + (y,\eta)\} = \{(x + y), (\mu, \eta)/\ (x,\mu) \in f and\ (y,\eta) \in g\}$$

and

$$kf = \{(kf, \mu)/\ (x,\mu) \in f\}$$

where $k \in K$.

The linear space $F(X)$ is said to be normed space if for every $f \in F(X)$ there is associated a non-negative real number $\|f\|$ called the norm of f in such a way,

(i) $\|f\| = 0$ if and only if $f = 0$.

For,

$$\|f\| = 0 \Leftrightarrow \{\|(x,\mu)\|/\ (x,\mu) \in f\} = 0$$
$$\Leftrightarrow x = 0, \mu \in (0,1] \Leftrightarrow f = 0$$

(ii) $\|kf\| = |k|\|f\|,\ k \in K$.

For ,

$$\|kf\| = \{\|k(x,\mu)\|/\ (x,\mu)f, k \in K\}$$

$$= \{|k|\|x,\mu\|/\ (x,\mu) \in f\} = |k|\|f\|$$

(iii) $\|f + g\| < \|f\| + \|g\|$for every $f, g \in F(X)$.

For,

$$\|f + g\| = \{\|(x,\mu) + (y,\eta)\|/\ \mathrm{x, y} \in \mathrm{X}, \mu\ , \eta \in (0,1]\}$$

$$= \{\|(x + y), (\mu \wedge \eta)\|/\ \mathrm{x, y} \in \mathrm{X}, \mu\ , \eta \in (0,1]\}$$

$$\leq \{\|(x, \mu \wedge \eta)\| + \|(y, \mu \wedge \eta)\|/\ (\mathrm{x}, \mu) \in \mathrm{f\ and} (\mathrm{y}, \eta) \in \mathrm{g}\}$$

$$= \|f\| + \|g\|$$

Then $(F(X), \|.\|)$ is a normed linear space.

Definition 2.3

A 2-fuzzy set on X is a fuzzy set on $F(X)$.

Definition 2.4

Let $F(X)$ be a linear space over the real field K. A fuzzy subset N of $F(X) \times F(X) \times R$ (R, the set of real numbers) is called a 2-fuzzy 2-norm on X (or fuzzy 2-norm on $F(X)$) if and only if,

(N1) for all $t \in R$with $t \leq 0$, $N(f_1, f_2, t) = 0$.

(N2) for all $t \in R$ with $t \geq 0, N(f_1, f_2, t) = 1$ if and only if f_1 and f_2 are linearly dependent.

(N3) $N(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2.

(N4) for all $t \in R$, with t ≥ 0, N(f_1, cf_2,t) = N($f_1, f_2, t/|c|$) if $c \neq 0, c \in K$ (field).

(N5) for all $s, t \in R$, N($f_1, f_2 + f_3$,$s + t$) ≥min{N($f_1, f_{2,} s$),N($f_1, f_{3,} t$)}.

(N6) N($f_1, f_2, .$): (0,∞) → $[0, 1]$is continuous.

(N7) $\lim_{t\to\infty} \mathrm{N}(f_1, f_2, \mathrm{t}) = 1$.

Then (F(X),N) is a fuzzy 2-normed linear space or (X,N) is a 2-fuzzy 2-normed linear space.

Definition 2.5

A sequence $\{f_n\}$ in a 2-fuzzy normed linear space (F(X),N) is said to be a convergent sequence if for a given t $>$ 0 and 0 $<$ r $<$ 1 there exist a positive number $n_0 \in N$such that

$N(f_n - f, g, t) > 1 - r$ for $g \in F(X)$ and for every $n \geq n_0$.

Definition 2.6

A sequence $\{f_n\}$ is said to be a Cauchy sequence in a 2-fuzzy normed linear space F(X) if for a given r > 0 with 0 < r < 1, t > 0 there exist a positive number n_0 such that

$\mathrm{N}(f_n - f_m, g, t) > 1 - r$ for g $\in$ F(X) and for every n, m ≥ n_0.

Definition 2.7

A 2-fuzzy 2-normed linear space (X,N) is said to be complete if every Cauchy sequence in X converge to some point in X.

3. 2-Fuzzy 2-Rectangular Metric Space

Definition 3.1

A 3-tuple $(X, M,*)$ is called a 2-fuzzy 2-metric space if $\Im(X)$ is the set of all fuzzy sets on X a non empty set, $*$ is a continuous t-norm and M is a fuzzy set on $\Im(X) \times \Im(X) \times (0, \infty)$satisfying the following conditions for each $f, g, h \in \Im(X)$ and $t, s > 0$,

(M1) $M(f, g, h, t) > 0$.

