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Univariate Maintenance Model for a Deteriorating System under Two Monotone Processes

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Abstract

The maintenance model for a decaying system is investigated in this study, in which successive running times and restore times follow partial sum process and alpha- series process separately is considered. Assume that the system after restore cannot be "as good as new", and the deterioration of the system is stochastic. Under these assumptions, we use a replacement strategy N based on the failure quantity of the system. An analytically determined optimal replacement strategy N^* for minimizing the long run mean cost for every unit time is derived from an explicit expression for the long run mean cost for every unit time under Nstrategy. A numeric illustration is also given.

Keywords: Partial Sum Process, Alpha –Series Process, Replacement Strategy.

1 Introduction

The analysis of maintenance problem is always an essential topic in reliability. Since Lotka [1939] presented it, the replacement problem for a repairable system has gotten a lot of attention. Sridhar and Allah Pitchai (2015) concentrate on a M/M/2 queuing system with two heterogeneous servers and working vocation. They infer the steady – state solution of the model utilizing matrix geometric method. Barlow and Proschan(1965) involved the replacement strategy in which h the system will have a preventive replacement when the age of the system arrives at T, and a failure replacement when it fizzles, whichever happens prior, and the replacement time is thought to be unimportant. The issue is to pick an optimal replacement strategy T^* to such an extent that the long run mean cost for every unit time is minimized. Sridhar and Allah Pitchai (2014) concentrate on two server queuing system with heterogeneous bunch service and the articulation for the expected queue length. Sridhar and Allah Pitchai(2013) have considered a two server queuing system with a solitary and bunch service and permit the later arriving customers to join the groups of continuous help in server-1. When the system size is not exactly or equivalent as far as possible.

Park (1979) concentrated on the replacement strategy in which the system will have a replacement when the quantity of failures of the system arrives at N. The goal is to find the optimal replacement strategyN^* that minimizes the long run mean cost for every unit time. A typical suspicion in the replacement problem is that the system after failure will be all around great. Yet, many systems are decaying in practice because of maturing impact and the gathered wear. In like manner, the number of consecutive running times after failure will be diminishing, while the numbers of consecutive restore times after failure will increment. Subsequently a monotone model would be a generally suitable model for a decaying system. For such peculiarities, Braun(2005) presented an alpha- series process, and Babu[2020] presented a partial sum process. In Lam (1988b) paper, both running and restore times are accepted to follow geometric process. We assume that in this study for a further developing system , the successive running times follow a partial sum process and that consecutive restore times follow a alpha-series process.

The long run mean cost for every unit time under N strategy is given an explicit solution , and the optimal replacement strategy N^* for minimizing the long run mean cost for every unit time is resolved systematically. A mathematical model is likewise given toward the end.

The preliminary definitions and results about alpha- series process and partial sum process are given underneath.

Definition 1.1 Let $\{X_m,\ m = 1,2,3,4 \ldots\ldots\}$ be a sequence of independent non-negative random variables and let $F(x)$ be the distribution function of X_1. Then $\{X_{m+1},\ m = 1,2,3,4 \ldots\ldots\}$ is known as a partial sum process, in the event that the distribution function of

X_{m+1} is $F(\eta_m x, m = 1,2,3 \dots \dots\}$where $\eta_m > 0$are constants with $\eta_m = \eta_0 + \eta_1 + \eta_2 + \cdots \eta_{m-1}$ and $\eta_m = \eta > 0$.

Lemma 1.1. For real $\eta_m\,(m = 1,2,3, \dots \dots.)\,, \eta_m = 2^{m-1}\eta$. (See e.g. Babu[2020])

Proof. When$m = 1\,, \eta_1 = \eta_0 = \eta$, As a result, the result is correct for $m = 1$.

Assume that the result is correct for $m = n$.

$$\eta_{n+1} = (\eta_0 + \eta_1 + \eta_2 + \cdots \eta_{n-1}) + \eta_n$$

$$= 2\eta_n$$

$$= 2(2^{n-1}\eta)(by\ induction\ assumption)$$

$$= 2^{(n+1)-1}\eta$$

As, a result, the result is correct for $m = n + 1$ also .

