Mathematical Statistician and Engineering Applications
ISSN: 2094-0343
2326-9865

Analysis of Semilocal and Order of Convergence on Riemannian Manifolds for Secant Method

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Article Info
Page Number: 1988-1994
Publication Issue:
Vol. 71 No. 4 (2022)

Article History
Article Received: 25 March 2022
Revised: 30 April 2022
Accepted: 15 June 2022
Publication: 19 August 2022

Abstract
This paper deals the analysis of semilocal and order of convergence on Riemannian Manifolds for Secant Method. Furthermore, it has divided difference geodesic points on euclidean spaces. Also, we have used the fact that ω is monotonic in its two arguments under invertible.This paper deals the analysis of semilocal and order of convergence on Riemannian Manifolds for Secant Method. Furthermore, it has divided difference geodesic points on euclidean spaces. Also, we have used the fact that ω is monotonic in itstwo arguments under invertible.

Keywords: Riemannian Manifolds , geodesic points, invertible.

Introduction

Let R_M be a Riemannian manifold and λ a graphical line in R_M and $[s, s+h] \subset dom(\lambda)$,$X \in \chi(R_M)$ is belongs to the field R_M. A vector space has defined as follows in the following linear combination:

$$V : L_{\lambda(s+h)}R_M \to L_{\lambda(s+h)}R_M$$

is the Newton difference of operator of initial variable X on $\lambda(s), \lambda(s+h)$ in $\lambda'(s)$ it contains

$$V(F_{\lambda s,s+h}(\lambda'(s))) = \frac{1}{h}(X(\lambda(s+h)) - F_{\lambda,s,s+h}(X(\lambda(s)))). \tag{1}$$

Let $f, g \in M$, $[f, g; X]$ be a divided difference (1) with λ a neighboring values $\lambda(s) = f$ and $\lambda(s+h) = g$, $[s, s+h] \subset dom(\lambda)$. let R_M be and Eulerian with neighboring values $x, y \in R_M$ is defined by

$$\lambda(s) - x - s(y - x) = 0, \qquad s \in \mathrm{R}.$$

Then (1), let $s = V$ and $h = 1$, we have

$$[x, y; X](y - x) = V(y - x) = X(y) - X(x),$$

Therefore, the solutions is (1) for $F = X$. Hence, (1).

Let $\lambda : \mathrm{R} \to R_M$ be a graphical fit, the line parallel in λ, given by F_λ, .,. and calculated by

$$F_{\lambda,a,b} : L_{\lambda(a)}R_M \to L_{\lambda(b)}R_M$$
$$v \mapsto V(\lambda(b)),$$

which belongs to a, $binR$; here V is a variable under both magnitude and direction in λ with$\nabla_{\lambda'(t)}V = 0$ and $V(\lambda(a)) = v$.

Let $F_{\lambda,a,b}$ is 1-1, with $F_{\lambda,a,b}$ be straight lines at slope $L_{\lambda(a)}R_M$ and $L_{\lambda(b)}R_M$. Its reciprocal has λ in $V(\lambda(b))$ to $V(\lambda(a))$. Let $F_{\lambda,a,b}$ be identical lies $L_{\lambda(a)}R_M$ and $L_{\lambda(b)}R_M$ for any $a, b, d \in$ R the following conditions:

$$F_{\lambda,b,d} \circ F_{\lambda,a,b} = F_{\lambda,a,d}, F^{-1}_{\lambda,b,a} = F_{\lambda,a,b}, \text{ and } F_{\lambda,a,b}(\lambda'(a)) = \lambda'(b).$$

Preliminaries

Let us consider the equation

$$v_n = -[f_{n-1}, f_n, X]^{-1}(X(f_n)) \tag{2}$$
$$f_{n+1} = \exp_{f_n}(v_n),$$

for each $n = 1, 2 \ldots$, with f_0 and f_1 given.

Let us assume that $\omega : R_+ \times R_+ \to R_+$ be angles. Here R_M is vector identity and $\Omega \subset R_M$ is interval at open. The variable $X \in \mathrm{X}(R_M)$ true ω-properties in Ω,

$$\|[f_1, f_2, X] \circ (f_{\lambda,0,1}) - (f_{\lambda,0,1}) \circ [g_1, g_2, X]\| \leq \omega(d(f_1, f_1), d(f_2, f_2)) \tag{3}$$

for all $f_1, f_2, g_1, g_2 \in \Omega$, here λ is a neighboring values $\lambda(0) = g_2$ and $\lambda(1) = f_2$.

