Investigation of Water Pollution in the River with Second-Order Explicit Finite Difference Scheme of Advection-Diffusion Equation and First-Order Explicit Finite Difference Scheme of Advection-Diffusion Equation

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Abstract
The perseverance of this research article is to investigate water pollution in rivers with a second-order explicit finite difference scheme of advection-diffusion equation (ADE) and a first-order explicit finite difference scheme of ADE. For investigation, two numerical schemes exploit here FTCSCS and second-order Lax-Wendroff type of ADE which is our new proposed one. In earlier Lax-Wendroff, type scheme existed only for hyperbolic partial differential equation (PDE), here a new second-order Lax-Wendroff type scheme is proposed for parabolic PDE and in addition assist to investigate water pollution with an expectation of better yield compared to the existing one. We implement numerical schemes to estimate the pollutant in water at different times and different points of water bodies. We investigate the numerical behaviour of water pollution by implementing the explicit centred difference scheme (FTCSCS) for advection-diffusion and for our proposed second-order Lax-Wendroff type scheme. Our computational result verifies the qualitative behavior of the solution of ADE for various considerations of the parameters.

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Introduction

ADE is a parabolic linear PDE and combination of the advection equation and diffusion equation. It is a parabolic type partial differential equation and is derived on the principle of conservation of mass using Fick's law (Socolofsky and Jirka 2002). Many investigators have been studied analytical and numerical solutions for higher-dimensional and higher order ADE from many years. In numerical analysis, numerical stability is usually a desired property of numerical algorithms. Stability analysis of finite difference schemes for the Navier-Stokes equations is obtained (Rigal 1979). Stability analysis of finite difference schemes for the advection-diffusion equation is studied (Chan 1984). A comparison of some numerical methods for the advection-diffusion equation is presented (Thongmoon and Mckibbin 2006). An analytical solution of the advection diffusion equation for a ground level finite area source is presented (Park and Baik 2008). An analytical solution is obtained of the one dimensional ADE by reducing the original ADE into a diffusion equation by introducing another dependent variable (Al-Niami and Ruston 1977). Analytical solution of 1D ADE with variable coefficients is presented in a finite domain by using Laplace transformation technique. In that process new independent space and time variables have been introduced (Kumar, A., D. K. Jaiswal and N. Kumar. 2010). Two explicit finite difference schemes such as FTBSCS and FTCSCS, for solving the ADE numerically are studied in this article. A numerical technique was proposed in 1960 by P.D. Lax and B. Wendroff for solving, approximately, systems of hyperbolic conservation laws. Here in this article a new explicit second order Lax-Wendroff type scheme is proposed for solving ADE numerically where we discretise the first order terms of ADE in second order similarly as Lax-Wendroff schemes for hyperbolic partial differential equation.

By exploiting ADE, the physical phenomena where particles, energy or other physical quantities are transferred inside a physical system due to diffusion and advection can be described. ADE is applicable in many disciplines like groundwater hydrology, chemical engineering bio sciences, environmental sciences and petroleum engineering to describe the behaviour of solute concentration. Water pollution has been one of the major environmental problems in fronts of governments and world leaders for decades. With the development of economy and improvement way of living standard, environmental problems have aroused wide attention. Water pollution is one of the most important concerns and may cause many accompanying problems. Actually, various kind of pollution caused various perilous effect in our daily life, environmental pollution is the breakneck risk in this 21st century. The pollution affects human life and its surrounding environment and sometimes pollutants can travel to areas very far from the source of emission thereby affecting livings organisms in that area. One of the ways to understand how pollutants disperse in the environment is through mathematical simulation and stated that simulation of water pollution is useful in providing information about the spread of pollution in area, the scale and level of pollution and estimation.

Water pollution can be demonstrated by one-dimensional advection-diffusion equation (ADE). It is derived on the principle of conservation of mass using Fick's 1st law. This equation considers physical phenomena where in the diffusion process particles are moving with certain velocity from higher concentration to lower concentration. The analytical and numerical solutions along with an initial condition and two boundary conditions help to realize the contaminant or pollutant concentration, distribution and behaviour through an open medium say rivers, lakes and porous medium. With the above discussion in view, our intension is to investigate a second order explicit finite difference scheme to solve ADE.

Review of Related Studies

LeVeque, R. J., & Leveque, R. J. (1992) showed a study on numerical methods for conservation laws. Febi Sanjaya and Sudi Mungkasi. (2017) conducted a study on a simple but accurate explicit finite difference method for the advection-diffusion equation. The way to find the numerical solution of advection-diffusion equation was showed on that article. P.D Lax; B. Wendroff (1960) conducted a study on systems of conservation laws, where Lax-Wendroff scheme for hyperbolic partial differential equation was showed. Azad, T.M.A.K., M. Begum and L.S.Andallah. (2015) conducted a study on an explicit finite difference scheme for advection diffusion equation, where they studied an explicit finite difference scheme for advection-diffusion equation. Ahmed S.G. (2012) conducted a study on a Numerical Algorithm for Solving Advection-Diffusion Equation with Constant and sVariable Coefficients. Murat Sari, Gurhan Gurarslan, and Asuman Zeytinoglu. (2010) showed a study on higher order finite difference approximation for solving advection-diffusion equation. Leon, L. F., & Austria, P. M. (1987) conducted a study on stability Criterion for Explicit