(M2) $M(f, g, h, t) = 1$if and only if f and g are linearly dependent.

(M3) $M(f, g, h, t) = M(g, f, h, t)$.

(M4) $M(f, g, h, t) * M(g, k, h, s) \leq M(f, k, h, t + s)$.

(M5) $M(f, g, h, .): (0, \infty) \to [0,1]$ is continuous.

Definition 3.2

Let $(X, M,*)$ be a 2-fuzzy 2-metricspace. For $t > 0$, the open ball $B(f, h, r, t)$ with centre $f \in \Im(X)$ and radius $0 < r < 1$ is defined by

$$B(f, h, \varepsilon, t) = \{g \in \Im(X): M(f, g, h, t) > 1 - \varepsilon\}$$

Definition 3.3

Let $(X, M,*)$ be a 2-fuzzy 2-metric space. Let τ be the set of all $A \subset \Im(X)$ with $f \in A$ if and only if there exists $t > 0$ and $0 < \varepsilon < 1$ such that $B(f, h, \varepsilon, t) \subset A$. Then τ is a topology on $\Im(X)$ (induced by the fuzzy metric M).

Definition 3.4

The 2-fuzzy 2-metric space $(X, M,*)$ is said to be complete if every cauchy sequence in $\Im(X)$ is convergent.

Definition 3.5

A sequence $\{f_n\}$ in $\Im(X)$ converges to f if and only if $M(f_n, f, h, t) \to 1$ as $n \to \infty$, for each $t > 0$. The sequence $\{f_n\}$ is a cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(f_n, f_m, h, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.

Definition 3.6

A subset A of $\Im(X)$ is said to be 2-fuzzy 2-F-bounded if there exists $t > 0$ and $0 < \varepsilon < 1$ such that $M(f, g, h, t) > 1 - \varepsilon$ for all $f, g \in A$.

Definition 3.7

Let $(X, M,*)$ be a 2-fuzzy 2-metric space, $f \in \Im(X)$ and $\emptyset \neq A \subseteq \Im(X)$. Define $D(f, A, h, t) = \sup\{M(f, g, h, t): g \in A\}\,(t > 0)$. Here $D(f, A, h, t)$is a degree of closeness of $f$to A.

Definition 3.8

A topological space is called a (topologically complete) 2-fuzzy 2-metrizable space if there exists a (topologically complete) 2-fuzzy 2-metric inducing the given topology on it.

Definition 3.9

Self maps A and S on a 2-fuzzy 2-metric space $(X, M,*)$ are said to be 2-fuzzy 2-compatible if $M(SAf_n, ASf_n, h, t) \to 0$ whenever $\{f_n\}$ is a sequence in $\Im(X)$ such that $\lim_{n\to\infty} Sf_n = \lim_{n\to\infty} Af_n = f$.

Definition 3.10

Self mappings A and S of a 2-fuzzy 2-metric space $(X, M,*)$ are said to be 2-fuzzy 2-weakly commuting if $M(SAf, ASf, h, t) \geq M(Sf, Af, h, t)$ for all $f \in \Im(X)$,

$t > 0$.

Definition 3.11

A point $f \in \Im(X)$ is called a coincidence point of S and A if and only if $Sf = Af$ and h is said to be the point of coincidence of A and S if $h = Sf = Af$.

Definition 3.12

Self maps$A$ and S of a 2-fuzzy 2-metric space $(X, M,*)$ are said to be 2-fuzzy 2-weakly compatible (or coincidentally commuting) if they commute at their coincidence points,

(i,e) if $Af = Sf$ for some $f \in \Im(X)$ then $ASf = SAf$

(i,e) $M(ASf, SAf, h, t) = 1$ if $M(Af, Sf, h, t) = 1$.

Definition 3.13

A 3-tuple $(X, M,*)$ is called a 2-fuzzy 2-rectangular metric space if $\Im(X)$ is the set of all fuzzy sets on X a non-empty set, $*$ is a continuous t-norm and let $M: \Im(X) \times \Im(X) \to [0, +\infty)$, satisfying the following conditions for each $f, g, k, s \in \Im(X)$ and $t > 0$,

(M1) $M(f, g, h, t) > 0$.

(M2) $M(f, g, h, t) = 1$ if and only if f and g are linear dependent.

(M3) $M(f, g, h, t) = M(g, f, h, t)$.