Let Ω be Riemannian manifold on an open connected of R_M and $f_0, f_{-1} \in \Omega$, we have

$$a(u) = \frac{\beta\omega(\alpha, u)}{1 - \beta\omega(\alpha, u)}, \quad b(u) = \frac{\beta\omega(u, 2u)}{1 - \beta\omega(\alpha + u, u)}, \quad c(u) = \frac{\beta\omega(2u, 2u)}{1 - \beta\omega(\alpha + u, u)},$$

where

$$\alpha = d(f_0, f_{-1}) \quad \text{and} \quad \beta = \|[f_{-1}, f_0, X]^{-1}\|. \tag{4}$$

Lemma 1. *Prove that*

$$d(f_3, f_0) \leq (ba + a + 1)\eta < R,$$

where $a = a(R)$ and $b = b(R)$.

Proof. We first prove that $f_1 \in B(f_0, R)$. In fact, by (2) one has

$$v_0 = -[f_{-1}, f_0, X]^{-1}(X(f_0))$$
$$f_1 = \exp_{f_0}(v_0),$$

hence, $\lambda_0(t) = \exp_{f_0}(tv_0)$,

$$\begin{aligned} d(f_0, f_1) &= \int_0^1 \|\lambda_0'(t)\| dt \\ &= \|v_0\| \\ &= \|[f^{-1}, f_0, X]^{-1}(X(f_0))\|. \end{aligned}$$

From hypothesis (ii), thus, we have
$$d(f_0, f_1) \leq \eta < R. \tag{5}$$

Conversely, the ω-properties (3) satisfies

$$\|F_{\lambda_0,0,1} \circ [f_{-1}, f_0, X] - [f_0, f_1, X] \circ F_{\lambda_0,0,1}\| \leq \omega(d(p_0, f_{-1}), d(f_1, f_0)) \leq \omega(\alpha, R), \tag{6}$$

from (4), $d(f_0, f_1) < R$, and the fact that ω is non-decreasing in its two arguments. Since the parallel transport is an isometry, it follows by hypothesis (ii) that

$$\|F_{\lambda_0,0,1} \circ [f_{-1}, f_0, X] - [f_0, f_1, X] \circ F_{\lambda_0,0,1}\| < \frac{1}{\beta} = \frac{1}{\|(F_{\lambda_0,0,1} \circ [f_{-1}, f_0, X])^{-1}\|}.$$

Thus, a classical result of linear operator theory, see Theorem 2.3.5 in [38], shows that $[f_0, f_1, X] \circ F_{\lambda_0,0,1}$ is invertible and moreover,

$$\|[f_0, f_1, X]^{-1}\| \leq \frac{\|[f_{-1}, f_0, X]^{-1}\|}{1 - \|[f_{-1}, f_0, X]^{-1}\| \|F_{\lambda_0,0,1} \circ [f_{-1}, f_0, X] - [f_0, f_1, X] \circ F_{\lambda_0,0,1}\|}$$

We conclude from (4) and (6) that

$$\|[f, f, X]^{-1}\| = \frac{\beta}{1-\beta\omega(\alpha, R)}. \tag{7}$$

and (6) we get,

$$\begin{aligned}\|X(f_1)\| &= \|([f_0, f_1, X] \circ F_{\lambda_0,0,1} - F_{\lambda_0,0,1} \circ [f_{-1}, f_0, X])(v_0)\| \\ &\le \|([f_0, f_1, X] \circ F_{\lambda_0,0,1} - F_{\lambda_0,0,1} \circ [f_{-1}, f_0, X])\|\|v_0\| \\ &\le \omega(\alpha, R)\|v_0\|\end{aligned}$$

s

The following procedure shows at $d(f_1, f_2) \le ad(f_0, f_1)$ and $d(f_0, f_2) < R$. To prove the first inequality, we use (2) toobtain

$$\begin{aligned}v_1 &= -[f_0, f_1, X]^{-1}(X(f_1)) \\ f_2 &= \exp_{f_1}(v_1),\end{aligned}$$

which gives, by (7) and (8),

$$\begin{aligned}d(f_2, f_1) &= \|v_1\| \\ &\le \|[f_0, f_1, X]^{-1}\|\|X(f_1)\| \\ &\le \|[f_0, f_1, X]^{-1}\|\|X(f_1)\| \\ &\le \frac{\beta\omega(\alpha, R)}{1-\beta\omega(\alpha, R)} d(f_1, f_0).\end{aligned}$$

$$d(f_2, f_1) \le ad(f_1, f_0), \tag{9}$$

By definition of a it follows that

which gives us that

$$\begin{aligned}d(f_2, f_0) &= d(f_2, f_1) + d(f_1, f_0) \\ &\le (a+1)d(f_1, f_0) \\ &< (a+1)\eta.\end{aligned} \tag{10}$$

