Oscillation of Second Order nonlinear Neutral Difference Equations

S. Sindhuja[1], J. Daphy Louis Lovenia[2*], A. P. Lavanya[3]

S. Sindhuja[1], Assistant Professor, Department of Mathematics, Karpagam Institute of Technology, Coimbatore, India, sindhusivaraj2211@gmail.com

J. Daphy Louis Lovenia[2*], Professor, Department of Mathematics, Karunya Institute of Technology and Sciences, Corresponding Author, Coimbatore, India, **daphy@karunya.edu**

A.P. Lavanya[3],Assistant Professor, Department of Mathematics, Sri Krishna college of Engineering and Technology, Coimbatore, India, algebralavanya@gmail.com

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Abstract
In this paper, we study the oscillatory behavior of secondorder nonlinear neutraldifference equation of the form

$$\Delta^2(x(s)-ax(s-m))=q(s)x(s-\sigma)+r(s)x(s+\tau)$$

Examples are provided to illustrate the results.

Keywords: Nonlinear neutral difference equation, oscillatory behavior.

1 Introduction

Consider the second order difference equation,

$$\Delta^2(x(s)-ax(s-m))=q(s)x(s-\sigma)+r(s)x(s+\tau) \quad (1.1)$$

where $s\in S(s_0), s_0$ is positive integer, a are real positive constant, $\{q_s\},\{r_s\}$ are real sequences m,σ,τ are positive integers.

By the solution of (1.1), we mean real sequence $\{x(s)\}$ is defined for $s\in S(s_0)$. A non-trivial solution $\{x(s)\}$ is oscillatory if it is neither eventually positive nor eventually negative and non-oscillatory otherwise.

In the past few years, there has been an increasing interest in the study of oscillatory solutions of difference equations see [1 –5]. For example Grace S R [1],MałgorzataMigdaa and JanuszMigdab [3] have done an extensive work on this topic. The observative have motivated to study the oscillatory behavior of solutions of second order nonlinear neutraldifference equation

2 Main results

Theorem 2.1

Suppose $\sigma>\tau$, $\{q(s)\}$ and $\{r(s)\}$ where $q(s)>0, r(s)>0$ are non-increasing sequences, then

i) $\Delta^2\mu(s)-r(s)\mu(s+\tau)\geq 0$ (2.1) has no eventually nonnegative increasingsolution,

ii) $\Delta^2 \mu(s) - q(s)\mu(s-\sigma) \geq 0$ (2.2)

has no eventually nonnegative decreasingsolution,

iii) $\Delta^2 u(s) + \frac{q(s)}{a} u(s-\sigma+m) + \frac{r(s)}{a} u(s+\tau+m) < 0$ (2.3)

has no eventually nonnegative solution. Therefore all solutions of (1.1) oscillate.

Proof:Assume $\{x(s)\}$ is anonoscillatory solution for (1.1).With no loss of generality, we get $s_1 \in S(s_0)$ and $x(s-\theta) > 0$ for all $s \geq s_1$. Suppose $z(s) = x(s) - ax(s-m)$. Then

$$\Delta^2 z(s) = q(s)(x(s-\sigma) + r(s)(x(s+\tau) > 0,\ s \geq s_1. \quad (2.4)$$

where $\{x(s)\}$ is non-oscillatory solution, according to (2.4), we claim $z(s) \geq 0$

By using the contradiction, assume that $z(s) < 0$ such that

$$0 < u(s) = -z(s) = -x(s) + ax(s-m) \leq ax(s-m)$$

Then

$$x(s) \geq \frac{1}{a} u(s+m) \ \ s \geq s_2$$

By (1.1), we get

$$0 = \Delta^2 u(s) + q(s)(x(s-\sigma)) + r(s)(x(s+\tau)) \geq \Delta^2 u(s) + \frac{q(s)}{a} u(s-\sigma+m) + \frac{r(s)}{a} u(s+\tau+m)$$

which contradicts to (2.3)