Thus the distribution function of X_{m+1} is $F(2^{m-1}\eta)\ \ for\ m = 1,2,3, \dots ..$

Therefore the density function of X_{m+1} is $f_{m+1}(x) = \eta_m f(\eta_m x)$.

Confirming the accompanying lemma is simple.

Lemma 1.2. Let $E(X_1) = \gamma$ then for $m = 1,2,3, \dots \dots$ then $E(X_{m+1}) = \frac{\gamma}{2^{m-1}\eta}$

Proof. $E(X_{m+1}) = \int_{-\infty}^{\infty} x f_{m+1}(x)dx$

$$= \int_{-\infty}^{\infty} x\ \eta_m f(\ \eta_m x)dx$$

Put $y = \eta_m x \Longrightarrow x = \frac{y}{\eta_m}$

$$dx = \frac{1}{\eta_m}dy$$

$$E(X_{m+1}) = \int_{-\infty}^{\infty} \frac{y}{\eta_m} f(y)\, \eta_m \frac{1}{\eta_m} dy$$

$$= \frac{1}{\eta_m} \int_{-\infty}^{\infty} y f(y) dy$$

$$= \frac{1}{\eta_m} E(Y)$$

$$= \frac{1}{\eta_m} E(X_1)$$

$$= \frac{\gamma}{\eta_m} = \frac{\gamma}{2^{m-1}\eta}$$

Therefore

$$E(X_{m+1}) = \frac{\gamma}{2^{m-1}\eta}$$

for, $m = 1,2,3 \dots \dots$

Lemma 1.3. The partial sum process $\{X_m, m = 1,2,3, \dots \dots\}$ with parameter $\eta(> 0)$ stochastically decreasing and hence it is a monotone process. (See e.g. Babu[2020])

Proof. Note that for any $\delta \geq 0$,

$$F(\delta) \leq F(\eta\delta) \leq F(2\eta\delta) \leq \cdots \dots \dots \leq F(2^{m-1}\eta\delta)$$

$$P(X_1 > \delta) \geq P(X_2 > \delta) \geq P(X_3 > \delta) \dots \dots \dots . P(X_m > \delta)$$

This implies that $\{X_m, m = 1,2,3, \dots \dots\}$ is stochastically decreasing.

Definition 1.2. Given a sequence of non-negative random variables $\{Y_m, m = 1,2,3, \dots \dots\}$, if they are independent and the distribution function of Y_m is given by $G(m^{\alpha}y)$ for $m = 1,2,3, \dots \dots$ where α is a real number, then $\{Y_m, m = 1,2,3, \dots \dots\}$ is called an alpha-series process.

We now state below the following results which are required for our discussion.

Result 1.1 Given a alpha-series process $\{Y_m, m = 1,2,3, \dots \dots\}$

If $\alpha < 0$, then $\{Y_m, m = 1,2,3, \dots \dots\}$ is stochastically increasing.

Result 1.2. Let $E(Y_1) = \lambda$. Then $E(Y_m) = \frac{\lambda}{m^\alpha}$ (see e.g. Braun[2005])

2 Model Presumptions.

Under the accompanying presumptions, we consider the maintenance model for a decaying system.

A1. A fresh system is initially installed. When a system fails, it can be replaced with a new one that is identical.

A2. Let X_1 be the running time before the 1^{st} failure and let $F(x)$ be the distribution function of X_1. Let X_{m+1} be the running time after the m repair for $m = 1,2,3, \ldots ...$ Then the distribution function of X_{m+1} is $(\eta_m x)$, where $\eta_m > 0$ are constant. That is the successive running times $\{X_{m+1}, m = 1,2,3, \ldots \ldots ..\}$ after restore constitute a decreasing partial sum process. Also assume that $E(X_1) = \gamma > 0$ and $\gamma_{m+1} = E(X_{m+1}) = \frac{\gamma}{2^{m-1}\eta}$.

