# A Review Paper on Radial Basis Function for Solving Differential Equations

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**Abstract**
In this review paper, we will know how the radial basis function is used for solving the ordinary differential equation using neural network techniques. Also, how radial basis function is used for solving partial differential equations. Radial basis functions are commonly popular in solving PDEs as we can say that it is the most powerful method in the case of solving partial differential equations. Well, many radial functions are used for solving PDEs like the Kansas method, differential quadrature method, etc. there is always ongoing research in the field of development and a few new effective methods are also being used nowadays like global to local approximations which are known as localized methods and also to hybrid methods. But here we will majorly discuss the differential quadrature method for solving partial differential equations.

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## INTRODUCTION

If we start from the first topic, we are going to discuss in this paper which is the radial basis function neural network commonly known as RBFNN for solving our ordinary differential equations, what do we understand from RBFNN? What is RBFNN? Why are we using this method for solving our differential equations? Above all what is the radial basis function? you will get all your answers in this paper in the below sections. Firstly, let's discuss what is radial basis function. If we say in simple words radial basis function is nothing but a real-valued function that can be denoted by φ. The value of these real-valued functions is subject to only the distance. This distance is measured between the input value and one fixed point, this fixed point can be of two types: -

**i.** we can take origin as the fixed point which we satisfy the following equation i.e., $\varphi(x) = \hat{\varphi}(\|x\|)$.

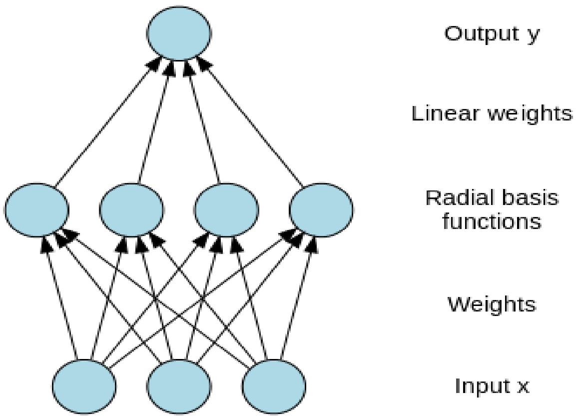
**ii.** We can also take any other fixed point let that point be c, which we can call center, this will also satisfy the following equation i.e., $\varphi(x) = \hat{\varphi}(\|x-c\|)$.

From the above definition and equations, we can conclude that any function taken as φ that will satisfy the equation mentioned in the first case is a radial basis function.

The radial basis function highly used due to its meshfree nature, not only it's a meshfree method but also it is very easy to implement. Above all, it is independent of dimensions due to which it can be widely used in many problems.

Now, a Radial basis function neural network is a type of feed-forward network in which we will have two layers in which we are excluding an input layer, the two layers are:

i. non-linear radial basis function activated layer, this layer is a hidden layer
ii. output layer




The above image shows the basic design of the radial basis function network. From this image, we can easily understand the functioning of our network. Also, we can say that a radial basis network is a hypothetical neural network as it uses the radial basis function as an activation function.

We will be using this method for solving our first order differential equations or ordinary differential equations as in recent papers published, the meshfree techniques are available to solve the first order differential equations based on the multiquadric radial basis function neural network. Straight as well as little complicated methods have already been discovered to solve ordinary differential equations. Also, the radial basis function neural network method is a bit different but a very efficient approach in the case of solving partial as well as ordinary differential equations.

If we take various radial basis functions and add them, their sum will be most commonly used for approximating specified functions. The above approximation method can simply be understood as the simplest form of radial basis function neural network.

This radial basis function procedure which we use today was due to its first introduction by hardy in 1971, he introduced this procedure about the topological applications it had on the quadric surfaces. Later, he also introduced the multiquadric approximation scheme. After many years of research on this topic, around 2003 'Shu' gave an idea of a hybrid method radial basis function- differential quadrature method (RBF-DQ). Which we are going to discuss in detail in this paper. As we have discussed earlier radial basis functions have meshfree nature and the differential quadrature method was combined due to its uncomplicatedness and high precision.

This procedure was then widely used by scientists in solving the partial differential equations of fluids.

**Radial basis function neural network**

As we have discussed earlier the radial basis neural network structure and also saw how neural networks have three layers if we include the input layer. But in this section, we will discuss how the radial basis function neural network methods are used for solving our set of differential equations. In this section, you will be provided with a short explanation of recent articles that have been published by researchers in solving the differential equations using the radial basis function neural network.