Scheme on the solution of Advection Diffusion Equation. Chan, T. F. (1984) conducted a study on stability analysis of finite difference schemes for the advection diffusion equation. Charney, J. G., Fjortoft, R., & Neumann, J. V. (1950) showed a study on numerical integration of the barotropic vorticity equation, a way for stability analysis was shown on this article. Researchers (Kumar et.al 2009) [5] presented an analytical solution of one- dimensional advection-diffusion equation with variable coefficients in a finite domain using Laplace transformation technique. Agusta and Bamingbola [24] studied on the numerical treatment of the mathematical model for water pollution. They used the implicit centered difference scheme in space and a forward difference method in time for the evaluation of the generalized transport equation. Changiun Zhu, Liping Wa and Sha [26] made a numerical simulation on river water pollution by using grey differential model. They corrected the model in finding the truncation error and found that the obtained results from the grey model are excellent and reasonable. M.M. Rahman. L.S. Andallah [20] presented a simulation of water pollution by finite difference method. They estimated and analyzed the extent of water pollution at different time and points.

Mathematical Model

The simplest one-dimensional ADE is

$$\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} \tag{1}$$

Where $x\epsilon[a,b], t\epsilon[0,T]$

With initial condition, $c(0,x) = c_0(x);$

Boundary condition $c(t,a) = c_a(t);$

And $c(t,b) = c_b(t);$

Where c is the concentration of the transference elements; *D* is the diffusion co-efficient and *v* is the speed of field.

Numerical Method

With the assistance of finite difference method ADE is solved numerically. Here explicit FTCSCS and proposed second order Lax-Wendroff type scheme of ADE are applied for numerical solutions.

Finite difference formulae

Derivatives in equation (1) are approximated by truncated Taylor Series expansions,

1st order forward difference formula in terms of time,

$$\frac{\partial c(x_i^n)}{\partial t} \approx \frac{c_i^{n+1} - c_i^n}{\Delta t} \tag{2}$$

1st order backward difference formula in terms of space,

$$\frac{\partial c(x_i^n)}{\partial x} \approx \frac{c_i^n - c_{i-1}^n}{\Delta x} \tag{3}$$

1st order central difference formula in terms of space,

$$\frac{\partial c(x_i^n)}{\partial x} \approx \frac{c_{i+1}^n - c_{i-1}^n}{2\Delta x} \tag{4}$$

2nd order central difference formula in terms of space,

$$\frac{\partial^2 c(x_i^n)}{\partial x^2} \approx \frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{(\Delta x)^2} \tag{5}$$

Explicit centered difference scheme (FTCSCS)

Substituting equations (2), (4), (5) into equation (1), we get

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} + v\frac{c_{i+1}^n - c_{i-1}^n}{2\Delta x} = D\frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{(\Delta x)^2}$$

$$\Rightarrow c_i^{n+1} = c_i^n - \frac{v\Delta t}{2\Delta x}(c_{i+1}^n - c_{i-1}^n) + \frac{D\Delta t}{(\Delta x)^2}(c_{i+1}^n - 2c_i^n + c_{i-1}^n) \tag{6}$$

Taking $\alpha = \frac{v\Delta t}{\Delta x}$ and $\gamma = \frac{D\Delta t}{(\Delta x)^2}$

$$\Longrightarrow c_i^{n+1} = c_i^n - \frac{\alpha}{2}(c_{i+1}^n - c_{i-1}^n) + \gamma(c_{i+1}^n - 2c_i^n + c_{i-1}^n)$$

$$\Longrightarrow c_i^{n+1} = \left(\frac{\alpha}{2} + \gamma\right)c_{i-1}^n + (1 - 2\gamma)c_i^n + \left(\gamma - \frac{\alpha}{2}\right)c_{i+1}^n \tag{7}$$

Which is known as the **explicit centered difference scheme** for ADE and it is also known as **FTCSCS** technique.

Stability condition of FTCSCS

The above scheme (7) satisfies the convex combination,

We can conclude that the FTCSCS is stable for

$$0 \le \alpha \le 1 \text{ and } 0 \le \gamma \le \tfrac{1}{2}$$

$$0 \le \tfrac{v\Delta t}{\Delta x} \le 1 \text{ and } 0 \le \tfrac{D\Delta t}{(\Delta x)^2} \le \tfrac{1}{2} \tag{8}$$

Explicit second order Lax-Wendroff type Scheme of ADE

For Explicit second order Lax-Wendroff type scheme of ADE, we discretize advective part in half time-step Lax-Friedrich scheme, then substituting that value in half-step Leapfrog scheme and combining with centered diffusion part explicit second order Lax-Wendroff type scheme of ADE is found.