Suppose,

$$\mu(s) = z(s) - \frac{a}{2} z_{s-m} \quad (2.5)$$

$$\Delta^2 \mu(s) = \Delta^2 z(s) - \Delta^2 \frac{a}{2} z_{s-m}$$

$$\Delta^2 \mu(s) = q(s)(x(s-\sigma) + r(s)(x(s+\tau) - \frac{a}{2}(q(s-m)x(s-\sigma-m)$$
$$+ r(s-m)(x(s+\tau-m))$$

$$\Delta^2 \mu(s) = q(s)(x(s-\sigma)) + r(s)(x(s+\tau)) - \frac{a}{2} q(s-m)x(s-\sigma-m)$$
$$- \frac{a}{2} r(s-m)(x(s+\tau-m)) \quad (2.6)$$

$$\Delta^2\mu(s) \geq q(s)[x(s-\sigma)] - a(q(s-m))(x(s-\sigma-m)) + r(s)(x(s+\tau))$$
$$- ar(s-m)(x(s+\tau-m))$$

$$\Delta^2\mu(s) \geq q(s)z(s-\sigma) + r(s)z(s+\tau) > 0 \qquad (2.7)$$

$\mu(s) < 0,$

$$0 < v(s) = -\mu(s) = -z(s) + \frac{a}{2}z(s-m) \leq \frac{a}{2}z(s-m)$$

Since

$$z(s) \leq \frac{2}{a}v(s+m)$$

Using (2.7) we get,

$$0 \geq \Delta^2 v(s) + \frac{q(s)}{a}v(s-\sigma+m) + \frac{r(s)}{a}v(s+\tau+m)$$

This contradicts (2.3).

Case: (i)

Suppose $\Delta z(s) < 0$ for $s \geq s_3 \geq s_2$ such that $\Delta x(s) < 0$ for $s \geq s_2$,

By using the contradiction, let $x(s) > 0, \Delta x(s) > 0,$ and $\Delta^2 x(s) > 0$ which gives $\lim_{m\to\infty} x(s) = \infty$

Since $x(s) > 0,$ which implies $z(s) > 0$ but $\Delta z(s) < 0,$ gives $\lim_{m\to\infty} z(m) = c < 0.$

By limits on (2.5), we get a contradiction. Using the monotonicity of $\{z(s)\}$ gives,

$$\mu_{s-\sigma} = z(s-\sigma) - \frac{a}{2}z(s-m-\sigma) \leq z(s-\sigma)$$

By (2.7), we get

$$\Delta^2\mu(s) \geq q(s)\mu(s-\sigma)$$

Therefore $x(s)$ is nonnegative decreasing solution ofequation (2.2), then it contradicts the proof.

Case: (ii)

Suppose $\Delta z(s) > 0$ for $s \geq s_4$. Two cases occur

Case:1

Suppose $\Delta x(s) < 0$ for $s \geq s_4$, the result is similar to case (i).

Case 2:

Suppose $\Delta x(s) > 0$ for $s \geq s_2$,

Since $\mu(s+\tau) = z(s+\tau) - \frac{a}{2} z(s-m+\tau) \leq z(s+\tau)$

By (2.7), we get

$\Delta^2 \mu(s) \geq r(s)\mu(s+\tau)$

which contradicts (2.1). The proof of the theorem is complete.

Theorem 2.2

Suppose $\sigma > m, \tau > m$

Let $\limsup_{s\to\infty} \sum_{t=s}^{s+\sigma-m} (s+\sigma-t-1)q > 1$ (2.8)

$\limsup_{s\to\infty} \sum_{t=s-\tau+m}^{s} (s-t+\tau-m+1)r > 1$ (2.9)

and the inequality (2.3) hasno eventually negative increasing solutionconsequently, then all solutions of (1.1) oscillate.

Proof: According to the conditions (2.8) and (2.9),inequalities (2.1) hasno eventually decreasing solution and (2.2) has no eventually increasing solution.

and (2.3) has eventually negative solution.

Therefore by theorem (2.1), the equation (1.1) is oscillatory.

3. Example

Example 1

Consider theequation

$$\Delta^2 (x(s) - 2(3)^m x(s-m)) = -6(3)^\sigma x(s-\sigma) + 2(3)^{-\tau} x(s+\tau) \quad (3.1)$$

Here $a = 2(3)^m, q(s) = -6(3)^\sigma, r(s) = 2(3)^{-\tau}$

Theorem 2.2 is not satisfied for the condition (2.8). Then the condition (3.1)isnon-oscillatory solution for $\{x(s)\} = \{(3)^s\}$.

Example 2

Consider the equation

$$\Delta^2 (x(s) - 4(2)^m x(s-m)) = -6(2)^\sigma x(s-\sigma) + 3(2)^{-\tau} x(s+\tau) \tag{3.2}$$

Here $a = 4(2)^m, q(s) = -6(2)^\sigma, r(s) = 3(2)^{-\tau}$

Theorem 2.2 is not satisfied for the condition (2.9). Then thecondition(3.2)isnon-oscillatory solution for $\{x(s)\} = \{(2)^s\}$.

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