**Ordinary differential equations (ODEs) using radial basis function neural network (RBFNN)**

Two researchers named Mai-Duy and Tran-Cong provided us with a meshfree method by which we can solve the ordinary differential equation. This method was derived from the multiquadric radial basis function neural network. Together they proved the straight and some twisted network methods that can be used in solving our differential equations. In this, we will take the equation of 2D Poisson over a certain domain say $\Omega$. We will take $\nabla^2$ as our operator of Laplacian, y will be taken as the position of spatial and a will be its known function and b will be taken as y's unknown function.

$$\nabla^2 b = a(y), \qquad \text{where y belongs to the given domain } \Omega$$

We have to find u about either Neumann boundary conditions or by Dirichlet boundary condition taking the boundary as $\Gamma$.

Also, after taking the boundary conditions the above equation can be represented as,

$$b = a_1(y), \qquad \text{now we will take y belonging to our boundary say } \Gamma_1$$

If we take n as the normal vector representing the outward unit and $\nabla$ as our operator of gradience we can represent the equation as

$$n. \nabla b = a_2(y), \qquad \text{here y will belong to our boundary say } \Gamma_2$$

as we have already discussed $\Gamma$ is our boundary based on Neumann boundary conditions or Dirichlet conditions. We will take our two boundaries $\Gamma_1$ and $\Gamma_2$ from our given domain i.e., $\Omega$ such that they fulfill a certain condition according to which the union of two boundaries should be equal to boundary $\Gamma$ and the intersection of our two boundaries from the domain should be equal to null set which is represented by $\Phi$ i.e.,

$\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \Phi$

Similarly, $a_1$ and $a_2$ will be the known functions of y just as a.

From this, it can be concluded that our unknown function which was earlier taken as b can be solved and the solution of b and the derivative of b can be estimated as our radial basis function.

In the above method that we have used above, the only constraint which can be adjusted is the width of our radial basis function about $m^{(j)} = \beta n^{(i)}$, we will be taking $n^{(i)}$ as the gap in the two centers i.e., $i^{th}$ center and the center which is closest to it.

From the method we have used above, it can be observed that the methods which were not straight commonly known as the indirect radial basis function neural network method can give the solutions up to certain order of differential equations magnitude as compared to the methods which were straight functions commonly known as direct radial basis function neural network method. If the two methods are indirect radial basis function neural network method and the direct radial basis function neural network method are compared together it can easily be observed that the indirect radial basis function neural network method is much more exact as compared to the direct radial basis function neural network method on a broad area of β

As a result, choosing the width of our only adjustable constraint which was our radial basis function become less difficult. To obtain these results various combinations of radial basis function centers, as well as several combinations of collocation points, are tried at regular intervals on the regularly shaped domains as well as irregularly shaped domains.The radial basis function neural network method of solving linear differential equations is a non-iterative method and therefore, it can be concluded that these are much more efficient when compared to the iterative method like the feed-forward neural network method. Also, the indirect radial basis functions neural network method using ideal β makes this method much more efficient as compared to the old methods used for solving the same like FDM, and BEM. Also, as we have already discussed radial basis function neural network method can be used on both regular and irregular-shaped domains.

In this method, we have to concentrate on choosing our activation function and how the total number of nodes can be increased by taking nodes as few as possible. The radial basis function neural network helps us in obtaining more precise and exact results than too with a faster speed solution of the ordinary differential equations. We know that the radial basis function network has an always increasing manner due to which we can attain better-approximated outcomes that too with less memory space as it does not waste space and the time of computing is also less. If we take the trigonometric functions in place of the radial basis function, then it is not necessary that we have to give the values between -1 and 1 only, also in the case of trigonometric functions we do not need to choose a certain center of our radial basis function. If we use differential reconstruction in the radial basis function, it also helps in making the choice of parameter easy, which in return increases the dampness of our system.Using the radial basis function network, we can also present the volumetric integral for the partial differential equation which will be dependent on time by using the radial basis function invention.

**Radial basis function-Differential quadrature method**

In the differential quadrature method, we follow the idea of integral quadrature. Firstly, in this method, we will take a linear combination of weight values and functional values. Now the partial derivative of the plane function can be estimated on a complete domain. Further, we can calculate the value of $f^{n}(y)$ concerning x at a point $y_{a}$ by the following equation:

$$^{n}(y_{a}) = \sum_{b=1}^{N} i_{ab}^{n} f(y_b) \qquad a=1,2,3\ldots\ldots\ldots N$$

where $i_{ab}^{n}$ are weight coefficient. We can also calculate the weight coefficient by different methods such as Fourier series expansion, Moving least square, and Interpolation polynomials.

Previously Lagrange's interpolation was used for differential quadrature but in 2002 researchers named Wu and Shu used the radial basis function method. Solely, the differential quadrature method had a few problems which could be resolved with the help of the meshfree method. They used the radial basis function which naturally provides a meshfree environment. This method was combined with the differential quadrature method to overcome its problems. In this method, the coefficient is found with the help of. Also, at each node different dependent variable values are represented, due to which radial basis function method is also applicable for the non-linear problem, and he named it as radial basis function-differential quadrature (RBF-DQ) method. The radial basis function differential quadrature method was of two types first was named as global method and the other one was the local method. In the first approach i.e., the Global approach we find coefficients at all the already defined nodes. This method was easy to implement and also quite operative. But the number of nodes becomes very large increasing the cost of total computation and hence the other method i.e., the local method was used, in the local approach we only find the coefficients of the node which are present in the neighborhood of the node already being observed. As we are only using the nodes which are present in the neighborhood it reduces the cost of computation as well as the shape parameter also stays stable and does not fluctuate much.