Half-time step lax-Friedrich scheme at the point (t^n, x_i):

$$c_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(c_{i+1}^n + c_i^n) - \frac{v\Delta t}{2\Delta x}(c_{i+1}^n - c_i^n) \tag{9}$$

$$c_{i-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(c_i^n + c_{i-1}^n) - \frac{v\Delta t}{2\Delta x}(c_i^n - c_{i-1}^n) \tag{10}$$

Half-step Leapfrog scheme at the point (t^n, x_i):

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} + u\left[\frac{c_{i+\frac{1}{2}}^{n+\frac{1}{2}} - c_{i-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x}\right] = 0 \tag{11}$$

By centred difference discretization of $\frac{\partial^2 c}{\partial x^2}$ at the point (t^n, x_i), we have

$$\frac{\partial^2 c(x_i^n)}{\partial x^2} \approx \frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{(\Delta x)^2} \tag{12}$$

Combining equation $(11), (12)$ in (1) we obtain,

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} + v\left[\frac{c_{i+\frac{1}{2}}^{n+\frac{1}{2}} - c_{i-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x}\right] = D\frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{(\Delta x)^2}$$

$$\Rightarrow c_i^{n+1} - c_i^n + \frac{v\Delta t}{\Delta x}\left[c_{i+\frac{1}{2}}^{n+\frac{1}{2}} - c_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right] = D\Delta t\left[\frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{(\Delta x)^2}\right]$$

$$\Rightarrow c_i^{n+1} = c_i^n - \frac{v\Delta t}{\Delta x}\left[c_{i+\frac{1}{2}}^{n+\frac{1}{2}} - c_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right] + D\Delta t\left[\frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{(\Delta x)^2}\right] \tag{13}$$

Now substituting the value of $c_{i+\frac{1}{2}}^{n+\frac{1}{2}}$ and $c_{i-\frac{1}{2}}^{n+\frac{1}{2}}$ in equation (13) ,we have

$$\Rightarrow c_i^{n+1} = c_i^n - \frac{v\Delta t}{\Delta x}\left[\frac{1}{2}(c_{i+1}^n + c_i^n) - \frac{v\Delta t}{2\Delta x}(c_{i+1}^n - c_i^n) - \frac{1}{2}(c_i^n + c_{i-1}^n) + \frac{v\Delta t}{2\Delta x}(c_i^n - c_{i-1}^n)\right] + \frac{D\Delta t}{(\Delta x)^2}[c_{i+1}^n - 2c_i^n + c_{i-1}^n]$$

$$\Rightarrow c_i^{n+1} = c_i^n - \frac{v\Delta t}{\Delta x}\left[\frac{1}{2}(c_{i+1}^n + c_{i-1}^n) - \frac{v\Delta t}{2\Delta x}(c_{i+1}^n - 2c_i^n + c_{i-1}^n)\right] + \frac{D\Delta t}{(\Delta x)^2}[c_{i+1}^n - 2c_i^n + c_{i-1}^n]$$

Taking $\alpha = \frac{v\Delta t}{\Delta x}$ and $\gamma = \frac{D\Delta t}{(\Delta x)^2}$ we have

$$\Rightarrow c_i^{n+1} = c_i^n - \alpha\left[\frac{1}{2}(c_{i+1}^n + c_{i-1}^n) - \alpha(c_{i+1}^n - 2c_i^n + c_{i-1}^n)\right] + \gamma[c_{i+1}^n - 2c_i^n + c_{i-1}^n]$$

$$\Rightarrow c_i^{n+1} = \frac{1}{2}(\alpha^2 + \alpha + 2\gamma)c_{i-1}^n + (1 - 2\gamma - \alpha^2)c_i^n + \frac{1}{2}(2\gamma - \alpha + \alpha^2)c_{i+1}^n \quad (14)$$

Which is required **second order Lax-Wendroff type scheme** of ADE.

Stability Condition second order Lax-Wendroff scheme with max-principle

The above equation(14) satisfies the convex combination, we obtain,

$$0 \leq \frac{1}{2}(\alpha^2 + \alpha + 2\gamma) \leq 1 \quad (15)$$

$$0 \leq (1 - 2\gamma - \alpha^2) \leq 1 \quad (16)$$

$$0 \leq \frac{1}{2}(2\gamma - \alpha + \alpha^2) \leq 1 \quad (17)$$

Then the new solution is a convex combination of the two previous solutions. That is, the solution at new time-step $(n + 1)$ at a spatial node i is an average of the solution at the previous time-step at the spatial-nodes $i - 1$, i *and* $i + 1$. This means that the extreme value of the new solution is the average values of the previous two solutions at the three consecutive nodes. Therefore, the new solution continuously depends on the initial value $c_i^0, i = 1,2,3, \dots \dots \dots \dots .., M$.

Therefore, from $(15), (16), (17)$ we have

$$0 \leq \alpha^2 + 2\gamma \leq 1, \;\; 0 \leq \gamma < 1 \text{ and } 0 \leq \alpha < 1 \quad (18)$$

Which is required stability condition for **second order Lax-Wendroff type scheme** of ADE.

