Loai Distribution: Properties, Parameters Estimation and Application to Covid-19 Real Data

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Abstract
In this paper, we propose a new two parameter continuous distribution. It is called Loai distribution. Some statistical properties of this distribution are derived such as: the moment generating function, the moments and related measures, the reliability analysis, and associated functions. Also, the distribution of order statistics and the quantile function are presented. Shannon, R*e*'nyi and Tsallis entropies are derived. The method of maximum likelihood and some other methods of estimation are used to estimate the distribution parameters. A simulation study is performed to investigate the performance of different methods of estimation. Covid-19 real data applications show that the proposed distribution can provide a better t than several well-known distributions.

Keywords: Mixing distribution, Loai distribution, gamma distribution, moments and moment generating function, reliability analysis, entropy, maximum likelihood estimate, and Lindley distribution.

1 Introduction and Literature Review

In statistics, modelling lifetime data is an important issue in many elds including biomedical sciences, economics, finance, engineering and many others. A lot of continuous distributions have introduced for modelling such data, because they can contribute better t than the base distributions. Many ways are recently used to propose new models such as the mixture of two or more distributions. These distributions are used in many elds of life such as: medicine, environment, biostatistics, and many others. Several distributions have been proposed from mixing distributions, for example, [39] suggested Darna distribution as a mixture of$Exp\left(\frac{\theta}{\alpha}\right)$ and $gamma\left(3,\frac{\theta}{\alpha}\right)$ with mixing proportion $\frac{2\alpha^2}{2\alpha^2+\theta^2}$, [36] suggested Rama distribution by mixing $Exp(\theta)$ and $gamma(4,\ \theta)$ using mixing proportion $\frac{\theta^3}{\theta^3+6}$. Another two components mixture of $Exp(\theta)$ and $gamma(3,\theta)$ is proposed using mixing proportion $\frac{\theta^3}{\theta^3+2}$ by [37] named Ishita distribution. [34] suggested Aradhana distribution by mixing $Exp(\theta)$, $gamma(2,\ \theta)$ and $gamma(3,\ \theta)$ with proportions $\frac{\theta^2}{\theta^2+2\theta+2}$, $\frac{2\theta}{\theta^2+2\theta+2}$and $\frac{2}{\theta^2+2\theta+2}$. [33] used the mixture weight $\frac{\theta^2}{\theta^2+1}$

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with $Exp(\theta)$ and $gamma(2\ ,\theta)$ to propose Shanker distribution. [19] proposed Gharaibeh distribution as a four components mixture of $exp(\theta)$, $gamma(2,\theta)$, $gamma(4,\theta)$ and $gamma(6,\theta)$ with proportions $\frac{\theta^6}{\theta^6+\theta^4+\theta^2+1}$, $\frac{\theta^4}{\theta^6+\theta^4+\theta^2+1}$, $\frac{\theta^2}{\theta^6+\theta^4+\theta^2+1}$ and $\frac{1}{\theta^6+\theta^4+\theta^2+1}$. [12] employed the concept of mixture distributions using the $Exp(\theta)$ and $gamma(\alpha-1,\theta)$, with mixture proportions $\frac{1}{\alpha\theta+1}$ and, to suggest a new two parameters distribution called $\frac{\alpha\theta}{\alpha\theta+1}$ Alzoubi distribution. [13] used the same concept to suggest a new two parameters Benrabia distribution.

On the other hand, the transmutation map is another method for proposing new distributions. It had been used by many authors. For example, [3] used this map to propose the transmuted Mukherjee-Islam distribution. Some properties are studied as well. The transmuted Janardan distribution has been proposed by [2]. [20] proposed transmuted Aradhana distribution. This map is also used to propose a generalization of the new Weibull-Pareto distribution [1]. [9] employed this map to suggest the transmuted Shanker distribution as a generalization to Shanker distribution. [11] generalized gamma-Gompertz distribution using the quadratic transmutation map. Some other distributions using this map were generated by [6]; [7]; [8]; [29]; [31].

Another method of generating new distributions by using the kernel function is employed by [4] by compounding the Biweight kernel function with the exponential distribution to propose the Biweight Exponential distribution and [24] compounded the Epanechnikov kernel function with the exponential distribution to propose the Epanechnikov-exponential distribution.

This paper will employ the mixture of Gamma distribution with parameters $\alpha = 3$ and θ, gamma$(3,\ \theta)$, and Lindley distribution with parameter θ, $Lin(\theta)$, with mixture proportions $\frac{\alpha}{\alpha+1}$ and $\frac{1}{\alpha+1}$. We will call this distribution Loai distribution with parameters α and θ, denoted as $LoaiD(\alpha,\ \theta)$. Also, we want to prove that the suggested distribution is more flexible than some other distributions based on some real lifetime data.

A random variable X is said to have a mixture of two or more distributions $f_1(x),\cdots,f_k(x)$, if its probability density function (pdf) $g(x)=\sum_{i=1}^{k}a_i f_i(x)$ with $0\le a_i\le 1$ is the mixing weight, such that $\sum_{i=1}^{k}a_i=1$.

A random variable X follows Lindley distribution with parameter $\theta>0$, if its probability density (pdf) and cumulative distribution (cdf) functions are defined as:

$$f(x,\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x},\ \ x>0,\ \theta>0, \tag{1.1}$$

$$F(x,\theta) = 1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x},\ \ x\ge 0,\ \theta>0 \tag{1.2}$$

The gamma distribution with parameters $\alpha = 3,\ \theta$ if the (pdf) and (cdf) functions are defined as:

$$f(x,\theta) = \frac{1}{2}\theta^3 x^2 e^{-\theta x}, \ x>0, \ \theta>0 \tag{1.3}$$

$$F(x,\theta) = 1-\frac{1}{2}(2+\theta x(\theta x+2))e^{-\theta x}, \ x\geq 0, \ \theta>0 \tag{1.4}$$

This paper is organized as follows; in Section 2 the probability density and the cumulative distribution functions of Loai distribution are derived. In Section 3, some statistical properties are considered including the moment generating function, the moments, and some measures related to these properties. In Section 4 the stochastic ordering and the reliability analysis are defined. The probability density functions of order statistics and the quantile function are derived in Section 5. In Sections 6 and 7 the Bonferroni and Lorenz curves are derived. In Section 7, the Shannon, R*e'*nyi and Tsallis entropies are derived, as well. The mean deviation about mean and median are computed in Section 8. In Section 9 we estimate the distribution parameters using some methods of estimation. In Section 10 the stress-strength reliability is derived. Section 11 provides a simulation study to assure the validity of the methods of estimation. A real Covid-19 dataset is applied to our distribution in Section 12. At the end, Section 13 sums up the article.

2 Loai Distribution

In this section, the pdf and the cdf of the proposed distribution are defined with graphic illustration for both.

Definition: A random variable *X* is said to have a LoaiD if it has a pdf defined as:

$$g(x|\alpha,\theta) = \frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\alpha\theta x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}, \ x>0, \ \theta>0, \ \alpha>0 \tag{2.1}$$

The function defined in (2.1) represents a pdf because $g(x|\alpha,\theta)\geq 0$, $\forall x>0$, and

$$\begin{aligned}
\int_0^\infty g(x|\alpha,\theta)dx &= \int_0^\infty \frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\alpha\theta x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}dx \\
&= \frac{\theta^2}{\alpha+1}\int_0^\infty \left[\frac{1}{2}\alpha\theta x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}dx \\
&= \frac{\theta^2}{\alpha+1}\left[\frac{\alpha+1}{\theta^2}\right] \\
&= 1 \qquad \square
\end{aligned}$$

The cdf of Loai distribution is obtained as

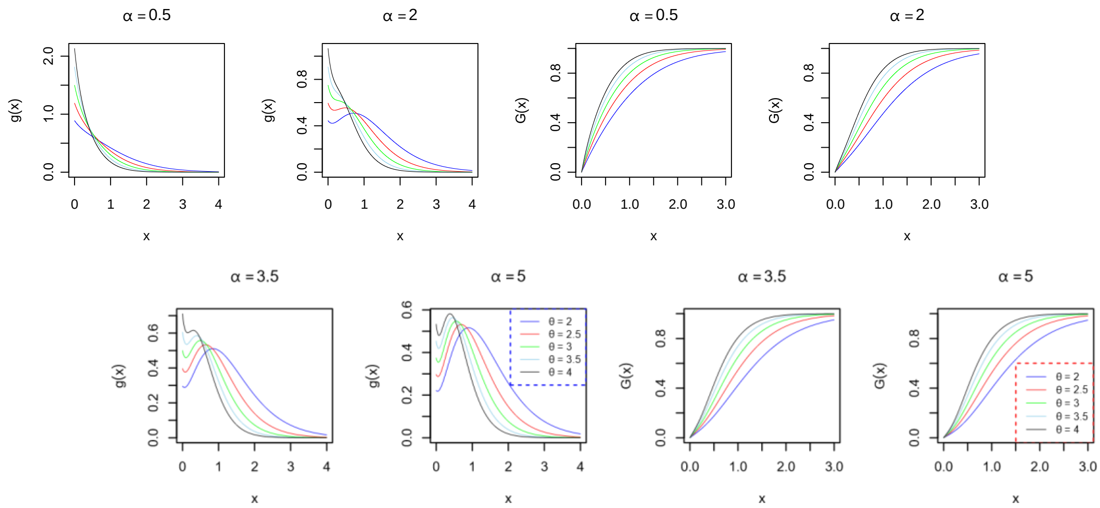
$$G(x|\alpha,\theta) = \int_0^x \frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\alpha\theta u^2+\frac{1}{\theta+1}(1+u)\right]e^{-\theta u}du$$

$$G(x|\alpha,\theta) = 1-\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}x^2+\left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta x\right]e^{-\theta x}, x\geq 0, \theta>0, \alpha>0 \tag{2.2}$$

This function satisfies the conditions of cumulative distribution function. That is

$$
\begin{aligned}
\lim_{x\to 0} G(x|\alpha,\theta) &= \lim_{x\to 0}\left(1-\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}x^2+\left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta x\right]e^{-\theta x}\right)\\
&= 1-1=0
\end{aligned}
$$

$$
\begin{aligned}
\lim_{x\to \infty} G(x|\alpha,\theta) &= \lim_{x\to \infty}\left(1-\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}x^2+\left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta x\right]e^{-\theta x}\right)\\
&= 1-0=1
\end{aligned}
$$



(a) The pdf of LoaiD when α =0.5, 2, 3.5 and 5 for different values of θ.