By definition of R, we obtain the second inequality. Thus

$$F_2 \in B(f_0, R). \tag{11}$$

Now, we consider a sequence of geodesics (ϕ_n) satisfying $\phi_n(0) = f_0$, $\phi_n(1) = f_n$, and $\phi_1 = \lambda_0$. We conclude from (3), (4), (11), and $d(f_0, f_1) < R$ that

$$\begin{aligned}\|[f_1, f_2, X] \circ F_{\phi_2,0,1} - F_{\phi_2,0,1} \circ [f_{-1}, f_0, X]\| &\le \omega(d(f_1, f_{-1}), d(f_2, f_0)) \\ &\le (d(f_0, f_1) + d(f_0, f_{-1}), d(f_2, f_0)) \\ &\le \omega(R+\alpha, R),\end{aligned}$$

hence that

$$\|[f_1, f_2, X] \circ F_{\phi_2,0,1} - F_{\phi_2,0,1} \circ [f_{-1}, f_0, X]\| \le \frac{1}{\beta},$$

by hypothesis (ii). Proceeding as in the proof of (7) one obtains that the operator $[f_1, f_2, X]$ is invertible and

$$\|[f, f, X]^{-1}\| = \frac{\beta}{1-\beta\omega(R+\alpha, R)}. \tag{12}$$

We claim that $d(f_3, f_2) \leq bd(f_2, f_1)$.

$$X(f_2) = ([f_1, f_2, X] \circ F_{\lambda_1,0,1} - F_{\lambda_1,0,1} \circ [f_0, f_1, X])(v_1),$$

from which we infer, by using (3) and $d(f_2, f_1) = \|v_1\|$, that

$$\begin{aligned}\|X(f_2)\| &\leq \|([f_1, f_2, X] \circ F_{\lambda_1,0,1} - F_{\lambda_1,0,1} \circ [f_0, f_1, X])\|\|(v_1)\| \\ &\leq \omega(d(f_1, p_0), d(f_2, p_1))d(f_2, f_1) \\ &\leq \omega(d(f_1, f_0), d(f_2, f_0) + d(f_1, f_0))d(f_2, f_1).\end{aligned}$$

Therefore, according to what we have proved above,

$$\|X(f_2)\| \leq \omega(R, R + R)d(f_2, f_1) \leq \omega(R, 2R)d(f_2, f_1). \tag{13}$$

Here we have used the fact that ω is non-decreasing in its two arguments.

On the other hand, by (2),

$$\begin{aligned}v_2 &= -[f_1, f_2, X]^{-1}(X(f_2)) \\ f_3 &= \exp_{f_2}(v_2).\end{aligned}$$

We conclude from (12) and (13) that

$$\begin{aligned}d(f_3, f_2) &= \|v_2\| \\ &\leq \|[f_1, f_2, X]^{-1}\|\|X(f_2)\| \\ &\leq \frac{\beta\omega(R, 2R)}{1 - \beta\omega(R + \alpha, R)} d(f_2, f_1),\end{aligned}$$

hence that,

$$d(f_3, f_2) = bd(f_2, f_1). \tag{14}$$

Finally, by (5), (9), (10), and (14), we deduce that

$$\begin{aligned}d(f_3, f_0) &\leq d(f_3, p_2) + d(f_2, f_0) \\ &\leq b\, d(f_2, f_1) + (a + 1)d(f_1, f_0) \\ &\leq ab\, d(f_1, f_0) + (a + 1)d(f_1, f_0) \\ &\leq (ba + a + 1)\eta,\end{aligned}$$

whence, in virtue of the equality

$$R = \left(\frac{ba}{1 - c} + a + 1\right)\eta,$$

we see that

$$d(f_3, f_0) = (ba + a + 1)\eta < R,$$

since $0 < c < 1$. □

Lemma 2. *Prove that*

(1) $d(f_n, f_0) < N1$;
(2) The function $[f_{n-1}, f_n, X]$ *is inverses*

$$\begin{aligned}\|[f_{n-1}, f_n, X]^{-1}\| &\leq \frac{\beta}{1 - \omega(d(f_{n-1}, f_{-1}), d(f_n, f_0))} \\ &\leq \frac{\beta}{1 - \beta\omega(R + \alpha, R)};\end{aligned} \tag{15}$$

(3) $\|X(f_n)\| \leq \omega(2N1, 2N1)d(f_n, f_{n-1})$;

(4) $d(f_{n+1}, f_n) \leq cd(f_n, f_{n-1})$,

for all $n \geq 3$.