Results and Discussion:

Imputation of Data into the Numerical Scheme

To incorporate the data for the different variable and parameters
Time $t = 60 * 60 seconds$;
Length $x = 50m$;
Velocity $v = 0.02m/s$;
Diffusion co-efficient $D = 0.01m^2/s$;
Initial concentration $c(x, 10) = \frac{1}{\sqrt{4\pi D}} exp\left(\frac{-(x-10v)^2}{4D}\right)$
The concentration on left boundary $c(1, t) = 1mg/m^3$
The concentration on right boundary $c(end, t) = \frac{1}{\sqrt{4\pi Dt}} exp\left(\frac{-(xf-vt)^2}{4Dt}\right) mg/m^3$;
Number of temporal grid points $nt = 1600$;
Number of spatial grid points $nx = 80$;
The temporal grid size $\Delta t = \frac{t-t_0}{nt}$;

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The spatial grid size $\Delta x = \frac{x - x_0}{nx}$;

Numerical solution according to input data for FTCSCS

Here in this part, numerical simulation results of FTCSCS of ADE is represented for pollutant transportation according to increase of time. The following **figure 1 to 6** shows how the pollutant concentration dispersed in a river water with increase of time. Water pollution in case of river the pollutants are discharged directly into water bodies without treating it first.

The following **figure 1** shows that the solution surface for the pollutant transportation at $t = 1$ minute. From this figure, the pollutant transportation increases due to time is noticeable.

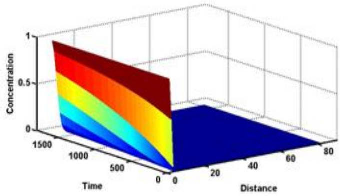

Figure 1(a) Solution surface for pollutant transportation in river at time t=1min with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

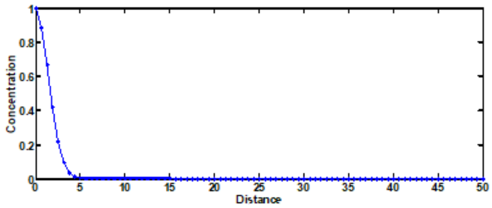

Figure 1(b) Spatial pollutant transportation in river at time t=1min with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

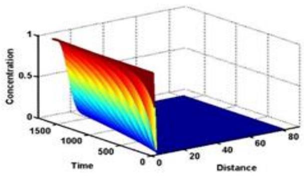

Figure 2(a) Solution surface for pollutant transportation in river at time $t = 5min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

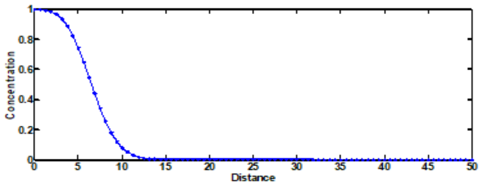

Figure 2(b) Spatial pollutant transportation in river at time $t = 5min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

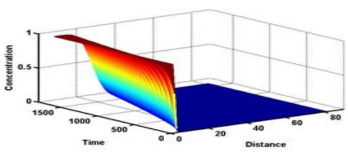

Figure 3(a) Solution surface for pollutant transportation in river at time $t = 10min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

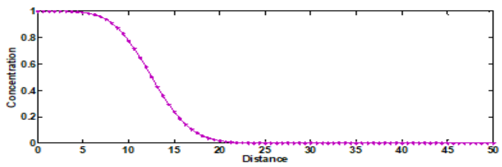

Figure 3(b) Spatial pollutant transportation in river at time $t = 10min$ with velocity $v = 0.02m/$

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s, Diffusion co-efficient $D = 0.01m^2/s$;

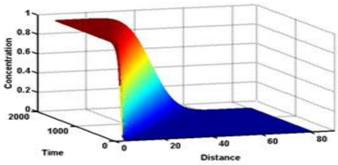

Figure 4(a) Solution surface for pollutant transportation in river at time $t = 20min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

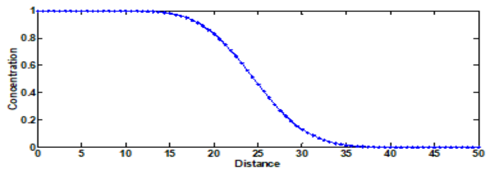

Figure 4(b) Spatial pollutant transportation in river at time $t = 20min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

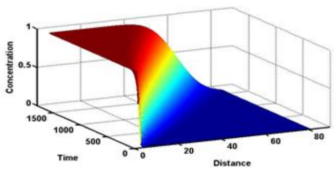

Figure 5(a) Solution surface for pollutant transportation in river at time $t = 30min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

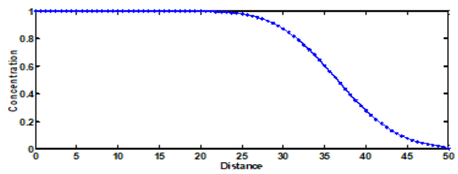

Figure 5(b) Spatial pollutant transportation in river at time $t = 30min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