(b) The pdf of LoaiD when α =0.5, 2, 3.5 and 5 for different values of θ.

3 Moments and Moment Generating Function

In this section, the r^{th} moment and the moment generating function are presented. Also, the mean, variance, kurtosis, skewness, and coefficient of variation are also calculated.

3.1 Moments and related measures

Theorem 1: The r^{th} moment of Loai distribution can be expressed as:

$$
E(X^r)=\frac{1}{\theta^r(\alpha+1)}\left[\frac{\alpha\Gamma(r+3)}{2}+\frac{\Gamma(r+2)}{(\theta+1)}+\frac{\theta\Gamma(r+1)}{(\theta+1)}\right] \tag{3.1}
$$

Proof

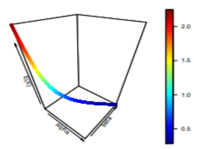
$$
\begin{aligned}
E(X^r) &= \int_0^\infty x^r g(x)dx\\
&= \frac{\theta^2}{\alpha+1}\int_0^\infty x^r\left[\frac{1}{2}\alpha\theta x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}dx\\
&= \frac{\theta^2}{\alpha+1}\left[\int_0^\infty \frac{1}{2}\alpha\theta x^{r+2}e^{-\theta x}+\frac{1}{\theta+1}x^{r+1}e^{-\theta x}+\frac{1}{\theta+1}x^r e^{-\theta x}dx\right]\\
&= \frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\alpha\theta\frac{\Gamma(r+3)}{\theta^{r+3}}+\frac{1}{\theta+1}\frac{\Gamma(r+2)}{\theta^{r+2}}+\frac{1}{\theta+1}\frac{\Gamma(r+1)}{\theta^{r+1}}\right]\\
&= \frac{1}{\theta^r(\alpha+1)}\left[\frac{\alpha\Gamma(r+3)}{2}+\frac{\Gamma(r+2)}{(\theta+1)}+\frac{\theta\Gamma(r+1)}{(\theta+1)}\right]
\end{aligned}
$$

Substituting r =1, 2, 3 and 4 in (3.1) we get first, second, third and fourth moments of LoaiD. They are defined as follows:

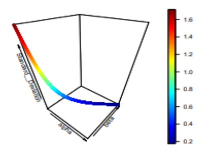
$$\begin{aligned}
E(X) &= \mu = \frac{3\alpha(\theta+1)+\theta+2}{\theta(\theta+1)(\alpha+1)} \\
E(X^2) &= \frac{12\alpha(\theta+1)+2\theta+6}{\theta^2(\theta+1)(\alpha+1)} \\
E(X^3) &= \frac{60\alpha(\theta+1)+6\theta+24}{\theta^3(\theta+1)(\alpha+1)} \\
E(X^4) &= \frac{360\alpha(\theta+1)+24\theta+120}{\theta^4(\theta+1)(\alpha+1)}
\end{aligned}$$

These moments can be used to obtain the variance (σ^2), standard deviation (σ), coefficients of variation (cv), skewness (sk) and kurtosis (ku) of a random variable that follows LoaiD. These measures are:

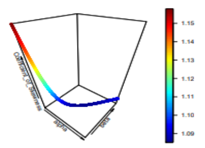
$$\begin{aligned}
Var(X) = \sigma^2 &= E(X^2) - (E(X))^2 \\
&= \frac{3\alpha^2\theta^2 + 6\alpha^2\theta + 3\alpha^2 + 8\alpha\theta^2 + 14\alpha\theta + 6\alpha + \theta^2 + 4\theta + 2}{[\theta(\theta+1)(\alpha+1)]^2} \\
\sigma &= \frac{\sqrt{3\alpha^2\theta^2 + 6\alpha^2\theta + 3\alpha^2 + 8\alpha\theta^2 + 14\alpha\theta + 6\alpha + \theta^2 + 4\theta + 2}}{\theta(\theta+1)(\alpha+1)} \\
cv(X) = \frac{\sigma}{\mu} &= \frac{\sqrt{3\alpha^2\theta^2 + 6\alpha^2\theta + 3\alpha^2 + 8\alpha\theta^2 + 14\alpha\theta + 6\alpha + \theta^2 + 4\theta + 2}}{3\alpha(\theta+1)+\theta+2} \\
sk(X) &= \frac{E(X^3) - 3\mu E(X^2) + 2\mu^3}{\sigma^3} \\
&= \frac{\begin{pmatrix} 6\alpha^3\theta^3 + 18\alpha^3\theta^2 + 18\alpha^3\theta + 6\alpha^3 + 18\alpha^2\theta^3 + 54\alpha^2\theta^2 + 54\alpha^2\theta \\ +18\alpha^2 + 30\alpha\theta^3 + 72\alpha\theta^2 + 60\alpha\theta + 18\alpha + 2\theta^3 + 12\theta^2 + 12\theta + 4 \end{pmatrix}}{\left(\sqrt{3\alpha^2\theta^2 + 6\alpha^2\theta + 3\alpha^2 + 8\alpha\theta^2 + 14\alpha\theta + 6\alpha + \theta^2 + 4\theta + 2}\right)^3} \\
ku(X) &= \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4}{\sigma^4} \\
&= \frac{\begin{bmatrix} 45\alpha^4\theta^4 + 180\alpha^4\theta^3 + 270\alpha^4\theta^2 + 180\alpha^4\theta + 45\alpha^4 + 216\alpha^3\theta^4 + 828\alpha^3\theta^3 \\ +1188\alpha^3\theta^2 + 756\alpha^3\theta + 180\alpha^3 + 306\alpha^2\theta^4 + 1188\alpha^2\theta^3 + 1710\alpha^2\theta^2 \\ +1080\alpha^2\theta + 252\alpha^2 + 192\alpha\theta^4 + 708\alpha\theta^3 + 996\alpha\theta^2 \\ +624\alpha\theta + 144\alpha + 9\theta^4 + 72\theta^3 + 132\theta^2 + 96\theta + 24 \end{bmatrix}}{[3\alpha^2\theta^2 + 6\alpha^2\theta + 3\alpha^2 + 8\alpha\theta^2 + 14\alpha\theta + 6\alpha + \theta^2 + 4\theta + 2]^2}
\end{aligned}$$



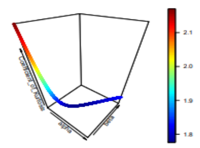
(a) The three dimensions plot of the mean of LoaiD for different values of α and θ.



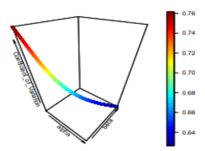
(b) The three dimensions plot of the standard deviation of LoaiD for different values of α and θ.



(c) The three dimensions plot of the coefficient of skewness of LoaiD for different values of α and θ.



(d) The three dimensions plot of the coefficient of kurtosis of LoaiD for different values of α and θ.



(e) The three dimensions plot of the coefficient of variation of LoaiD for different values of α and θ.

Figures 2(a) - 2(e) show the three dimensions plots of the mean, standard deviation, skewness, excess kurtosis, and the coefficient of variation of Loai distribution for different values of α and θ. The figures show that the distribution is skewed right as all the coefficients of skewness are positive. The values of excess kurtosis= kurtosis - 3 [22] are all positive, which means that the tails of the distribution are heavier than the normal distribution tails. It also shows that all measure values decrease as the values of parameter θ increases.

3.2 Moment generating function

Theorem 2 The moment generating function of Loai distribution can be expressed as follows

$$M_X(t) = \frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\alpha\theta\left(\frac{2}{\theta-t}\right)^3 - \left(\frac{t-\theta-1}{(\theta+1)(\theta-t)^2}\right)\right], \theta > t \tag{3.2}$$

Note that $\frac{\partial^r M_X(t)}{\partial t^r}|_{t=0} = E(X^r)$.

Proof

$$\begin{aligned} M_X(t) &= E\left(e^{tX}\right) = \int_0^\infty e^{tx} g(x)dx \\ &= \int_0^\infty e^{tx}\frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\alpha\theta x^2 + \frac{1}{\theta+1}(1+x)\right]e^{-\theta x}dx \\ &= \int_0^\infty \frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\alpha\theta x^2 + \frac{1}{\theta+1}(1+x)\right]e^{(t-\theta)x}dx \\ &= \frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\alpha\theta\left(\frac{2}{\theta-t}\right)^3 - \left(\frac{t-\theta-1}{(\theta+1)(\theta-t)^2}\right)\right], \theta > t \end{aligned}$$

4 Stochastic Ordering and Reliability analysis

4.1 Stochastic ordering

Stochastic ordering is an important tool in finance and reliability theory to assess the comparative performance of the models or systems. Let X and Y be two random variables with pdf, cdf, and reliability functions: $g(x)$, $g(y)$, $G(x)$, $G(y)$, $\bar{G}(x) = 1 - G(x)$ and $\bar{G}(y) = 1 - G(y)$; respectively. Then

1. Mean residual life order denoted by $X \leq_{MRLO} Y$, if $m_x(x) \leq m_y(y)$, $\forall x$.
2. Hazard rate order denoted as $X \leq_{HRO} Y$, if $\frac{\bar{G}_X(x)}{\bar{G}_Y(x)}$ is decreasing if $x \geq 0$.
3. Stochastic order denoted as $X \leq_{SO} Y$, if $\bar{G}(x) \leq_{SO} \bar{G}_Y(x)$, $\forall x$.
4. Likelihood ratio order denote as $X \leq_{LRO} Y$, if $\frac{f_X(x)}{f_Y(x)}$ is decreasing for $x \geq 0$.

All these orders are related to each other as follows [32]:

$$X \leq_{LRO} Y \Rightarrow X \leq_{HRO} Y \Rightarrow X \leq_{MRLO} Y$$

$$\Downarrow$$

$$X \leq_{SO} Y$$

Theorem 3: Assume that X and Y are two independent random variables with pdfs $g_x(x, \alpha, \theta)$ and $g_y\,(x, \eta, \tau)$; respectively. If $\eta < \theta$ and $\tau < \alpha$, then $X \leq_{LRO} Y$, $X \leq_{HRO} Y$, $X \leq MRLO\ Y$ and $X \leq SO\ Y$.

