Cohomology of groups

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Abstract
let G be a group and ZG be its integral group ring. Thus an additive group ZG is the free abelian group with the element of G, a ZG Module M is the same thing as specifying an abelian group M on which G acts, An abelian group H^n (G, M) where n= 0,1,2,3…… called n^{th}Cohomology of G with the coefficients in the ZG - module M. In this paper we explore the notion of Cohomology of finite groups and infinite groups.
Keywords : Algebraic groups, Noetherian rings, modules, projective modules.

1. Introduction

To understand this we need to know what the group ring ZG, Thus an additive group ZG is free abelian group with basic elements of G, the ring is generated by multiplication of the basis elements in G. An element of ZG is a sum $\sum_{x \epsilon g} \lambda_x$ x with $\lambda_x \epsilon$ Z where not all λ_x are zero. A ZG- module M is similar thing as an abelian group M on which G acts i.e there is homomorphism G $\rightarrow$ Aut(M) and denote by additive Z the ZG-module if gn=n for all nϵZ and gϵG i.e action of G is trivial. The n^{th} cohomology group of G with coefficients in ZG – module M defined as H^n(G,M) := Ext^n_{ZG}(Z,M) .The cohomology groups may be defined topologically and also algebraically.

2. Topological approach of cohomology

Now we will say something about Topological approach of cohomology by Hurewicz theorem if X be path –connected space with π_nX= 0 for all n ≥2 such X is called aspherical then X determined upto**.** If cohomology an aspherical space X is locally path connected the universal cover $\widetilde{X}$ is contractible and X= $\widetilde{X}/G$. Also $H^n(X)$ depend only on π_1(X) if G= π_1(X) We mustt thus define $H^n(G, Z) = H^n(X)$ and because X is determined upto cohomology equivalence the definition does not depend on X The Hurewics theorem gives what the group cohomology is if there exist an aspherical space with the fundamental group , but if does not clearly defined that always such space exist.

3. Algebraical approach of cohomology

Various of the low – dimensional cohomology group had been studied earlier than the topologically defined groups or the general definition of group cohomology . in 1932 Baer studied $H^2(G,A)$ as a group of equivalence classes of extensions . it was in 1945 that Eilenberg and Maclane introduced an algebraic approach which included these groups as special cases. let G be a group and ZG be its integral group ring .Thus an additive group ZG is the free abelian group with the element of G, now for each group G and representation M of G there are abelian group H^n (G,M) Where n = 0.1,2,3……….. called the n^{th} cohomology of G with coefficient in M.

(3.1) COROLLARY.

If G is a finite group and M is a ZG module then for all $n \geq 1$, $H^n(G, M)$ is finite abelian group of exponent dividing $|G|$.

proof. An abelian group A is uniquely divisible by an integer n if for all $a \in A$ there exists a unique $b \in A$ with $a = nb$. This happens if the homomorp- -hism $n : A \to A$ is an isomoprhism. we say that A is uniquely divisible if it is uniquely divisible by each positive integer n. For example, Q and R are uniquely divisible , Q/Z divisible but not uniquely. At last we have If A is finite and g.c.d $(|A|, n)=1$ then A is uniquely divisible by n.

(3.2) COROLLARY

If G is a finite group and M is a finitely generated ZG module which is uniquely divisible by $|G|$ as an abelian group then $H^n(G, M)=0$ for all $n \geq 1$.

Proof. Since multiplication $|G|:M \to M$ is an isomorphism,so $|G| : H^n(G,M) \to H^n(G,H)$ By functoriality of cohomology. This map is zero for each $n \geq 1$, by Proposition, we have $H^n(G,M)=0$ for each $n \geq 1$.

(3.3) COROLLARY.

(1) $H^n(G,Z) \cong H^{n-1}(G, Q/Z) \cong H^{n-1}(G,C^\times)$for each $n \geq 2$, with similar isomorphism's in cohomology.

(2) If M is finitely generated RG module in which $|G|$ is invertible then $H^n(G,M)=0$ for all $n \geq 1$.