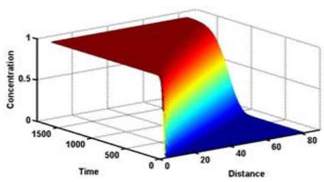

Figure 6(a) Solution surface for pollutant transportation in river at time $t = 1hour$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

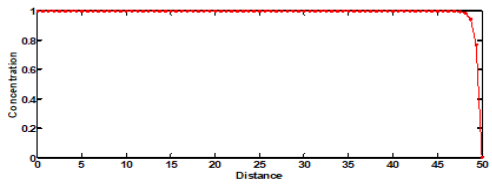

Figure 6(b) Spatial pollutant transportation in river at time $t = 1hour$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

From the above **figure 1(a), 2(a),3(a), 4(a), 5(a) and 6(a)** shows the solution surface for the pollutant transportation at different time. From **figure 1(b), 2(b), 3(b),** we notice that at time $t = 1min$ the concentration distribution is nebulous; at $t = 5min$ the pollutant concentration is transported on the boundary in very small; at $t = 10min$ the pollutant concentration distribution increase along boundary, similarly when time is increased in figure **4(b), 5(b)** with time $t = 20min$ and $t = 30min$ gradually transportation of pollutants along boundary is noticeable. Finally, in **figure 6(b),** when time $t = 1hour$, the pollutant concentration in a river are transported all along the boundary of the river water. The stability condition of one-dimensional ADE for FTCSCS are $0 \leq \alpha = \frac{v\Delta t}{\Delta x} \leq 1$ and $0 \leq \gamma = \frac{D\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$. This will be continued until the stability condition are satisfied.

The following **figure 7,** concentration profile is discussed with respect to different time. In this figure the concentration change with increase of time is demonstrated; from starting time $t = 1min$ the rate of concentration of pollutant is very smaller than the last time $t = 30min$.

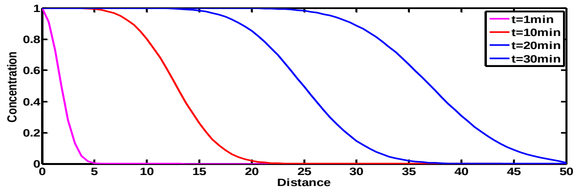

Figure 7: Concentration Distribution at different time for FTCSCS.

The following **figure 8,** is described the concentration with respect to space. In this figure the concentration distribution at different distance and concentration parameter are delineated; the curve identified the change of concentration in the position x=5meter, x=10meter, x=20meter, x=30meter, x=40meter, x=50meter. Finally, it can be said that the pollutant concentration is increased in a still position with respect to time.

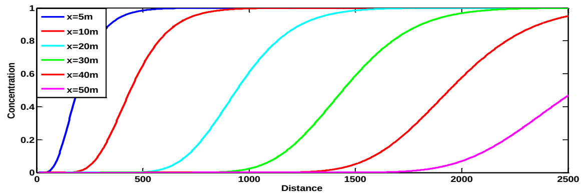

Figure 8: Concentration distribution for FTCSCS at different position.

Numerical solution according to input data for second order Lax-Wendroff type

Here in this part, numerical simulation results of second order Lax-Wendroff type scheme of ADE is represented for pollutant transportation according to increase of time. The following **figure 9 to 14** shows how the pollutant concentration dispersed in river water with increase in time. Water pollution in case of river the pollutants are discharged directly into water bodies without treating it first.

The following **figure 9** shows that the solution surface for the pollutant transportation at $t = 1minute$. From this figure, the pollutant transportation increase due to time is noticiable,

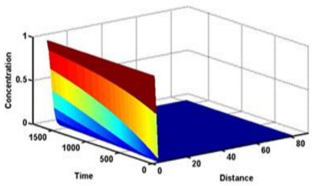

Figure 9(a) Solution surface for pollutant transportation in river at time t=1min with velocity $v = 0.02m/$

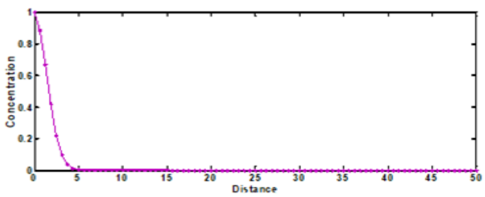

Figure 9(b) Spatial pollutant transportation in river at time t=1min with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

s, Diffusion co-efficient $D = 0.01m^2/s$;

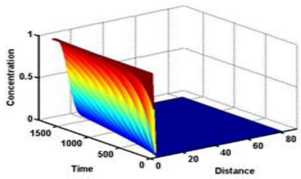

Figure 10(a) Solution surface for pollutant transportation in river at time $t = 5min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