Proof: Consider $\Lambda = \frac{g_X(x,\alpha,\theta)}{g_Y(x,\eta,\tau)}$. Thus,

$$\begin{aligned}\Lambda &= \frac{\frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\alpha\theta x^2 + \frac{1+x}{\theta+1}\right]e^{-\theta x}}{\frac{\eta^2}{\tau+1}\left[\frac{1}{2}\tau\eta x^2 + \frac{1+x}{\eta+1}\right]e^{-\eta x}} \\ &= \frac{\theta^2(\tau+1)\left[\frac{1}{2}\alpha\theta x^2 + \frac{1+x}{\theta+1}\right]}{\eta^2(\alpha+1)\left[\frac{1}{2}\tau\eta x^2 + \frac{1+x}{\eta+1}\right]}e^{(\eta-\theta)x}\end{aligned}$$

Thus,

$$\begin{aligned}ln(\Lambda) &= ln\left[\frac{\theta^2(\tau+1)\left[\frac{1}{2}\alpha\theta x^2 + \frac{1+x}{\theta+1}\right]}{\eta^2(\alpha+1)\left[\frac{1}{2}\tau\eta x^2 + \frac{1+x}{\eta+1}\right]}e^{(\eta-\theta)x}\right] \\ &= ln\left[\frac{\theta^2(\tau+1)}{\eta^2(\alpha+1)}\right] + ln\left[\frac{1}{2}\alpha\theta x^2 + \frac{1+x}{\theta+1}\right] - ln\left[\frac{1}{2}\tau\eta x^2 + \frac{1+x}{\eta+1}\right] + (\eta-\theta)x\end{aligned}$$

Deriving with respect to x, we get:

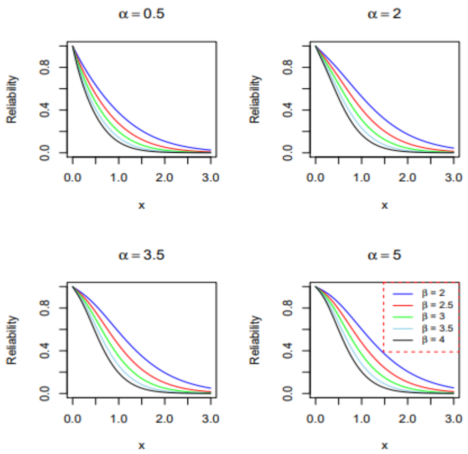
$$\frac{\partial ln(\Lambda)}{\partial x} = \frac{\alpha\theta x + \frac{1}{\theta+1}}{\frac{1}{2}\alpha\theta x^2 + \frac{1+x}{\theta+1}} - \frac{\tau\eta x + \frac{1}{\eta+1}}{\frac{1}{2}\tau\eta x^2 + \frac{1+x}{\eta+1}} + (\eta-\theta)$$

$\frac{\partial ln(\Lambda)}{\partial x} < 0\ if\ \eta < \theta,\ \tau < \alpha$. Thus, $X \leq_{LRO} Y$, $X \leq_{HRO} Y$, $X \leq_{MRLO} Y$ and $X \leq_{SO} Y$.

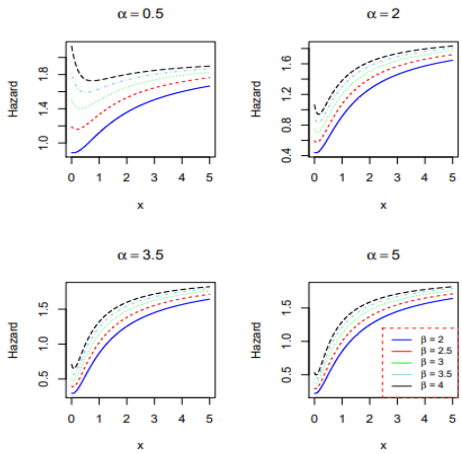
4.2 Reliability analysis

The reliability, hazard rate, cumulative hazard rate, reversed hazard rate and odds rate functions of a random variable that follows LoaiD are defined to be

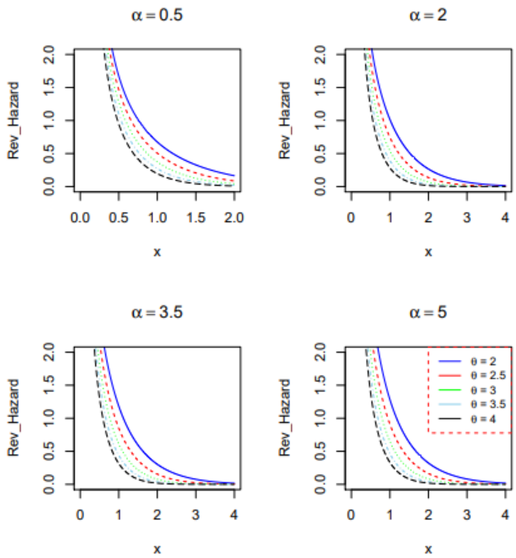
$$
\begin{aligned}
R(t) &= 1 - G(t) \\
&= \left[1 + \frac{\alpha\theta^2}{2(1+\alpha)}t^2 + \left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta t\right] e^{-\theta t} \\
h(t) &= \frac{g(t)}{1-G(t)} \\
&= \frac{\theta^2\left[\frac{1}{2}\alpha\theta t^2 + \frac{1}{\theta+1}(1+t)\right]}{\alpha+1+\frac{1}{2}\alpha\theta^2 t^2 + \left[\frac{\alpha\theta+\alpha+1}{1+\theta}\right]\theta t} \\
H(t) &= -ln(1-G(t)) \\
&= \theta t - ln\left[1 + \frac{\alpha\theta^2}{2(1+\alpha)}t^2 + \left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta t\right] \\
rh(t) &= \frac{g(t)}{G(t)} \\
&= \frac{\frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\alpha\theta t^2 + \frac{1}{\theta+1}(1+t)\right]e^{-\theta t}}{1-\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}t^2 + \left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta t\right]e^{-\theta t}} \\
O(t) &= \frac{G(t)}{1-G(t)} = \frac{1-\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}t^2 + \left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta t\right]e^{-\theta t}}{\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}t^2 + \left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta t\right]e^{-\theta t}}
\end{aligned}
$$



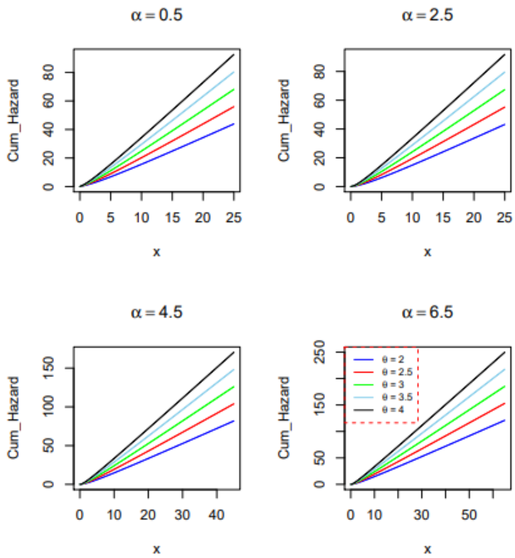
(a) The reliability function of LoaiD for $\alpha = 0.5,\ 2,\ 3.5,\ 5$



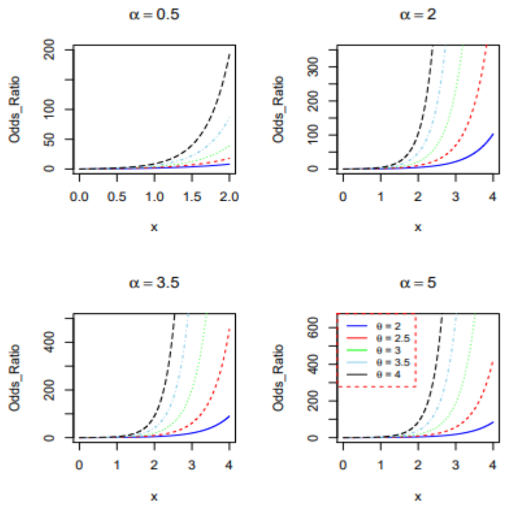
(b) The hazard rate function of LoaiD for $\alpha = 0.5,\ 2,\ 3.5,\ 5$



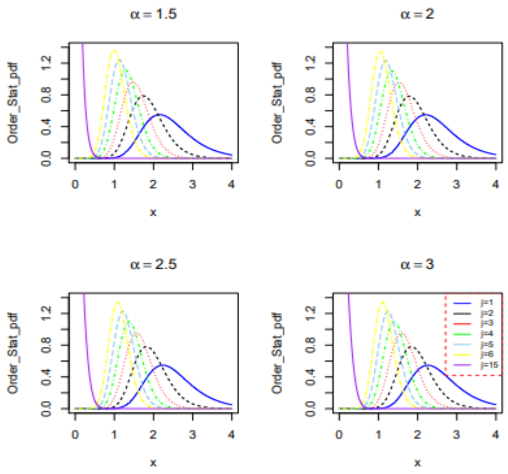
(a) The reversed hazard rate function of LoaiD for $\alpha = 0.5,\ 2,\ 3.5,\ 5$



(b) The cumulative hazard rate function of LoaiD for $\alpha = 0.5,\ 2,\ 3.5,\ 5$



(a) The odds ratio function of LoaiD when $\alpha = 0.5$, 2, 3.5 and 5 for different values of α



(b) The Order statistics pdf of LoaiD when $\theta = 2.5$

5 Order Statistics and Quantile function

In this section, we will derive the pdfs of first, n^{th} and j^{th} order statistics and the quantile function of LoaiD.