(M4) $M(f, g, h, t) \geq M(f, k, h, t) * M(k, s, h, t) * M(s, g, h, t)$.

(M5) $M(f, g, h, .): (0, \infty) \to [0,1]$ is continuous.

Theorem 3.14

Let $(X, M,*)$ be a 2-fuzzy 2-rectangular metric space. Let $T: \Im(X) \to \Im(X)$ be a map satisfying the condition $M(f, g, h, t) > 1 - t \Rightarrow M(Tf, g, h, \lambda t) > 1 - \lambda t$ for all $f \in \Im(X), t > 0$ and $\lambda \in (0,1)$ then

(i) for any real number $\delta \in (0,1)$,there exists $M_0(\delta)$ such that $T^m f \to 0$.
(ii) T has almost one fixed point which is the null vector of $\Im(X)$.

Proof

Consider the mapping $T: \Im(X) \to \Im(X)$ satisfying the condition $M(f, g, h, t) > 1 - t \Rightarrow M(f, g, h, \lambda t) > 1 - \lambda t$

Given $\delta \in (0,1)$

$$M(f,g,h,\delta) > 1-\delta \;\Rightarrow M(Tf,g,h,\lambda\delta) > 1-\lambda\delta$$

Consider

$$M(T^2 f,g,h,\lambda^2\,\delta) = M(T(Tf),g,h,\lambda(\lambda\delta))$$

$$> 1-\big(1-\;\lambda(\lambda\delta)\big)$$

$$= \lambda^2\,\delta$$

Continuing in this way, $M(T^m f,g,h,\lambda^m\delta) > \lambda^m\delta$

For each $\lambda \in (0,1)$and $\delta \in (0,1)$,$\lambda^m\delta < \delta$ (1)

Therefore, $M(T^m f,g,h,\delta) > M(T^m f,g,h,\lambda^m\delta) > \lambda^m\delta$

Since δ is arbitrary, we get $M(T^m f,g,h,\delta) \to 1$ as $m \to \infty$

Therefore $T^m f \to 0$, satisfying (1)

Assume $Tf = f$

Again from (1),

$M(f,g,h,\delta) > 1-\delta$ implies $M(T^m f,g,h,\lambda\delta) > 1-\lambda\delta$

Thus, $M(f,g,h,\lambda\delta) > 1-\lambda\delta$ implies T has the fixed point f and Since $T^m f = 0 \implies f = 0$.

Theorem 3.15

Let $\{f_n\}$ be a 2-fuzzy 2-rectangular metric space if for every $t > 0$ there exists a constant $\lambda \in (0,1)$ such that$M(f_n,f_{n+1},g,t) \geq M(f_{n-1},f_n,g,t/\lambda)$ for all $g \in \Im(X)$ then $\{f_n\}$ is a cauchy sequence.

Proof

Let $t > 0$ and $\lambda \in (0,1)$ then for $m \geq n$,

$$M(f_n,f_m,g,t) \geq M\left(f_n,f_{n+1},g,\frac{\lambda}{2}t\right) * M\left(f_{n+1},f_{n+2},g,\frac{\lambda}{2}t\right) * M(f_{n+2},f_m,g,(1-\lambda)t)$$

$$\geq M\left(f_{n-1},f_n,g,\frac{t}{\lambda}\left(\frac{\lambda}{2}\right)\right) * M\left(f_n,f_{n+1},g,\left(\frac{\lambda}{2}\right)\frac{t}{\lambda}\right) * M(f_{n+2},f_m,g,(1-\lambda)t)$$

$$\geq M\left(f_{n-1},f_n,g,\frac{t}{2}\right) * M\left(f_n,f_{n+1},g,\frac{t}{2}\right) * M(f_{n+2},f_m,g,(1-\lambda)t)$$

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$$\geq M\left(f_1,f_0,g,\frac{1}{2}\left(\frac{t}{\lambda^n}\right)\right) * M\left(f_1,f_2,g,\frac{1}{2}\left(\frac{t}{\lambda^{n-1}}\right)\right) * M(f_{n+2},f_m,g,(1-\lambda)t)$$

$$\geq M\left(f_1, f_0, g, \frac{1}{2}\left(\frac{t}{\lambda^n}\right)\right) * M\left(f_1, f_0, g, \frac{1}{2}\left(\frac{t}{\lambda^n}\right)\right) * M(f_{n+2}, f_m, g, (1-\lambda)t)$$