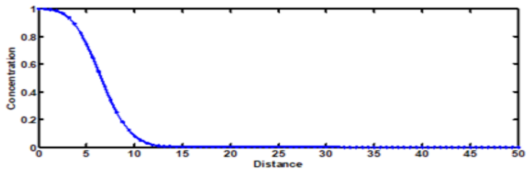

Figure 10(b) Spatial pollutant transportation in river at time $t = 5min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

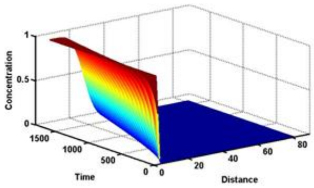

Figure 11(a) Solution surface for pollutant transportation in river at time $t = 10min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

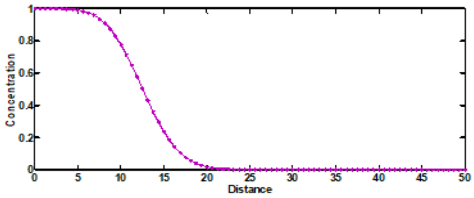

Figure 11(b) Spatial pollutant transportation in river at time $t = 10min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

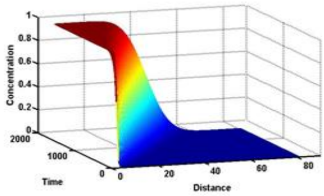

Figure 12(a) Solution surface for pollutant transportation in river at time $t = 20min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

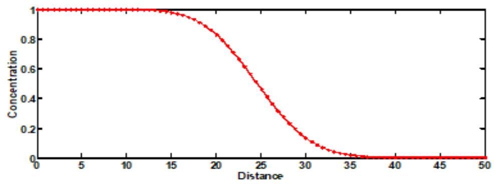

Figure 12(b) Spatial pollutant transportation in river at time $t = 20min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

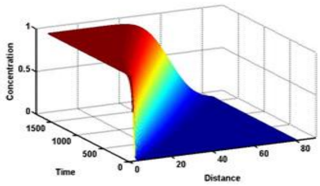

Figure 13(a) Solution surface for pollutant transportation in river at time $t = 30min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

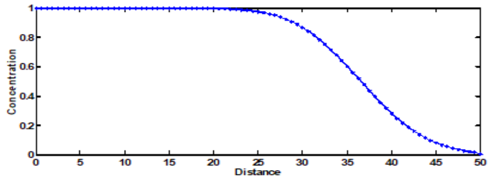

Figure 13(b) Spatial pollutant transportation in river at time $t = 30min$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

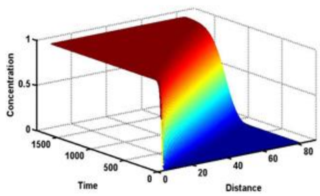

Figure 14(a) Solution surface for pollutant transportation in river at time $t = 1hour$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

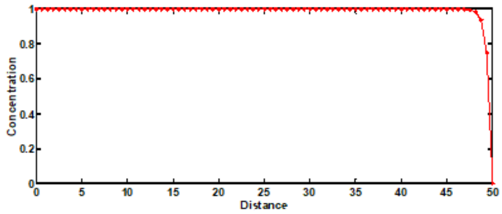

Figure 14(b) Spatial pollutant transportation in river at time $t = 1hour$ with velocity $v = 0.02m/s$, Diffusion co-efficient $D = 0.01m^2/s$;

From the above **figure 9(a), 10(a), 11(a), 12(a), 13(a) and 14(a)** shows the solution surface for the pollutant transportation at different time. From **figure 9(b), 10(b), 11(b),** we notice that at time $t = 1min$ the concentration distribution is nebulous; at $t = 5min$ the pollutant concentration is transported on the boundary in very small; at $t = 10min$, the pollutant concentration distribution increase along boundary, similarly when time is increased in **figure 12(b), 13(b)** with time $t = 20min$ and $t = 30min$ gradually transportation of pollutants along boundary is noticeable. Finally, in **figure 14(b),** when time $t = 1hour$, the pollutant concentration in river are transported all along the boundary of the river water. The stability condition of 1-D ADE for Lax-Wendroff type are $0 \leq \alpha^2 + 2\gamma \leq 1$, $0 \leq \gamma = \frac{D\Delta t}{(\Delta x)^2} < 1$ and $0 \leq \alpha = \frac{v\Delta t}{\Delta x} < 1$. This will be continued until the stability condition are satisfied.

The following **figure 15,** concentration profile is discussed with respect to different time. In this figure the concentration change with increase of time is demonstrated; from starting time $t = 1min$ the rate of concentration of pollutant is very smaller than the last time $t = 30min$.

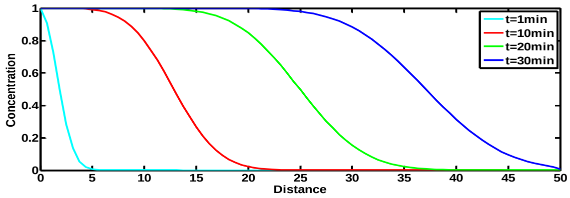

Figure 15: Concentration Distribution at different time for Lax-Wendroff type.