5.1 Order statistics

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of the random sample $X_1, X_2, \cdots, X_n$ selected from LoaiD. Then the pdf of the j^{th} order statistics $X_{(j)}$ is defined as

$$g_{(j)}(x) = j\binom{n}{j}[G(x)]^{j-1}[1-G(x)]^{n-j}g(x) \tag{5.1}$$

By replacing (2.1) and (2.2) in (5.1) and using binomial theorem, we get

$$\begin{aligned}
g_{(j)}(x) &= j\binom{n}{j}\left[1-\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}x^2+\left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta x\right]e^{-\theta x}\right]^{j-1} \\
&\quad \times\left[\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}x^2+\left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta x\right]e^{-\theta x}\right]^{n-j} \\
&\quad \times\frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\alpha\theta x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}
\end{aligned}$$

The distribution of the minimumfirst order statistic $X_{(1)} = min(X_1, X_2, \cdots, X_n)$ and the largest order statistic $X_{(n)} = max(X_1, X_2, \cdots, X_n)$ can be computed by replacing j in the previous equation by 1 and n; respectively. So, we get

$$\begin{aligned}
g_{(1)}(x) &= \frac{n\theta^2}{\alpha+1}\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}x^2+\left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta x\right]^{n-1}\left[\frac{1}{2}\alpha\theta x^2+\frac{1}{\theta+1}(1+x)\right]e^{-n\theta x} \\
g_{(n)}(x) &= \frac{n\theta^2}{\alpha+1}\left[1-\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}x^2+\left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta x\right]e^{-\theta x}\right]^{n-1} \\
&\quad \times\left[\frac{1}{2}\alpha\theta x^2+\frac{1}{\theta+1}(1+x)\right]e^{-\theta x}
\end{aligned}$$

Figure 5(b) represents the plot of the pdf of j^{th} order statistics for values of α of 1.5, 2, 2.5 and 3 and values of θ of 2, 2.5, 3, 3.5 and 4 and values of j of 1-6 and 15. It shows that the peak of the plot gets sharper as the value of j gets larger.

5.2 Quantile function

The quantile function of a distribution with cdf, $G(x)$, is defined by $q = G(x_q)$, where $0 < q < 1$. Thus, the quantile function of Loai distribution is the real solution of the following equation:

$$1-q=\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}x_q^2+\left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta x_q\right]e^{-\theta x_q} \tag{5.2}$$

This equation has no close form of solutions. Figure 6 shows the quantile plot for different values of q. The selected values are q = 0.05, 0.10, 0.15, …, 0.95.

6 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves have importance in many domains such as economics, demography [23]. The Bonferroni and Lorenz curves for a random variable X are, respectively, defined as:

$$\begin{aligned} B(p) &= \frac{1}{p\mu}\int_0^p xg(x)dx \\ L(p) = &= \frac{1}{\mu}\int_0^p xg(x)dx, \end{aligned}$$

where $q = F^{-1}(p)$; $p \in [0,1]$ *and* $\mu = E(X)$. Hence the Bonferroni and Lorenz curves of our distribution are, respectively, given by:

$$\begin{aligned} B(p) &= \left[\frac{1}{p}-\frac{[\alpha(\theta+1)[6+\theta p[6+\theta p(3+\theta p)]]+2[2+\theta p(2+\theta p)]+\theta(2+\theta p)]\,e^{-\theta p}}{12\alpha p(\theta+1)+2(3+\theta)p}\right] \\ L(p) &= \left[1-\frac{[\alpha(\theta+1)[6+\theta p[6+\theta p(3+\theta p)]]+2[2+\theta p(2+\theta p)]+\theta(2+\theta p)]\,e^{-\theta p}}{12\alpha(\theta+1)+2(3+\theta)}\right] \end{aligned}$$

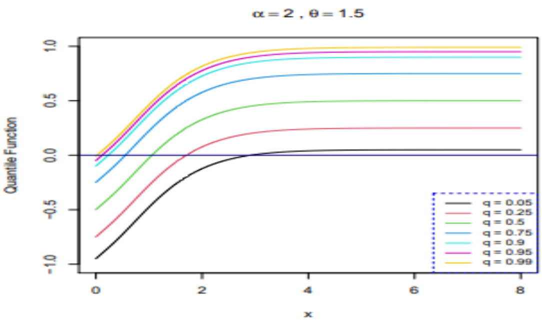


Figure 6: The quantile function of Loai distribution when α = 2, θ = 1.5

7 Entropy

The entropy was first introduced by [38]. It describes the amount of information in a signal or event in information theory. Entropy defined as a measure of uncertainty of the probability distribution of a random variable *X* in statistics [44]. It is used in many elds such as statistics and engineering. [38]; [30]; [42] defined the Shannon, R*e'*nyi and Tsallis entropies of a random variable *X* as

$$S_\rho = -\int_0^\infty g(x) log(g(x)) dx$$

$$R_\rho = \frac{\rho}{1-\rho} log \int_0^\infty [g(x)]^\rho dx; \quad \rho > 0 \quad \rho \neq 1$$

$$T_\rho = \frac{1}{\rho-1}\left[1 - \int_0^\infty [g(x)]^\rho dx\right]; \quad \rho > 0 \quad \rho \neq 1$$

Theorem 4: Shannon entropy of the random variable *X* such that $X \sim LoaiD(\alpha, \theta)$ is defined by

$$S_\rho = -\int_0^\infty \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \log\left[\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\right] dx$$

The R*e´*nyi and Tsallis entropies of the random variable $X \sim Loai(\alpha,\theta)$ is defined by

$$\begin{aligned} R_\rho &= \frac{\rho}{1-\rho}\ln\left(\frac{\theta^2}{\alpha+1}\right) + \frac{1}{1-\rho}\sum_{i=1}^{\rho}\sum_{k=1}^{i}\binom{\rho}{i}\binom{i}{k}\left(\frac{1}{\theta+1}\right)^i \\ &\times \left(\frac{\alpha\theta}{2}\right)^{\rho-i}\left(\frac{1}{\theta\rho}\right)^{2(\rho-i)+k+1}\Gamma(2(\rho-i)+k+1) \\ T_\rho &= \frac{1}{\rho-1}\left[1 - \left[\ln\left(\frac{\theta^2}{\alpha+1}\right) + \frac{1}{1-\rho}\sum_{i=1}^{\rho}\sum_{k=1}^{i}\binom{\rho}{i}\binom{i}{k}\left(\frac{1}{\theta+1}\right)^i\right.\right. \\ &\left.\left.\times \left(\frac{\alpha\theta}{2}\right)^{\rho-i}\left(\frac{1}{\theta\rho}\right)^{2(\rho-i)+k+1}\Gamma(2(\rho-i)+k+1)\right]\right] \end{aligned}$$

Proof: Using (2.1) and the definition of Re'nyi entropy above, we have

$$\begin{aligned} R_\rho &= \frac{1}{1-\rho}\log\int_0^\infty [g(x)]^\rho dx \\ &= \frac{1}{1-\rho}\log\left(\frac{\theta^2}{\alpha+1}\right)^\rho \int_0^\infty \left[\frac{\alpha\theta}{2}x^2 + \frac{1+x}{1+\theta}\right]^\rho e^{-\theta\rho x} dx \\ &= \frac{\rho}{1-\rho}\log\left(\frac{\theta^2}{\alpha+1}\right) + \frac{1}{1-\rho}\log\left[\int_0^\infty \sum_{i=1}^{\rho}\sum_{k=1}^{i}\binom{\rho}{i}\binom{i}{k}\left(\frac{1}{\theta+1}\right)^i\right. \end{aligned}$$

$$\left.\times \left(\frac{\alpha\theta}{2}\right)^{\rho-i} x^{2(\rho-i)+k} e^{-\theta\rho x} dx\right]$$

$$\begin{aligned} &= \frac{\rho}{1-\rho}\log\left[\log\left(\frac{\theta^2}{\alpha+1}\right) + \sum_{i=1}^{\rho}\sum_{k=1}^{i}\binom{\rho}{i}\binom{i}{k}\left(\frac{1}{\theta+1}\right)^i\right. \\ &\left.\times \left(\frac{\alpha\theta}{2}\right)^{\rho-i}\left(\frac{1}{\theta\rho}\right)^{2(\rho-i)+k+1}\Gamma(2(\rho-i)+k+1)\right] \end{aligned}$$

The proof of Tsallis entropy is like Re'nyi entropy's proof.

Table 1: Numerical results for Shannon, Re´nyi and Tsallis entropies for Sameera distribution using different values of α and θ with δ=5.

α	θ	Shannon	Re'nyi	Tsallis	α	θ	Shannon	Re'nyi	Tsallis
2.0	2.0	1.2530318	-0.008324	0.241676	3.5	2.0	0.565239	-0.080412	0.169588
2.0	2.5	1.2547316	-0.007776	0.242224	3.5	2.5	0.584865	-0.072348	0.177652
2.0	3.0	1.2557851	-0.007589	0.242411	3.5	3.0	0.597087	-0.069428	0.180572
2.0	3.5	1.2564640	-0.007549	0.242451	3.5	3.5	0.605009	-0.068582	0.181418
2.0	4.0	1.2569129	-0.007573	0.242427	3.5	4.0	0.610280	-0.068634	0.181366
2.0	4.5	1.2572139	-0.007625	0.242375	3.5	4.5	0.613845	-0.069088	0.180912
2.5	2.0	0.905024	-0.020488	0.229512	4.0	2.0	0.430432	-0.138639	0.111361
2.5	2.5	0.923114	-0.018870	0.231130	4.0	2.5	0.450615	-0.123562	0.126438
2.5	3.0	0.934349	-0.018304	0.231696	4.0	3.0	0.463199	-0.118041	0.131959
2.5	3.5	0.941605	-0.018164	0.231836	4.0	3.5	0.471364	-0.116364	0.133636
2.5	4.0	0.946415	-0.018209	0.231791	4.0	4.0	0.476806	-0.116355	0.133645
2.5	4.5	0.949652	-0.018336	0.231664	4.0	4.5	0.480492	-0.117095	0.132905
3.0	2.0	0.720900	-0.042932	0.207068	4.5	2.0	0.311561	-0.224284	0.025716
3.0	2.5	0.739843	-0.039050	0.210950	4.5	2.5	0.332209	-0.198257	0.051743
3.0	3.0	0.751626	-0.037665	0.212335	4.5	3.0	0.345093	-0.188641	0.061359
3.0	3.5	0.759251	-0.037289	0.212711	4.5	3.5	0.353462	-0.185616	0.064384
3.0	4.0	0.764316	-0.037350	0.212650	4.5	4.0	0.359046	-0.185456	0.064544
3.0	4.5	0.767734	-0.037605	0.212395	4.5	4.5	0.362835	-0.186584	0.063416

Table 1 shows the numerical results of the entropies used in this paper. For all results, we have used the values of α and θ 2, 2.5, 3, 3.5, 4 and 4.5. For Re´nyi and Tsallis entropies, only $\rho = 5$ is used. For all results, we have used the R programming software [28]. The table shows that all Re´nyi entropy values are negative. Shannon entropy has the largest values. The values of Shannon entropy gets larger for larger values of θ.