Proof. The verification of the conditions 1–4 of Lemma 2 follows by induction on n. For the case $n = 3$, the condition $d(f_n, f_0) < R$ is obvious from Lemma 1. To prove 2 when $n = 3$, we consider a geodesic ϕ_3 satisfying $\phi_3(0) = f_0$ and $\phi_3(1) = f_3$. Then, from (3) we have

$$\|[f_2, f_3, X] \circ F_{\phi_{3},0,1} - F_{\phi_{3},0,1} \circ [f_{-1}, f_0, X]\| \leq \omega(d(f_2, f_{-1}), d(f_3, f_0))$$
$$\leq \omega(d(f_2, f_0) + d(f_0, f-1), d(f_3, f_0))$$

and so, by (4), (11), and Lemma 1, it may be concluded that

$$\|[f_2, f_3, X] \circ F_{\phi_{3},0,1} - F_{\phi_{3},0,1} \circ [f_{-1}, f_0, X]\| \leq \omega(\alpha + R, R).$$

Proceeding as in the proof of (7) one obtains that $[f_2, f_3, X]$ is invertible and

$$\|[f_2, f_3, X] - 1\| \leq \frac{\beta}{1 - \omega(d(f_2, f_{-1}), d(f_3, f_0))} \tag{16}$$
$$\leq \frac{\beta}{1 - \beta\omega(R + \alpha, R)}, \tag{17}$$

which is (15) for $n = 3$, and (3), it follows that

$$\|X(f_3)\| \leq \|[f_2, f_3, X] \circ F_{\lambda_{3},0,1} - F_{\lambda_{3},0,1} \circ [f_1, f_2, X]\|\|v_2\|$$
$$\leq \omega(d(f_2, f_1), d(f_3, f_2))d(f_3, f_2)$$
$$\leq \omega(d(f_2, f_0) + d(f_1, f_0), d(f_3, f_0) + d(f_2, f_0))d(f_3, f_2),$$

and consequently
$$\|X(f_3)\| \leq \omega(2R, 2R)d(f_3, f_2).$$
Hence, (2), and (16), we obtain

$$d(f_4, f_3) = \|v_3\|$$
$$\leq \|[f_2, f_3, X]^{-1}\|\|X(f_3)\|$$
$$\leq \frac{\beta\omega(2R, 2R)}{1 - \beta\omega(R + \alpha, R)} d(f_3, f_2),$$

which, by definition of c, yields
$$d(f_4, f_3) = c\, d(f_3, f_2).$$
This completes the proof for $n = 3$; if new case $n > 3$, Lemma 2 satisfies in $k = 3, \ldots, n$, we have

(1) $d(f_k, f_0) < R$;

(2) The given function $[f_{k-1}, f_k, X]$ is reciprocal to each other

$$\|[f_{k-1}, f_k, X]^{-1} \leq \frac{\beta}{1 - \omega(d(f_{k-1}, f_{-1}), d(f_k, f_0))}$$
$$\leq \frac{\beta}{1 - \beta\omega(R + \alpha, R)};$$

(3) $\|X(f_k)\| \leq \omega(2R, 2R)d(f_k, f_{k-1})$;

(4) $d(f_{k+1}, f_k) = c\, d(f_k, f_{k-1})$,

in $k \in \{3ton\}$, the procedure for Mathematical Induction $k = n + 1$, thus obtain,

$$d(f_{k+1}, f_k) \leq c^{k-2} d(f_3, f_2),$$

with,

$$\begin{aligned} d(f_{n+1}, f_0) &\leq d(f_{n+1}, f_n) + \cdots + d(f_3, f_2) + d(f_2, f_0) \\ &\leq \frac{1-c^{n-1}}{1-c} d(f_3, f_2) + d(f_2, f_0). \end{aligned}$$

$$\begin{aligned} d(f_{n+1}, f_0) &\leq \frac{1-c^{n-1}}{1-c} ab\eta + (a+1)\eta \\ &< \left(\frac{ab}{1-c} + a + 1\right)\eta \\ &= R. \end{aligned} \tag{18}$$