The following **figure 16,** described the concentration with respect to space. In this figure the concentration distribution at different distance and concentration parameter are delineated; the curve identified the change of concentration in the position x=5meter, x=10meter, x=20meter, x=30meter, x=40meter, x=50meter. Finally, it can be said that the pollutant concentration is increased in a still position with respect to time.

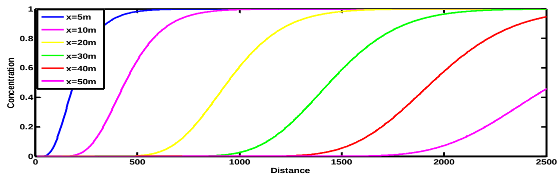

Figure 16: Concentration distribution for Lax-Wendroff type at different position.

Numerical Simulation for Different Velocity and Diffusion Rate for FTCSCS

This section represents the numerical simulation results for transportation of pollutant in ariver with increasing water flow velocity and increasing diffusion co-efficient. To check the accuracy of the numerical scheme by FTCSCS technique for the ADE, we implement the model for some artificial data for the transport of the pollutant in river. Our aim is to show that for the river pollution, any substance with bigger diffusion results a wider pollutant front.

In the following **figure 17,** the diffusion rate is fixed and consider the concentration distribution at different velocity.

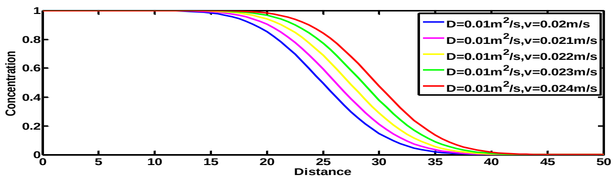

Figure 17: Concentration distribution of ADE(FTCSCS) at fixed diffusion and different velocity at time $t = 20min$.

The above **figure 17** shows the with fixed diffusion rate $D = 0.01m^2/s$ and increase of velocity from $v = 0.02m/s$, to $v = 0.021m/s$, to $v = 0.022m/s$, to $v = 0.023m/s$, to $v = 0.024m/s$, then concentration rate is increased along the boundary.

The following **figure18** represents the change of concentration with respects to fixed velocity and different diffusion rate.

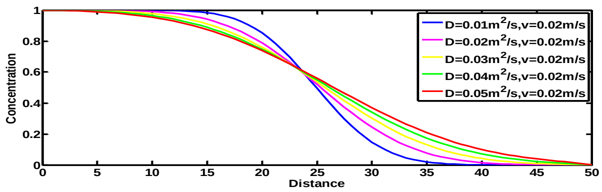

Figure 18: Concentration distribution of ADE(FTCSCS) with fixed velocity and different diffusion rate at time $t = 20min$.

In the above **figure 18** the velocity is fixed $v = 0.02m/s$ but the diffusion rate is changes from $D = 0.01m^2/s$, to $D = 0.02m^2/s$,to $D = 0.03m^2/s$,to $D = 0.04m^2/s$,to $D = 0.05m^2/s$. This have seen that with increase of diffusion rate with fixed velocity the both transportation and dispersion happened here along the boundary the concentration distribution is increased.

The following **figure 19** shows the changed of concentration distribution of pollutant at different velocity and different rate of diffusion.

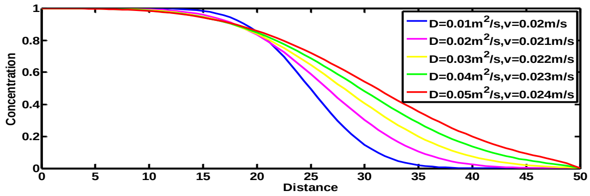

Figure 19: Concentration Distribution of ADE(FTCSCS) at different velocity and different rate of diffusion at $t = 20min$.

The above **figure 19** shows the change of concentration at different diffusion $D = 0.01m^2/s$ to $D = 0.02m^2/s$ to $D = 0.03m^2/s$ to $D = 0.04m^2/s$ to $D = 0.05m^2/s$ and different velocity $v = 0.02m/s$ to $v = 0.021m/s$ to $v = 0.022m/s$ to $v = 0.023m/s$ to $v = 0.024m/s$.

As this is known that the stability condition of the scheme by FTCSCS technique are $0 \leq \alpha = \frac{v\Delta t}{\Delta x} \leq 1$ and $0 \leq \gamma = \frac{D\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$. This will be continued until the stability condition are satisfied.

Numerical Simulation for Different Velocity and Diffusion Rate for second order Lax-Wendroff type

This section represents the numerical simulation results for transportation of pollutant in a river with increasing water flow velocity and increasing diffusion co-efficient. To check the accuracy of the numerical scheme by second order Lax-Wendroff type technique for the ADE, we implement the model for some artificial data for the transport of the pollutant in a river. Our aim is to show that for the river pollution, any substance with bigger diffusion results a wider pollutant front.

In the following **figure 20,** the diffusion rate is fixed and consider the concentration distribution at different velocity.