8 Mean Absolute Deviations about Mean and Median

The mean absolute deviations about mean or median are giving a better measure of dispersion from the average [18]. Hence the mean absolute deviation about mean and median for the LoaiD are defined respectively, to be

$$
\begin{aligned}
MD_{median} = E|X-M| &= \int_0^\infty |x-M| g(x)dx \\
&= \int_0^M (M-x)g(x)dx + \int_M^\infty (x-M)g(x)dx \\
&= 2MG(M) + \mu - M - 2\int_0^M xg(x)dx \\
&= \mu - 2\int_0^M xg(x)dx \\
&= \frac{1}{\alpha+1}\left[\frac{\alpha\theta}{2}\left[\frac{6-[6+\theta M[6+\theta M(3+\theta M)]]e^{-\theta M}}{\theta^4}\right]\right. \\
&\quad \left. + \frac{2-[2+\theta M(2+\theta M)]e^{-\theta M}}{(\theta+1)\theta^3} + \frac{1-(1+\theta M)e^{-\theta M}}{(\theta+1)\theta^2}\right],
\end{aligned}
$$

$$
\begin{aligned}
MD_{mean} = E|X-\mu| &= \int_0^\infty |x-\mu| g(x)dx \\
&= \int_0^\mu (\mu-x)g(x)dx + \int_\mu^\infty (x-\mu)g(x)dx
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int_0^\mu (\mu-x)g(x)dx \\
&= 2\mu G(\mu) - 2\int_0^\mu xg(x)dx \\
&= 2\left[\frac{3\alpha(\theta+1)+\theta+2}{\theta(\theta+1)(\alpha+1)}\right]\left[\frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\alpha\theta\mu^2 + \frac{1}{\theta+1}(1+\mu)\right]\right]e^{-\theta\mu} \\
&- \frac{2}{\alpha+1}\left[\frac{\alpha\left(-\mu^3\theta^3 e^{-\mu\theta} - 3\left(\mu^2\theta^2 e^{-\mu\theta} - 2\left(-\mu\theta e^{-\mu\theta} - e^{-\mu\theta}\right)\right) + 6\right)}{2\theta}\right. \\
&+ \left.\frac{\theta\left(-\mu\theta e^{-\mu\theta} - e^{-\mu\theta} + 1\right) - \mu^2\theta^2 e^{-\mu\theta} + 2\left(-\mu\theta e^{-\mu\theta} - e^{-\mu\theta}\right) + 2}{\theta(1+\theta)}\right],
\end{aligned}
$$

where $= \frac{3\alpha(\theta+1)+\theta+2}{\theta(\theta+1)(\alpha+1)}$.

9 Parameters Estimation

9.1 Maximum likelihood

Let $X_1, X_2, \ldots, X_n$ be a random sample from LoaiD, then the likelihood function $L(x; \alpha, \theta)$ is defined by

$$L(x,\alpha,\theta)=\prod_{j=1}^{n} g(x_j,\alpha,\theta)$$
$$=\left[\frac{\theta^2}{\alpha+1}\right]^n \prod_{j=1}^{n}\left[\frac{1}{2}\alpha\theta x_j^2+\frac{1}{\theta+1}(1+x_j)\right]e^{-\theta x_j},$$

then, the log-likelihood function is

$$\ell = ln\ L = 2nln(\theta) - nln(\alpha+1) + \sum_{j=1}^{n} ln\left(\frac{1}{2}\alpha\theta x_j^2+\frac{1}{\theta+1}(1+x_j)\right) - \theta\sum_{j=1}^{n} x_j \tag{9.1}$$

Deriving (9.1) partially with respect to α and θ we have

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha} &= \frac{-2}{\alpha+1}+\sum_{j=1}^{n}\frac{\theta(\theta+1)x_j^2}{\alpha\theta(\theta+1)x_j^2+2(1+x_j)}\\ \frac{\partial \ell}{\partial \theta} &= \frac{2n}{\theta}+\sum_{j=1}^{n}\frac{\alpha(\theta+1)^2x_j-2(1+x_j)}{\alpha\theta(\theta+1)^2x_j^2+2(\theta+1)(1+x_j)}-\sum\theta_{j=1}^{n}x_j\end{aligned} \tag{9.2}$$

The maximum likelihood estimates (MLEs) of the parameters of LoaiD are the solutions of the system of equations given in (9.2). But there is no exact solution for this system of equations. Thus, we can use numerical methods to find the MLEs of the distribution parameters.

9.2 Least square and weighted least square estimations

This subsection shows other methods for the estimation of the model parameters, which are the ordinary least squares (OLSE) and the weighted least squares (WLSE) estimators. They are suggested by [41].

Let X_1, X_2, ..., X_n be random sample with cdf G and $X_{(1)}$, $X_{(2)}$, ..., $X_{(n)}$ be the order statistics of the random sample. The OLSE of α and θ can be obtained by minimizing

$$\sum_{i=1}^{n}\left[G(x_{(i)};\alpha,\theta)-\frac{i}{n+1}\right]^2 \tag{9.3}$$

with respect to α and θ. In our case, the $\hat{\alpha}_{OLSE}$ and $\hat{\theta}_{OLSE}$ can be obtained by minimizing

$$\sum_{i=1}^{n}\left[\left[1-\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}x_{(i)}^2+\left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta x_{(i)}\right]e^{-\theta x_{(i)}}\right]-\frac{i}{n+1}\right]^2,$$

with respect to the two parameters.

The weighted least squares estimator (WLSE) of α and θ can be found by minimizing

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[G(x_{(i)};\alpha,\theta)-\frac{i}{n+1}\right]^2, \tag{9.4}$$

so, in the case of LoaiD, the $\hat{\alpha}_{WLSE}$ and $\hat{\theta}_{WLSE}$ can be obtained by minimizing

$$\sum_{i=1}^{n}\left[\left[1-\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}x_{(i)}^2+\left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta x_{(i)}\right]e^{-\theta x_{(i)}}\right]-\frac{i}{n+1}\right]^2,$$

respectively with respect to α and θ. Some results of those estimators are given by a simulation study in the next section.

9.3 Method of maximum product of spacings

[14] and [15] proposed the method of maximum product spacing (MPS) to be an alternative method to the maximum likelihood (ML) method of estimation. This method relies on maximizing the geometric mean of the spacings of the data with respect to the parameters. The MPS method provides consistent and asymptotically efficient estimators whether MLE exists or not.

For a random sample $X_1, X_2, ..., X_n$ of size n and $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be the order statistics of the random sample. The uniform spacings is defined as:

$$\Psi_i(\alpha,\theta) \quad = \quad G(x_{(i)}|\alpha,\theta) - G(x_{(i-1)}|\alpha,\theta),\ i = 1,...,n,$$

where $G(x_{(0)}|\alpha,\theta) = 0$ and $G(x_{(n+1)}|\alpha,\theta) = 1$. It is obvious that $\sum_{i=1}^{n+1}\Psi_i(\alpha,\theta)=1$.

The MPS estimators of the distribution parameters α and θ denoted by $\hat{\alpha}_{MPS}$ and $\hat{\theta}_{MPS}$ can be obtained by maximizing the geometric mean of the spacings, that is,

$$G(\alpha,\theta|x) \quad = \quad \left(\prod_{i=1}^{n+1}\Psi_i(\alpha,\theta)\right)^{\frac{1}{n+1}} \tag{9.5}$$

Now, the natural logarithm of (9.5) gives

$$NL(\alpha,\theta|x) \quad = \quad \frac{1}{n+1}\sum_{i=1}^{n+1} ln\left(\Psi_i(\alpha,\theta)\right) \tag{9.6}$$

The MPS estimators $\hat{\alpha}_{MPS}$ and $\hat{\theta}_{MPS}$ can be attained by solving the following nonlinear system of equations for α and θ.

$$\begin{aligned}\frac{\partial NL(\alpha,\theta|x)}{\partial\alpha} &= \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{\Delta_1(x_{(i)}|\alpha,\theta)-\Delta_1(x_{(i-1)}|\alpha,\theta)}{\Psi_i(\alpha,\theta)}=0\\ \frac{\partial NL(\alpha,\theta|x)}{\partial\theta} &= \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{\Delta_2(x_{(i)}|\alpha,\theta)-\Delta_2(x_{(i-1)}|\alpha,\theta)}{\Psi_i(\alpha,\theta)}=0\end{aligned} \tag{9.7}$$

(9.8)

where

$$\begin{aligned}\Delta_1(x_{(i)}|\alpha,\theta) &= \frac{\partial G(x_{(i)}|\alpha,\theta)}{\partial\alpha} \\ &= \left[\frac{2\,(\alpha+1)\,(\theta+1)+\alpha\theta^2\,(\theta+1)\,x_{(i)}^2+(\alpha\theta+\alpha+1)\,\theta x_{(i)}}{2\,(\alpha+1)\,(\theta+1)}\right]e^{-\theta x_{(i)}}\end{aligned} \quad (9.9)$$

$$\begin{aligned}\Delta_2(x_{(i)}|\alpha,\theta) &= \frac{\partial G(x_{(i)}|\alpha,\theta)}{\partial\theta} \\ &= \left[\frac{\alpha\theta^2(\theta+1)x_{(i)}^2+2\theta(\alpha\theta+\alpha+1)x_{(i)}+2(\alpha+1)(\theta+1)}{2(\alpha+1)(\theta+1)}\right]e^{-\theta x_{(i)}}\end{aligned} \quad (9.10)$$

9.4 Methods of minimum distances

[46] first proposed the minimum distance method to obtain strong consistent estimators. Consider the random sample of size n, say $X_1,\cdots, X_n$ with cdf $G(x|\alpha,\theta)$ and let $G_n(x)$ be the empirical distribution function based on the sample $\mathbf{x} = (x_1,\cdots, x_n)$. If $(\hat{\alpha},\hat{\theta})$ is the vector of estimators of (α, θ), then $G(x|\hat{\alpha},\hat{\theta}))$ is an estimator of $G(x|\alpha,\theta)$. Assuming $(\hat{\alpha},\hat{\theta})$ is exist, such that

$$d[G(x|\hat{\alpha},\hat{\theta})), G_n(x)] = inf\{d[G(x|\alpha, \theta), G_n(x)]\},$$

where $d[.,.]$ is the distance between $G(x|\hat{\alpha},\hat{\theta}))$ and $G_n(x)$, then $(\hat{\alpha},\hat{\theta})$ is called the minimum-distance estimate of (α,θ) (Drossos and Philippou, 1980).