Proof. let $C^\times$ be multiplicative group of nonzero complex numbers, which is isomorphic to $R^\times_{>0} \times S^1$ by the map $z \leftrightarrow (|z|, \arg(z))$. We find a short exact sequence$1 \to Z \to R^\times_{>0} \times R^+ \to C^\times \to 1$. Because$R^\times_{>0} \cong R^+$ by the natural logarithm, the middle term of this sequence is divisible uniquely and now the long exact sequence associated to the exact sequence gives the result.

Now we show an application of this and a result know as the integral duality theorem which tell us a finite group that $(G,Z) \cong H_n(G,Z)$ when $n \geq 1$, Putting this together we have $H_2(G,Z) \cong H^3(G,Z) \cong H^2(G,C^\times) \cong H^2(G, Q/Z)$
These groups are all isomorphic to the Schur multiplier.

(3.4) COROLLARY

Let $1 \rightarrow M \rightarrow E \rightarrow G \rightarrow 1$ be a short exact sequence of finite groups where g.c.d(|M|,|G|)=1. Then the extension is split, E≅M⋉G,and all subgroups of E of order |G| are conjugate.

Proof. we proof when M is abelian. Here H^2(G,M)=(G,M)=0 by Corollary, so the result follows from our interpretation of second and first cohomology.
Let C be an abelian group. We will call any module of the from ZG⊗z C an induced module, and any module of the form $HOM_Z(ZG,C)$ a coinduced module. The latter is made into a ZG-module using the right action on ZG.

(3.5) LEMMA.

If M is coinduced then H^n (G.M)=0 for each n≥1.
There is no restriction on G for this result.

Proof. if M=Home (ZG,C) is a coinduced module for any abelian group C we evaluate Cohomology with coefficients in M by applying the functor Hom_{ZG} (-,Hom_z(ZG,C)to a projective resolution . Now for some module P we have a natural isomorphism
functor Hom_z(-,C) to a projective resolution of Z we obtainacyclic complex because
Hom_{ZG} (P,Hom_z (ZG,C)≅Hom_z($ZG_{\otimes ZG}$ P,C) ≅ Hom_z(P ,C) and if we apply the
as abelian groups the projective resolution splits. Thus H^n(G,M)=0 for all n ≥1.

(3.6) PROPOSITION.

If G is finite then induced and coinduced modules coincide.
Hence cohomology vanishes on induced modules in degrees ≥ 1, If P is a projective RG-module for some commutative ring R then H^n(G,P)=0 for all n ≥1.

Proof. We define a mapping ZG ⊗z C → Hom_Z(ZG, C) by g ⊗c ↦ $\phi_{(g,c)}$where $\phi_{(g,c)}$:ZG→C is the homomorphism determined by by

$$\phi_{(g,c)}(h)=\begin{cases} c & \text{if } g = h \\ 0 & \text{otherwise} \end{cases}$$

We exanime that this is a homomorphism of ZG-modules which is injective and surjective if G is finite . Since free modules are induced we find that chomology vanishes on them as well as on projective modules as they are direct summands of free modules

4. cohomologically finite generation property of groups

Let G is a group and R is a ring of unit 1, The graded R-algebra of a group G is cohomologically finite if there are noetherian R-algebras and finitely produced H^n(G, M) modules in R-algebras. For any RG-module M which is generated finitely over R.

Several representational features of G's group theory are compromised with limited generation. Finitely generated cohomology rings can be found in [2,7]. Is the ring of invariants A^G a Noetherian one? This is an important question of invariance theory. All Noetherian associative rings A must have the finite generation property if a group meets this requirement. When a group is cohomological type, the finite generation feature can be discovered in its cohomological form. Finite generation properties are the same for all extensions of finite generation groups. Groups of the cohomological type that extend finite groups are cohomological of finite type when the second basic statement is true. [12]'s invariant subring finiteness criteria apply to numerous groups, and some are closed under group extensions.

For the major results to be shown, the cohomological generalization to limited generations of invariant subrings is necessary. Cohomological adaptation for group schemes over a field k was investigated by Van der Kallen in [17], $H^n(G, A)$ represents the finitely generated k-algebra G over k when A is a finitely generated (commutative) k-algebra with a (rational) G-action as k-algebra automorphisms? CFG is the acronym used in [22] to describe this type of G group (cohomologically finitely generated). As long as finite numbers generate A^G, whether A is countably generated by k-algebra, G can be called a group scheme. According to [19], all finite group schemes have the same attribute, CFG. Algebraic group schemes over a field and invariants and cohomology rings are discussed further in [25]. As recently proven in [31], an algebraic group G can only possess the characteristic CFG if it also possesses the property F.G. Many reductive groups, including all reductive groups, are CFG subgroups.