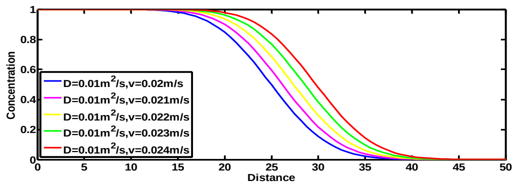

Figure 20: Concentration distribution of ADE (second order Lax-Wendroff type) at fixed diffusion and different velocity at time $t = 20min$.

The above **figure 20** shows the with fixed diffusion rate $D = 0.01m^2/s$ and increase of velocity from $v = 0.02m/s$, to $v = 0.021m/s$, to $v = 0.022m/s$, to $v = 0.023m/s$, to $v = 0.024m/s$, then concentration rate is increased along the boundary.

The following **figure 21** represents the change of concentration with respects to fixed velocity and different diffusion rate.

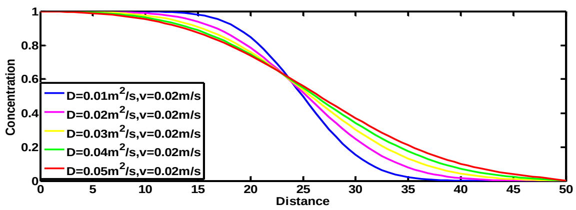

Figure 21: Concentration distribution of ADE (second order Lax-Wendroff type) with fixed velocity and different diffusion rate at time $t = 20min$.

In the above **figure 21** the velocity is fixed $v = 0.02m/s$ but the diffusion rate is changes from $D = 0.01m^2/s$, to $D = 0.02m^2/s$,to $D = 0.03m^2/s$,to $D = 0.04m^2/s$,to $D = 0.05m^2/s$. This have seen that with increase of diffusion rate with fixed velocity the both transportation and dispersion happened here along the boundary the concentration distribution is increased.

The following **figure 22** shows the change of concentration distribution of pollutant at different velocity and different rate of diffusion.

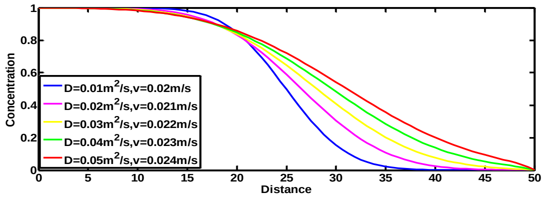

Figure 22: Concentration Distribution of ADE (second order Lax-Wendroff type) at different velocity and different rate of diffusion at $t = 20min$.

The above **figure 22** showS the change of concentration at different diffusion $D = 0.01m^2/s$ to $D = 0.02m^2/s$ to $D = 0.03m^2/s$ to $D = 0.04m^2/s$ to $D = 0.05m^2/s$ and different velocity $v = 0.02m/s$ to $v = 0.021m/s$ to $v = 0.022m/s$ to $v = 0.023m/s$ to $v = 0.024m/s$.

As this is known that the stability condition of the scheme by Lax-Wendroff type technique are $0 \le \alpha^2 + 2\gamma \le 1$, $0 \le \gamma = \frac{D\Delta t}{(\Delta x)^2} < 1$ and $0 \le \alpha = \frac{u\Delta t}{\Delta x} < 1$. This will be continued until the stability condition are satisfied.

Conclusion

In this paper, the different numerical schemes of ADE such as FTCSCS and proposed second order Lax-Wendroff type have been discussed. A second order Lax-Wendroff type scheme for ADE has been proposed like as Lax-Wendroff scheme of hyperbolic partial differential equation. Here for proposed new second order Lax-Wendroff type scheme of ADE the discretisation of first order terms are in second order same as Lax-Wendroff scheme of hyperbolic partial differential equation. The stability conditions have been determined for FTCSCS and second order Lax-Wendroff type scheme by maximum principle. Here the Advection-Diffusion equation is exploited to describe the real-life phenomena water pollution in river by using the FTCSCS and second order Lax-Wendroff type scheme. Here the numerical solutions have also been showed verifying with respect to the velocity, diffusion rate, distance, and time. The graphical exhibition is verifying the qualitative behaviour of the solutions of ADE for various considerations of the parameters. The results show that the water pollutions are being spreading with the varied the advection and diffusion co-efficient term with respect to time and space. The change of pollution concentration at different time with fixed space, at different position with fixed time, at different velocity with fixed diffusion coefficient, at different diffusion with fixed velocity, with both changing diffusion coefficient and velocity is observed. In the estimation of water pollution for FTCSCS and second order Lax-Wendroff type scheme of ADE, this is seen that both schemes can be used to describe the water pollution but as second order Lax-Wendroff type schemes is in second order discretisation for both time and space and has less error so second order Lax-Wendorff type scheme of ADE is the better one to describe the water pollution. So, after this comparison it can be concluded that second order Lax-Wendroff type scheme of ADE is better than FTCSCS scheme of ADE for estimation of water pollution in river.

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