9.5 Cramer-Von-Mises method

Cramer-Von-Mises method proposed by [16] and [43] usually called W^2, is a method used in one-sample applications to compare between the theoretical cdf $G^*(x)$ of a random variable and a given empirical distribution $G_n(x)$ using the goodness of fit. It is used as a part of the minimum distance method of estimation. It is defined as

$$\varrho^2 = \int_{-\infty}^{\infty}\left[G_n(x)-G^*(x)\right]^2 dG^*(x)$$

For a random sample of size n with observed values $x_1, \cdots, x_n$ sorted in an ascending order the Cramer-Von Mises test statistic value is [40],

$$W^2 = n\varrho^2 = \sum_{i=0}^{n}\left[G(x_{(i)},\alpha,\theta)-\frac{2i-1}{2n}\right]^2+\frac{1}{12n}$$

Thus

$$\begin{aligned}W^2 &= \frac{1}{12n}+\sum_{i=0}^{n}\left[G(x_{(i)},\alpha,\theta)-\frac{2i-1}{2n}\right]^2 \\ &= \frac{1}{12n}+\sum_{i=1}^{n}\left[1-\left[1+\frac{\alpha\theta^2}{2(1+\alpha)}x_{(i)}^2+\left[\frac{\alpha\theta+\alpha+1}{(1+\alpha)(1+\theta)}\right]\theta x_{(i)}\right]e^{-\theta x_{(i)}}-\frac{2i-1}{2n}\right]^2\end{aligned}$$

The Cramer-Von Mises estimators $\hat{\alpha}_{CV}$, and $\hat{\theta}_{CV}$ of α and θ can be calculated by minimizing W^2. Thus the estimators are the solutions of

$$\begin{aligned}\sum_{i=0}^{n}\left[2G(x_{(i)},\alpha,\theta)-\frac{2i-1}{n}\right]\Delta_1(x_{(i)}|\alpha,\theta) &= 0\\ \sum_{i=0}^{n}\left[2G(x_{(i)},\alpha,\theta)-\frac{2i-1}{n}\right]\Delta_2(x_{(i)}|\alpha,\theta) &= 0\end{aligned},$$

where Δ_1 and Δ_2 as defined in (9.9) and (9.10).

9.6 Method of Anderson-Darling

[10] introduced a method of estimating the distribution parameters. This method is called Anderson-Darling method of estimation, it is defined as

$$A(\alpha,\theta) = -n-\frac{1}{n}\sum_{i=0}^{n}(2i-1)\left\{log[G(x_{(i)};\alpha,\theta)]+log\bar{G}(x_{(n+1-i)};\alpha,\theta)\right\} \tag{9.11}$$

The estimators $\hat{\alpha}_{AD}$ and $\hat{\theta}_{AD}$ can be determined by minimizing (9.11), or by solving the following nonlinear system of equations.

$$\begin{aligned}\frac{\partial A(\alpha,\theta|x)}{\partial\alpha} &= \sum_{i=0}^{n}(2i-1)\left\{\left[\frac{\Delta_1(x_{(i)}|\alpha,\theta)}{G(x_{(i)};\alpha,\theta)}\right]-\frac{\Delta_1(x_{(i)}|\alpha,\theta)}{\bar{G}(x_{(n+1-i)};\alpha,\theta)}\right\}=0\\ \frac{\partial A(\alpha,\theta|x)}{\partial\theta} &= \sum_{i=0}^{n}(2i-1)\left\{log[\frac{\Delta_2(x_{(i)}|\alpha,\theta)}{G(x_{(i)})};\alpha,\theta)]-\frac{\Delta_2(x_{(i)}|\alpha,\theta)}{\bar{G}(x_{(n+1-i)};\alpha,\theta)}\right\}=0\end{aligned},$$

where $\bar{G}= 1 - G$ and Δ_1 and Δ_2 are defined in (9.9) and (9.10).

10 Stress-Strength Reliability

Let X and Y are two independent random variables from LoaiD, where X represents the strength of the system and Y is the stress applied to this system [5]. The item failed to work at the moment that the stress applied to it exceeds the strength and the item will operate reasonably whenever $X > Y$. The stress strength model is defined as $p(Y < X)$ [21].

$$\begin{aligned}p(Y<X) &= \int_0^\infty\int_0^x\left[\frac{\theta^2}{\alpha+1}\right]^2\left[\frac{1}{2}\alpha\theta x^2+\frac{1}{\theta+1}(1+x)\right]\left[\frac{1}{2}\alpha\theta y^2+\frac{1}{\theta+1}(1+y)\right]e^{-\theta x}e^{-\theta y}dydx\\ &= \left[\frac{\theta^2}{\alpha+1}\right]^2\left[\int_0^\infty(x(\alpha\theta(\theta+1)x+2)+2)(\alpha(\theta+1))(\theta x(\theta x+2))e^{-2\theta x}\right.\\ &\quad -2(x(\alpha\theta(\theta+1)x+2)+2)(\alpha(\theta+1))e^{-tx}+2(x(\alpha\theta(\theta+1)x+2)+2)e^{-2\theta x}\\ &\quad +2\theta x(x(\alpha\theta(\theta+1)x+2)+2)e^{-2\theta x}-2(x(\alpha\theta(\theta+1)x+2)+2)(\theta+1)e^{-\theta x}\\ &\quad \left.+(x(\alpha\theta(\theta+1)x+2)+2)(\theta+1)e^{-2\theta x}\right]\\ &= \frac{\theta^2\begin{bmatrix}3\alpha^2(\theta+1)^2+3\alpha(\alpha\theta^2+\theta+1+\alpha)+2\alpha(\theta+1)(\theta+2)\\ +4\alpha\theta(\theta+1)-(8\alpha+2)(\alpha+1)(\theta+1)^2+2(\alpha(\theta+1)+4\theta+2)\\ +2((\theta+1)(\alpha\theta+4\theta+\alpha+2))\end{bmatrix}+(\alpha+1)(3\alpha\theta(\theta+1)+4\theta^2(\theta+1))}{4(\alpha+1)^2}\end{aligned}$$

11 Simulation study

A simulation study is employed in this section to test the performance and precision of the methods of estimating the parameters of Loai distribution. The R software [28]is used to perform this study. N = 1500 samples are generated for this purpose, each of size 100, 150, 200, 300, 400 and 500 for the values of (α, θ) = (0.5,2) and (2, 3). For each sample, the estimates of the parameter space $\phi = (\alpha, \theta)$, average mean square error (AMSE) and the average bias (AB) are obtained as follows:

$$AB(\hat{\phi}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\phi}-\phi), \quad MSEs = \frac{1}{N}\sum_{i=1}^{N}(\hat{\phi}-\phi)^2$$

The results of this simulation are summarized in Tables 2 and 3.

Table 2: Parameter Estimates and their ABs and AMSEs, when $\alpha = 2$ and $\theta = 3$.

n	Method	$\hat{\alpha}$	$\hat{\theta}$	$AB(\hat{\alpha})$	$AMSE(\hat{\alpha})$	$AB(\hat{\theta})$	$AMSE(\hat{\theta})$
100	MLE	2.5001	3.0221	0.5001	5.4387	0.0221	0.0965
100	OLS	2.1200	3.0461	0.1200	0.0657	0.0461	0.0372
100	WLS	2.3916	2.9944	0.3916	3.2273	-0.0056	0.0893
100	MPS	2.9047	3.0992	0.9047	8.6444	0.0992	0.0935
100	CVM	2.6108	3.0028	0.6108	8.5894	0.0028	0.1111
100	AD	2.2906	2.9794	0.2906	2.2299	-0.0206	0.0973
150	MLE	2.2588	3.0139	0.2588	1.1200	0.0139	0.0587
150	OLS	2.1237	3.0413	0.1237	0.0534	0.0413	0.0238
150	WLS	2.2018	2.9935	0.2018	1.0178	-0.0065	0.0594
150	MPS	2.4756	3.0638	0.4756	1.7441	0.0638	0.0612
150	CVM	2.3035	3.0045	0.3035	1.4171	0.0045	0.0666
150	AD	2.2121	2.9925	0.2121	1.0493	-0.0075	0.0622
200	MLE	2.1846	3.0097	0.1846	0.7458	0.0097	0.0410
200	OLS	2.1266	3.0412	0.1266	0.0464	0.0412	0.0189
200	WLS	2.1474	2.9946	0.1474	0.7269	-0.0054	0.0449
200	MPS	2.3646	3.0575	0.3646	0.9700	0.0575	0.0431
200	CVM	2.1786	3.0077	0.1786	0.7838	0.0077	0.0487
200	AD	2.1473	2.9921	0.1473	0.6856	-0.0079	0.0470
300	MLE	2.1160	3.0075	0.1160	0.3668	0.0075	0.0256
300	OLS	2.1312	3.0376	0.1312	0.0418	0.0376	0.0131
300	WLS	2.0926	2.9970	0.0926	0.3653	-0.0030	0.0275
300	MPS	2.2155	3.0426	0.2155	0.4910	0.0426	0.0287
300	CVM	2.1424	3.0107	0.1424	0.4826	0.0107	0.0319
300	AD	2.0847	2.9971	0.0847	0.3854	-0.0029	0.0275
400	MLE	2.0582	2.9998	0.0582	0.2684	-0.0002	0.0190
400	OLS	2.1283	3.0418	0.1283	0.0363	0.0418	0.0105
400	WLS	2.0462	2.9939	0.0462	0.2745	-0.0061	0.0200
400	MPS	2.1434	3.0294	0.1434	0.3177	0.0294	0.0206
400	CVM	2.0932	3.0001	0.0932	0.3437	0.0001	0.0254