Now

$$M(f_{n+2}, f_m, g, (1-\lambda)t) \geq M\left(f_{n+2}, f_{n+1}, g, \frac{\lambda}{2}(1-\lambda)\frac{t}{\lambda}\right) * M\left(f_{n+1}, f_{n+3}, g, \frac{\lambda}{2}(1-\lambda)\frac{t}{\lambda}\right)$$

$$* M(f_{n+3}, f_m, g, (1-\lambda)^2 t)$$

$$\geq M\left(f_{n+2}, f_{n+1}, g, (1-\lambda)\frac{t}{2}\right) * M\left(f_{n+1}, f_{n+3}, g, (1-\lambda)\frac{t}{2}\right) * M(f_{n+3}, f_m, g, (1-\lambda)^2 t)$$

....

$$\geq M\left(f_1, f_0, g, \frac{1}{2}\frac{(1-\lambda)t}{\lambda^n}\right) * M\left(f_1, f_0, g, \frac{1}{2}\frac{(1-\lambda)t}{\lambda^n}\right) * M(f_{n+3}, f_m, g, (1-\lambda)^2 t)$$

On repeating the same argument

$M(f_{n+3}, f_m, g, (1-\lambda)^2 t) \geq$ ....

$$\geq M\left(f_0, f_1, g, \frac{(1-\lambda)^2 t}{\lambda^n}\right)$$

Since $\frac{(1-\lambda)t}{\lambda^{n+1}} \geq \frac{t}{\lambda^n}$ and $\frac{(1-\lambda)^2 t}{\lambda^n}$, it is obvious that $M(f_n, f_m, g, t) \geq M(f_0, f_1, g, \frac{t}{\lambda^n})$

As $n, m \to \infty$, it implies that $M(f_n, f_m, g, t) \to 1$ for every $f_n, f_m, g \in \mathfrak{I}(X)$, $t > 0$ and therefore $\{f_n\}$ is a cauchy sequence.

Definition 3.16

Let $(X, M, *)$ be a 2-fuzzy 2-rectangular metric space, a mapping T on $\mathfrak{I}(X)$ is said to be a 2-fuzzy 2-contraction if there exists $k \in (0,1)$ such that

$$kM(Tf, Tg, h, t) \geq M(f, g, h, t)$$

Definition 3.17

A mapping $T: \mathfrak{I}(X) \to \mathfrak{I}(X)$ is said to be 2-fuzzy 2-uniformly continuous if for $\varepsilon \in (0,1)$ there exists $\delta \in (0,1)$ such that$M(Tf, Tg, h, t) > 1 - \varepsilon$ provided $M(f, g, h, t) > 1 - \delta$.

Theorem 3.18

Let $(X, M, *)$ be a 2-fuzzy 2-rectangular metric space then every 2-fuzzy 2-contractive mapping is 2-fuzzy 2-continuous.

Proof

Assume T is a 2-fuzzy 2-contractive mapping, then there exists $k \in (0,1)$ such that

$$kM(Tf, Tg, h, t) \geq M(f, g, h, t)$$

Assume for a given $\varepsilon \in (0,1)$ there exists $\delta \in (0,1)$ such that

$$M(f, g, h, t) > 1 - \delta$$

Hence $kM(Tf, Tg, h, t) > 1 - \delta$ (2)

Choose k and δ in such a way that $\delta = \frac{1}{1+k}$

Then, define ε so that $\frac{1-\delta}{k} \geq 1 - \varepsilon$

Therefore from (2), $M(Tf, Tg, h, t) > 1 - \varepsilon$

So,T is 2-fuzzy 2-continuous.

Lemma 3.19

Let $\Im(X)$ be a set of all fuzzy sets on X. Suppose that the mappings $A, B: \Im(X) \to \Im(X)$ have a unique coincidence point h in $\Im(X)$. If A and B are 2-fuzzy 2-weakly compatible, then A and B have a unique common fixed point.

Proof

Let $h \in \Im(X)$ be the coincidence point of $A, B: \Im(X) \to \Im(X)$

$Ah = Bh = f$ (3)

Since A and B are 2-fuzzy 2-weakly compatible.

For $g \in \Im(X)$ $M(ABg, BAg, l, t) = 1$ when $M(Ag, Bg, l, t) = 1$

Therefore $M(ABh, BAh, l, t) = 1$ when $M(Ah, Bh, l, t) = 1$

(i,e)., $M(Af, Bf, l, t) = 1$ when $M(Ah, Bh, l, t) = 1$

Hence$f = h$, using (3) $Ah = Bh = h$

and so h is the fixed point.

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