The point lies between ϕ_{n+1} with $\phi_{n+1}(0) = f_0$ to $\phi_{n+1}(1) = f_{n+1}$. Thus, the ω-property, (18), from Mathematical Induction, if prove that

$$\begin{aligned} \|[f_n, f_{n+1}, X] \circ F_{\phi_{n+1},0,1} - F_{\phi_{n+1},0,1} \circ [f_{-1}, f_0, X]\| &\leq \omega(d(f_n, f_{-1}), d(f_{n+1}, f_0)) \\ &\leq (d(f_n, f_0) + d(f_0, f_{-1}), d(f_{n+1}, f_0)) \\ &\leq \omega(\alpha + R, R). \end{aligned}$$

The reference as (7), it defined $[f_n, f_{n+1}, X]$ is each other to inverses

$$\begin{aligned} \|[f_n, f_{n+1}, X]^{-1}\| &\leq \frac{\beta}{1 - \omega(d(f_n, f_{-1}), d(f_{n+1}, f_0))} \\ &\leq \frac{\beta}{1 - \beta\omega(R + \alpha, R)}. \end{aligned}$$

$$\|X(f_{n+1})\| \leq \|[f_n, f_{n+1}, X] \circ F_{\lambda_n,0,1} - F_{\lambda_n,0,1} \circ [f_{n-1}, f_n, X]\| \|v_n\|,$$

$$\|X(f_{n+1})\| \leq \omega(2R, 2R) d(f_{n+1}, f_n).$$

$$v_k = -[f_{k-1}, f_k, X]^{-1}(X(f_k)) \quad \text{and} \quad d(f_{k+1}, f_k) = \|v\|, \quad k = 0, 1, \ldots,$$

$$\begin{aligned} d(f_{n+2}, f_{n+1}) &= \|[f_n, f_{n+1}, X]^{-1}\| \|X(f_{n+1})\| \\ &\leq \frac{\beta\omega(2R, 2R)}{1 - \beta\omega(R + \alpha, R)} d(f_{n+1}, p_n) \\ &\leq c\, d(f_{n+1}, f_n), \end{aligned}$$

Hence the proof is complete. □

Applications. Prove that the convergence at (p_n) is a Cauchy sequence,

$$d(f_{k+1}, f_k) \leq c^k d(f_1, f_0), \qquad k = 0, 1 \ldots,$$

hence, if $m < n$,

$$\begin{aligned} d(f_n, f_m) &\le d(f_n, f_{n-1}) + d(f_{n-1}, f_{n-2}) + \cdots + d(f_{m+1}, f_m) \\ &\le (c^{n-1} + c^{n-2} + \cdots + c^m)d(f_1, f_0). \end{aligned}$$

Since $c < 1$, we deduce that (f_n) be a series solution of $B[f_0, R]$, hence it is in $f_* \in B[f_0, R]$ such that (f_n) converges to f_*.

Now we show that f_* is a root of X. This follows directly by taking limits of both sides of the inequality

$$\|X(f_n)\| \le \omega(2R, 2R)d(f_n, f_{n-1}),$$

which is part of the conclusion of Lemma 2. To finish the proof, we prove that f_* is the unique root of X in $B[f_0, R]$. If there existed a $g_* \in B[f_0, R]$ such that $X(g_*) = 0$, we would have

$$\begin{aligned} \|[g_*, f_*, X] \circ F_{\phi,0,1} - F_{\phi,0,1} \circ [f_{-1}, f_0, X]\| &\le \omega(d(g_*, g_{-1}), d(f_*, f_0)) \\ &\le \omega(d(g_*, f_0) + d(f_0, f_{-1}), d(f_*, f_0)) \\ &\le \omega(\alpha + R, R), \end{aligned}$$

here ϕ is neighboring values $\phi(0) = f_0$ to $\phi(1) = f_*$. It obtains that $[g_*, f_*, X]$ is inverse. The α is a $\alpha(0) = g_*$ with $\alpha(1) = f_*$. Then by (1),

$$[\alpha(0), \alpha(1), X] \circ F_{\alpha,0,1}(\alpha'(0)) = X(\alpha(1)) - F_{\alpha,0,1}(X(\alpha(0))) = 0$$

$$[g_*, f_*, X] \circ F_{\alpha,0,1}(\alpha'(0)) = 0.$$

Since $[f_*, g_*, X]$ and $F_{\alpha,0,1}$ is inverse to each other, we obtain $\alpha'(0) = 0$. Hence $f_* = g_*$, and the proof is over. □

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