Automorphisms of G on a finitely generated R-algebra are only possible when the fixed point subalgebra A^G is likewise an R-algebra. If we're talking about Noetherian rings that are commutative and finitely generated R-algebras, we call this F.G. An A^G algebra in particular, can be generated infinitely. G has the CFG property if the graded R-algebra $H^n(G, A)$ is a finitely generated R-algebra and the Noetherian RG-module homomorphism $H^n(G, M)$ is a Noetherian RG-module homomorphism for any Noetherian commutative ring R and commutative R-algebra A on which G acts as R-algebra automorphisms. G is Cohomological finiteness is a group property with the CFG property (by taking A and R with trivial G-action). Because H^0(G, A) is a quotient of H, A^G is a finitely generated R-algebra (G, A). In other words, the property CFG entails the property F.G.

It is said that group G has cohomological finiteness if the R-module H^n(G, M) is finitely constructed for any Noetherian multiplicative ring R and RG-module M.which is finitely generated as an R-module.

We find Groups that are finite in cohomological type are known as cohomologically finite groups. Contrary to popular belief, the opposite of this is not true by a result of Evens[15] and Venkov[32] They showed that if G is finite then G is cohomologically of finite type but a finite group G is not cohomologically finite because cohomology groups H^n (G, R) are infinite dimensions when R is a field with fundamental splitting equal to the order of G..

A mathematical group G has the attribute CFG if and only if the rational action in FG can be applied to it, as per Touzé and van der Kallen [35] According to the Mumford Conjecture, an algebraic group G can only have the attribute F.G. if it is reducible. Touzé and van der Kallen might simply assume that G is a reductive group. In the case of characteristic zero, all rational representations can be reduced to zero. The cohomological dimension of G is zero, which means that F.G. and CFG are accurate. There are no more reasons to examine Frobenius kernels and twists in spectral sequences. Thus we can now focus on the positive characteristic situation.

For finite cohomological dimensions, whether G entails CFG can be asked. A Noetherian ring A and a finitely generated A^G-module M about which H^k(G, M) is a finite number generated A^G-module are required for G to have the attribute CFG. When k is 0, then G has the property FG.

H^k(G, A) is not finitely formed over A^G. when the cohomological dimensionality is 1, such as the free group.

5. virtual finite projective resolution of Groups

Type VFP (virtual finite projective resolution) is applied to a group G when it contains a subgroup of finite index that is an F.P. group. When it comes to VFPs, all F.P.s are VFPs. There are finitely created abelian groups in all degrees of cohomology groups $H^n(G, Z)$. Groups with nontrivial torsion components can have cohomology groups $H^n(G, R)$ of arbitrarily high degrees. Therefore it's important to remember that these groups are not invariably cohomologically finite. An argument in [24] uses the stratification of cohomological variations in the perspective of elementary abelian p-subgroups. Ian Leary supplied the following rationale. If G is a finite subgroup and Z is a cyclic subgroup of prime order p, then H^n (Z, Fp) is a finitely generated module of H^n(G, Fp). A non-zero image can be found in infinitely many degrees, for example, using H^n(G, Fp), H^n(Z, Fp). If G has a finite VCD, it has a torsion-free subgroup of the finite index The finite cyclic subgroup Z = (g) is embedded in G/H via the composition G → G/H as a torsion-free normal subgroup H with a finite index (of prime order p). Because the constraint map H^k(G/H, Fp)→H^k((g), F_p) has an unlimited number of degrees, H^k(g, Fp) is never 0 for any degree in H^k(G, Fp). Because H^k(G, Fp) factors into the restriction map, the group of Z-rational points is affected. For every reductive algebraic group, G, GL(n, C) built over the field of rational numbers Q, G(Z), G(Q), GL(n, Z) is of type VFP and usually contains nontrivial torsion members. Other natural groups include the translating class group and the outer predictiveness group. Nontrivial torsion elements can be found in Coxeter groups and Out(Fn) of free groups Fn, but they are still type VFP. It was shown in the preceding section that they are not cohomologically finite.

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