400	AD	2.0812	3.0002	0.0812	0.2838	0.0002	0.0209
500	MLE	2.0601	3.0031	0.0601	0.2265	0.0031	0.0152
500	OLS	2.1266	3.0428	0.1266	0.0344	0.0428	0.0090
500	WLS	2.0467	2.9993	0.0467	0.2271	-0.0007	0.0164
500	MPS	2.1174	3.0226	0.1174	0.2420	0.0226	0.0158
500	CVM	2.0576	2.9993	0.0576	0.2371	-0.0007	0.0196
500	AD	2.0493	2.9944	0.0493	0.2204	-0.0056	0.0171

Table 3: Parameter Estimates and their ABs and AMSEs, when $\alpha = 0.5$ and $\theta = 2$.

n	Method	$\hat{\alpha}$	$\hat{\theta}$	$AB(\hat{\alpha})$	$AMSE(\hat{\alpha})$	$AB(\hat{\theta})$	$AMSE(\hat{\theta})$
100	MLE	0.6108	1.9853	0.1108	0.3173	-0.0147	0.1698
100	OLS	0.5593	2.0430	0.0593	0.0151	0.0430	0.0133
100	WLS	0.6018	1.9859	0.1018	0.2861	-0.0141	0.1247
100	MPS	0.7492	2.1005	0.2492	0.4571	0.1005	0.1903
100	CVM	0.6272	2.0256	0.1272	0.3279	0.0256	0.1371
100	AD	0.6066	2.0044	0.1066	0.2737	0.0044	0.1187
150	MLE	0.5536	1.9747	0.0536	0.1970	-0.0253	0.1261
150	OLS	0.5580	2.0481	0.0580	0.0116	0.0481	0.0105
150	WLS	0.5507	1.9775	0.0507	0.1784	-0.0225	0.0899
150	MPS	0.6633	2.0761	0.1633	0.2357	0.0761	0.1257
150	CVM	0.6062	2.0162	0.1062	0.2488	0.0162	0.1073
150	AD	0.5624	1.9895	0.0624	0.1812	-0.0105	0.0910
200	MLE	0.5387	1.9758	0.0387	0.1480	-0.0242	0.1011
200	OLS	0.5556	2.0450	0.0556	0.0097	0.0450	0.0080
200	WLS	0.5413	1.9804	0.0413	0.1426	-0.0196	0.0770
200	MPS	0.6247	2.0457	0.1247	0.1684	0.0457	0.0935
200	CVM	0.5745	2.0041	0.0745	0.1608	0.0041	0.0877
200	AD	0.5305	1.9732	0.0305	0.1270	-0.0268	0.0719
300	MLE	0.5211	1.9759	0.0211	0.0982	-0.0241	0.0670
300	OLS	0.5540	2.0448	0.0540	0.0082	0.0448	0.0064
300	WLS	0.5281	1.9838	0.0281	0.0963	-0.0162	0.0548
300	MPS	0.5852	2.0363	0.0852	0.1031	0.0363	0.0611
300	CVM	0.5369	1.9930	0.0369	0.1018	-0.0070	0.0626
300	AD	0.5077	1.9733	0.0077	0.0902	-0.0267	0.0552
400	MLE	0.5170	1.9816	0.0170	0.0666	-0.0184	0.0456
400	OLS	0.5528	2.0414	0.0528	0.0071	0.0414	0.0053
400	WLS	0.5194	1.9842	0.0194	0.0679	-0.0158	0.0427
400	MPS	0.5580	2.0302	0.0580	0.0705	0.0302	0.0454

400	CVM	0.5133	1.9792	0.0133	0.0758	-0.0208	0.0521
400	AD	0.5137	1.9839	0.0137	0.0710	-0.0161	0.0441
500	MLE	0.5099	1.9858	0.0099	0.0557	-0.0142	0.0392
500	OLS	0.5496	2.0441	0.0496	0.0063	0.0441	0.0051
500	WLS	0.5070	1.9821	0.0070	0.0596	-0.0179	0.0387
500	MPS	0.5476	2.0171	0.0476	0.0554	0.0171	0.0350
500	CVM	0.5126	1.9826	0.0126	0.0695	-0.0174	0.0481
500	AD	0.5150	1.9896	0.0150	0.0591	-0.0104	0.0380

Tables 2 and 3 show the values of the AB and the AMSE. The behaviour of the average biases and average mean squares errors is in the standard manner as they decrease with increasing the sample size. This proves that the parameters estimates are asymptotically unbiased and consistent. We recommend using smaller values of α based on the average bias. Whereas our recommendation based on the average mean squares errors is to use larger values of θ. Tables 4 and 5 show the preferences of the methods of estimation. They show that based on the AMSE the OLS method of estimation is preferred regardless the values of α and θ. The MLE and MPS methods are not preferred for AB and AMSE as their values are higher than their correspondence values for all sample sizes.

Table 4: Best methods based on simulation results based on α =2 and θ = 3

n	$AB(\hat{\alpha})$	$AMSE(\hat{\alpha})$	$AB(\hat{\theta})$	$AMSE(\hat{\theta})$
100	OLS	OLS	AD	OLS
150	OLS	OLS	AD	OLS
200	OLS	OLS	WLS	OLS
300	AD	OLS	AD	OLS
400	WLS	OLS	CVM	OLS
500	WLS	OLS	WLS or CVM	OLS

Table 5: Best methods based on simulation results based on α = 0.5 and θ = 2

n	$AB(\hat{\alpha})$	$AMSE(\hat{\alpha})$	$AB(\hat{\theta})$	$AMSE(\hat{\theta})$
100	OLS	OLS	AD	OLS
150	WLS	OLS	AD	OLS
200	AD	OLS	CVM	OLS
300	AD	OLS	CVM	OLS
400	CVM	OLS	AD	OLS
500	WLS	OLS	AD	OLS

12 Application of Loai Distribution to COVID-19 Data

In this section, we present an application of the LoaiD to COVID-19 data set. The data represents the number of new deaths in Jordan due to COVID-19 from 13/1/2021 to 10/3/2021. This data is available at the World Health Organization (WHO) website, (Organization, 2021) and given in Table 6.

Table 6: The number of new deaths in Jordan due to COVID-19 from 13/1/2021 to 10/3/2021

15	16	14	16	8	8	17	17	11	9
10	7	15	9	14	7	12	23	12	10
8	10	10	8	7	10	6	10	16	10
12	11	11	18	18	12	9	16	15	11
16	19	22	16	23	25	26	26	29	37
40	29	38	35	52	59	60			

The COVID-19 data set is fitted using the proposed distribution and the following considered distributions:

Weibull [45]: $f(x) = \frac{\alpha}{\theta^{\alpha}} x^{\alpha-1} e^{-(x/\theta)^{\alpha}}; x > 0, \alpha > 0, \theta > 0$

Alzoubi [12]: $f(x) = \frac{1}{1+\alpha\theta}(\theta + \frac{\theta^{\alpha} x^{\alpha-2}\alpha}{\Gamma(a-1)})e^{-\theta x}; x > 0, \theta > 0, \alpha > 1$

Lindley [26]: $f(x) = \frac{\alpha^2(1+x)e^{-\alpha x}}{1+\alpha}; x > 0,\ \alpha > 0$

Akshaya [35]: $f(x) = \frac{\alpha^4}{\alpha^3+3\alpha^2+6\alpha+6}(1+x)^3 e^{-x\alpha}; x > 0, \alpha > 0$

Shanker [33]: $f(x) = \frac{\alpha^2}{\alpha^2+1}(\alpha + x)e^{-\alpha x}; x > 0, \alpha > 0$

Gharaibeh [19]: $f(x) = \frac{\alpha^6(x^5+20x^3+120x+120\alpha)e^{-\alpha x}}{120(\alpha^6+\alpha^4+\alpha^2+1)}; x > 0; \alpha > 0$

Rama [36]: $f(x) = \frac{\alpha^4}{\alpha^3+6}(1+x^3)e^{-x\alpha}; x > 0, \alpha > 0$

Exponential [25]: $f(x) = \alpha e^{-\alpha x}; x > 0,\ \alpha > 0$

To compare the goodness of fit of the fitted distributions, the following criteria: -2logL, Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), Kolmogorov-Smirnov Statistic (KS) and its p-value are computed, and the results are given in Table 7.

From Table 7, the proposed Loai distribution has the smallest values of -2logL, AIC, CAIC, BIC, HQIC and KS with highest p-value. Consequently, Loai distribution has the superiority of fitting the COVID-19 data set over the other considered distributions.

Moreover, the last two columns in Table 7 presents the MLEs of the parameters of the fitted distributions with their corresponding standard error.

Table 7: Goodness of fit criteria and MLEs of the parameters of the fitted distributions

Distribution	$-2log L$	*AIC*	*CAIC*	*BIC*	*HQIC*	*KS*	*p-value*	*MLE*	*Standard Error*
Loai	414.871	418.871	419.093	422.957	420.459	0.154	0.136	$\hat{\alpha}$=0.166 $\hat{\theta}$=899542	0.013 11863.3
Weibull	423.115	427.115	427.337	431.201	428.703	0.163	0.098	$\hat{\alpha}$=1.638 $\hat{\theta}$=20.415	0.153 1.757
Alzoubi	430.619	434.619	434.842	438.705	436.207	0.296	0.000	$\hat{\alpha}$=2.462 $\hat{\theta}$=0.058	0.139 0.008
Lindley	423.183	425.183	425.256	427.226	425.977	0.185	0.040	0.105	0.010
Akshaya	417.358	419.358	419.431	421.401	420.152	0.172	0.069	0.210	0.014
Shanker	420.157	422.157	422.230	424.200	422.951	0.167	0.084	0.110	0.010
Gharaibeh	430.752	432.752	432.825	434.795	433.546	0.228	0.005	0.319	0.017
Rama	417.624	419.624	419.697	421.667	420.418	0.179	0.051	0.221	0.015
Exponential	443.946	445.946	446.019	447.989	446.740	0.304	0.000	0.055	0.007

13 Conclusion

This paper proposes a new two parameters distribution as a mixture of gamma and Lindley distributions. This distribution is called Loai distribution. Different estimation methods are used to estimate the distribution parameters. Different entropy measures are calculated with numerical results for some values of the parameters show that Re'nyi entropy has the lowest value. Stresss-trength reliability is calculated. Some other properties are derived. The simulation study reveals that the OLS method of estimation is the best way to estimate the distribution parameters based on AMSE. At the end an application to a Covid-19 real data set is performed, it shows that LoaiD fits better than the competitive distributions.