In the method of global radial basis function differential quadrature(GRBFDQ), the derivatives which were already found can be represented as the sum of the function at certain points whose values will be given in the domain. If we take the nth order derivative of a certain function say f(y, z) concerning y at a given node ($y_a$, $z_a$), also we will take N which will represent the total number of nodes present in the domain whereas $w_{ab}^{n}$ will provide us with the weight coefficients at taken as c which is taken as $(y_b, z_b)$ will be represented by the following equation:

$$f^{n}(z_a \ \cong \sum_{b=1}^{N} w_{ab}^{n} f(y_b, z_b)$$

The above equation whose parameters are already defined should satisfy the given radial basis function $\varphi_k(y,z)$ for determining the weight coefficient. Also, for determining the weight coefficients the above equation can be modified as,

$$\varphi_k^{n}(y_a, z_a) \cong \sum_{b=1}^{N} w_{ab}^{n} \varphi_k(y_b, z_b) \qquad \forall\ k=1,2,3\ldots\ldots.$$

The second version of RBF-DQ was local radial basis functional differential quadrature (LRBFDQ) which was used by Shu to solve the two-dimensional equations of Navier- Stokes which cannot be compressed and from the results of this he applied these local radial basis function differential quadrature method to solve the partial differential equation of compressible fluids. The LRBFDQ method was also used by another scientist to solve boundary layer problems. Shu also invented another method in which in place of using radial basis function he used indirect radical function network which we discussed in the above section with differential quadrature and named the resulting method as indirect radial basis functional differential quadrature method (iRBF – DQ).

Now with the help of the iRBF – DQ method he was successful in solving the 1-dimensional burger's equation. The indirect radial basis functional differential quadrature method (iRBF – DQ) also came in handy in the analysis of seepage. By analyzing seepage Hashemi and Hatam were able to justify how the indirect radial basis functional differential quadrature method (iRBF – DQ) works due to its meshfree properties working perfectly well with irregular domains as well. The analysis of Seepage was very important for the designing and proper working of the hydraulic structure. Let's discuss in little detail how Dehghan used the local radial basis functional differential quadrature method (LRBFDQ) for solving boundary value problems of second order. While solving boundary value problems he used the multiquadric radial basis function. We already know how the shape parameter plays a significant part in the radial basis function. So, he applied different techniques. These two techniques were optimal shape parameter techniques OCSP whereas the second one was the OVSP method.

## CONCLUSION

Over the past few years, the development in the number of numerical methods that are usually based on radial basis function has increased significantly mostly due to the meshfree nature of radial basis function. In this paper, we have tried to put some light on the increasing developments in the field of radial basis function in past years. We are already aware about the daily developments in different fields like biology, engineering, mathematics, physics, and chemistry is taking place vastly. With the vast increase in these fields, we have observed many phenomena in this field can only be described with the help of differential equations. Hence the development in the field of mathematics is not only limited to it but extends to all the other fields of science and technology as well. Hence, the radial basis function becomes an important part of the research of solving differential equations. Also, when an artificial neural network like radial basis function neural network is attached to the phenomena of various field it not only helps in just solving the equations but also add more striking features to it.

In this paper, we have mostly discussed two different ways of using the radial basis function for the solution of two different types of differential equations. In the first method, we used an artificial neural network combined with a radial basis function which resulted in a radial basis function neural network and this network helped us in solving ordinary or we can say linear differential equations. From this, we have concluded how the direct radial basis function network and indirect radial basis function network differ from each other and how the indirect method is better in many ways than the old methods used as well as the direct radial basis function method. Not only the functions of the indirect and direct network but also the vastness of using radial basis function network. Using this network can find a voluminous integral for partial differential equations which are time-dependent. We have also discussed how we can choose our activation functions and the space and time of computation. Taking trigonometric functions, we don't have to choose the center which results in the replication of parameters easily.

In the second method, we discussed how the radial basis function was combined with the differential quadrature method because of its different features. There was already developed Kansas method which was the simplest meshfree method out of all the other methods when it comes to solving the partial differential equations but just like any other thing with the pros it also had various cons such as high cost of computation, it usually failed while working on large problems. To overcome the problems of the Kansas method other hybrid methods were discovered. The radial basis function was combined with the differential quadrature method and became one of the hybrid methods to make the radial basis function more reliable. Further, the radial basis function can be made more precise by node refinement and by finding the effect of scaling and stability on the radial basis function.

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