References

[1] Al-Omari, A. Al-khazaleh, and L. Alzoubi. "A generalization of the new Weibull-Pareto distribution", Investigación Operacional, vol. 41, no. 1, pp. 138-146, 2020. *Gale OneFile: Informe Académico*, link.gale.com/apps/doc/A612476088/IFME?u=anon~46e938a8&sid=googleScholar&xid=1de7147f.

[2] A. Al-Omari I., A. Al-khazaleh, and L. Alzoubi. "Transmuted Janardan distribution: A Generalization of the Janardan distribution". Journal of Statistics applications & Probability, vol. 6, no. 2, pp. 259-269. DOI: 10.18576/jsap/060202.

[3] L. Al-Zou'bi, "Transmuted Mukherjee-Islam distribution: A generalization of Mukherjee-Islam distribution". Journal of Mathematics Research, vol. 9, no. 4, pp. 135-144, 2017. DOI: 10.5539/jmr. v9n4p135.

[4] A. Alkhazaalh, and L. Alzoubi, "Epanechnikov-exponential distribution: properties and applications". General Mathematics, vol. 29, no. 1, pp. 13-29, 2021. doi:10.2478/gm-2021-0002.

[5] A. Almarashi, A., Algarni, and M. Nassar, "On estimation procedures of stress-strength reliability for Weibull distribution with application". *PloS one*, *15*(8), e0237997. https://doi.org/10.1371/journal.pone.0237997

[6] N. Almawajdeh, "Generalization of Rani and generalized Akash distributions". Master's thesis, Al al-Bayt University, 2019.

[7] B. Almousa, "Generalization of Sujatha and Amarendra distributions using quadratic transmutation map". Master's thesis, Al al-Bayt University, 2019.

[8] H. Alsikeek, "Quadratic Transmutation Map for Reciprocal and Two-Parameter Weighted Exponential distributions". Master's thesis, Al al-Bayt University, 2017.

[9] L. Alzoubi, M. Gharaibeh, and R. Alzghool, "Transmuted Shanker distribution: properties and applications". Italian Journal of Pure and Applied Mathematics, in press.

[10] T. Anderson, and D. Darling, "Asymptotic theory of certain "goodness of fit" criteria based on stochastic processes". The Annals of Mathematical Statistics, vol. 23, no. 2, pp. 193-212, 1952. DOI: 10.1214/aoms/1177729437.

[11] R. AzZwideen, and L. Alzoubi (2020). "The transmuted gamma-Gompertz distribution". International Journal of Research -GRANTHAALAYAH, vol. 8, no. 10, pp. 236-248, 2020. https://doi.org/10.29121/granthaalayah.v8.i10.2020.1892.

[12] M. Benrabia, and L. AlZoubi, "Alzoubi distribution: properties and applications". Journal of Statistics applications & Probability, An International Journal, vol. 11, No. 2, pp. 625-640, 2022. DOI:10.1285/i20705948v15n2p300.

[13] M. Benrabia, and L. AlZoubi, "Benrabia distribution: properties and applications". Electronic Journal of Applied Statistical Analysis, vol. 15, no. 2, pp. 300-317. DOI: 10.1285/i20705948v15n2p300

[14] R. Cheng, and A. Amin. "Maximum product-of-spacings estimation with applications to the log-normal distribution". University of Wales IST, Math Report 79–1, 1979

[15] R. Cheng, and A. Amin. "Estimating Parameters in Continuous Univariate distributions with a Shifted Origin". Journal of the Royal Statistical Society. Series B

(Methodological), vol. 45, no. 3, pp. 394-403, 1983. https://doi.org/10.1111/j.2517-6161.1983.tb01268.x.

[16] H. Crame´r, “On the Composition of Elementary Errors”. Scandinavian Actuarial Journal, vol. 1, pp. 13-74, 1928. https://doi.org/10.1080/03461238.1928.10416862.

[17] C. Drossos, and A. Philippou, “A note on minimum distance estimates”. Annals of the Institute of Statistical Mathematics, vol. 32, no. 1, pp. 121-123, 1980. DOI: 10.1007/BF02480318.

[18] R. Geary, “The Ratio of the Mean Deviation to the Standard Deviation as a Test of Normality.” *Biometrika*, vol. 27, no. 3/4, pp. 310–32, 1935. *JSTOR*, https://doi.org/10.2307/2332693.

[19] M. Gharaibeh, “Gharaibeh distribution and its applications”. Journal of Statistics applications & Probability, vol. 10, No. 2, 441-452, 2021. http://dx.doi.org/10.18576/jsap/100214.

[20] M. Gharaibeh, “Transmuted Aradhana distribution: properties and applications”. Jordan Journal of Mathematics and Statistics, vol. 13, no. 2, pp. 287-304, 2020. Doi: https://doi.org/10.47013/13.2.8

[21] M. Hassan, “Comparison of different estimators of $P(Y < X)$ for two parameter Lindley distribution”. International Journal of Reliability and applications, Vol. 18, No. 2, pp. 83-98, 2017.

[22] D. Joanes, and C. Gill, “Comparing measures of sample skewness and kurtosis.” *Journal of the Royal Statistical Society. Series D (The Statistician)*, vol. 47, no. 1, pp. 183–189, 1998. *JSTOR*, http://www.jstor.org/stable/2988433.

[23] N. Kakwani, and N. Podder, “On the estimation of Lorenz curves from grouped observations.” *International Economic Review*, vol. 14, no. 2, 1973, pp. 278–92. *JSTOR*, https://doi.org/10.2307/2525921.

[24] A. Khazaleh, and L. Alzoubi, “New compound probability distribution using biweight kernel function and exponential distribution”. J. Math. Comput. Sci., vol. 11, no. 5, pp. 5878-5896, 2021. https://doi.org/10.28919/jmcs/6082.

[25] Kingman, J. (1982). “The coalescent”. Stochastic Processes and their applications, vol. 13, no. 3, pp. 235 248, 1982. https://doi.org/10.1016/0304-4149(82)90011-4.

[26] Merovci, F. and Elbatal, I. (2014). “Transmuted Lindley-geometric distribution and its applications”. Journal of Statistics applications & Probability, vol. 3, no. 1, pp. 77-91, 2014. https://doi.org/10.48550/arXiv.1309.3774.

[27] World Health Organization, “Who coronavirus (covid-19)”, 2021. Published Online.

[28] R Core Team, R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria, 2021.

[29] F. Rabaiah, “Generalizations of power function and type-I half logistic distributions using quadratic transmutation map”. Master’s thesis, Master’s thesis, Al al-Bayt University, 2018.

[30] A. Re’nyi, “On measures of entropy and information”. In: Proceedings of the 4th Berkeley Symposium on Mathematics, Statistics and Probability, Vol. 1, University of California Press, Berkeley, 547-561, 1960.

[31] F. Saadeh, "Generalization to some distributions using transmutation maps". Master's thesis, Master's thesis, Al al-Bayt University, 2019.
[32] M.Shaked, and J. Shanthikumar, "Stochastic orders and their applications, distribution and associated inference". Academic Press New York, 1994.
[33] R. Shanker, "Shanker distribution and its applications". International Journal of Statistics and applications, vol. 5, no. 6, pp. 338-348, 2015. doi:10.5923/j.statistics.20150506.08.
[34] R. Shanker, "Aradhana distribution and its applications". International Journal of Statistics and applications, vol. 6, no. 1, pp. 23-34, 2016. doi:10.5923/j.statistics.20160601.04.
[35] R. Shanker, "Akshaya distribution and its applications". American Journal of Mathematics and Statistics, vol. 7, no. 2, pp. 51-59, 2017. doi:10.5923/j.ajms.20170702.01.
[36] R. Shanker, "Rama distribution and its applications". International Journal of Statistics and applications, vol. 7, no. 1, pp. 26-35, 2017. doi:10.5923/j.statistics.20170701.04.
[37] R. Shanker, and K. Shukla, "Ishita distribution and its applications". Biometrics & Biostatistics International Journal, vol. 5, no. 2, pp. 1-9, 2017. DOI: 10.15406/bbij.2017.05.00126.
[38] C. Shannon, "A mathematical theory of communication". The Bell System Technical Journal, Vol. 27, pp. 379–423, 623–656, 1948.
[39] D. Shraa, and A. Al-Omari. "Darna distribution: properties and application". Electronic Journal of Applied Statistical Analysis, vol. 12, no. 2, pp. 520-541, 2019. DOI: 10.1285/i20705948v12n2p520.
[40] M. Stephens, "Tests based on EDF statistics. In goodness-of-fit techniques", chapter 4, pages 97-193. New York: Dekker, 1986.
[41] J. Swain, S. Venkatraman, and J. Wilson, Least Squares Estimation of distribution Function in Johnson's Translation System. Journal of Statistical Computation and Simulation, vol. 29, no. 4, pp. 271-297, 1988. https://doi.org/10.1080/00949658808811068.
[42] C. Tsallis, "Possible generalization of Boltzmann-Gibbs statistics". Journal of Statistical Physics, vol. 52, pp. 479-487, 1988. https://doi.org/10.1007/BF01016429.
[43] R. Von Mises," Wahrscheinlichkeit statistik und wahrheit". Julius Springer, 1928. https://doi.org/10.1007/978-3-7091-8287-1.
[44] Q. Wang, "Probability distribution and entropy as a measure of uncertainty". Journal of Physics: A Mathematical General, vol. 41, no. 6, pp. 1-12, 2008. http://dx.doi.org/10.1088/1751-8113/41/6/065004.
[45] W. Weibull, "A statistical distribution function of wide applicability". Journal of Applied Mechanics, vol. 18, no. 3, pp. 293-297, 1951. https://doi.org/10.1115/1.4010337.
[46] J. Wolfowitz, "The minimum distance method". The Annals of Mathematical Statistics, vol. 28, no. 1, pp. 75-88, 1957. http://www.jstor.org/stable/2237024.