A3. Let Y_1 be the restore time after the 1^{st} failure and let $G(y)$ be the distribution function of Y_1. Let Y_m be the restore times after m failure. Then the distribution function of Y_m is $G(m^\alpha y)$ for $m = 1,2,3, \ldots \ldots$ where $\alpha < 0$ is a real number and $\lambda_m = E(Y_m) = \frac{\lambda}{m^\alpha}$ .That is the consecutive restore times $\{Y_m, m = 1,2,3, \ldots \ldots\}$ form an increasing alpha-series process.

A4. The running times $\{X_m, m = 1,2,3, \ldots \ldots\}$ and restore times $\{Y_m, m = 1,2,3, \ldots \ldots\}$ are independent.

A5. Let Z be the time of replacement with $E(Z) = \tau$.

A6. The maintenance cost is c, the award rate is r and the replacement cost is R.

3 Replacement strategy N

Definition3.1. A replacement strategy N is an approach wherein we replace the system at N^{th} failure of the system. Our goal is to discover an optimal replacement strategy N^* at minimizes the long run mean cost for every unit time.

By the Renewal reward theorem, (see e.g. Ross[4]) , the long run mean cost for every unit time under strategy$N$ is given by

$$C(N) = \frac{The expected cost incurred in a cycle}{The expected length of a cycle}$$
$$= \frac{E(c\sum_{m=1}^{N-1} Y_m + R - r\sum_{m=1}^{N} X_m)}{E(\sum_{m=1}^{N} X_m + \sum_{m=1}^{N-1} Y_m + Z)}$$
$$= \frac{c\sum_{m=1}^{N-1} E(Y_m) + R - r\sum_{m=1}^{N} E(X_m)}{\sum_{m=1}^{N} E(X_m) + \sum_{m=1}^{N-1} E(Y_m) + E(Z)}$$

Thus,

$$C(N) = \frac{c\sum_{m=1}^{N-1} \lambda_m + R - r\sum_{m=1}^{N} \gamma_m}{\sum_{m=1}^{N} \gamma_m + \sum_{m=1}^{N-1} \lambda_m + \tau}$$

(1)

4 Optimal Replacement Strategy N^*.

We will decide N^* for minimizing $C(N)$ under N strategy. From equation (1)

$$C(N) = \frac{(c+r)\sum_{m=1}^{N-1} \lambda_m + R + r\tau}{\sum_{m=1}^{N} \gamma_m + \sum_{m=1}^{N-1} \lambda_m + \tau} - r \tag{2}$$

To get N^*, we want to explore the distinction between $C(N+1)$ and $C(N)$

$$C(N+1) - C(N) = \left(\frac{(c+r)\sum_{m=1}^{N} \lambda_m + R + r\tau}{\sum_{m=1}^{N+1} \gamma_m + \sum_{m=1}^{N} \lambda_m + \tau} - r\right) - \left(\frac{(c+r)\sum_{m=1}^{N-1} \lambda_m + R + r\tau}{\sum_{m=1}^{N} \gamma_m + \sum_{m=1}^{N-1} \lambda_m + \tau} - r\right)$$
$$= \left(\frac{(c+r)\sum_{m=1}^{N} \lambda_m + R + r\tau}{\sum_{m=1}^{N+1} \gamma_m + \sum_{m=1}^{N} \lambda_m + \tau} - \frac{(c+r)\sum_{m=1}^{N-1} \lambda_m + R + r\tau}{\sum_{m=1}^{N} \gamma_m + \sum_{m=1}^{N-1} \lambda_m + \tau}\right)$$

After Simplifications **,** we get

$$C(N+1) - C(N)$$
$$= \frac{\begin{pmatrix} (c+r)\lambda\left(2^{N-1}\eta\left(\gamma + \sum_{m=2}^{N} \frac{\gamma}{2^{m-2}\eta}\right) - N^\alpha \gamma \sum_{m=1}^{N-1} \frac{1}{m^\alpha} + \tau 2^{N-1}\eta\right) - \\ (R + r\tau)(\gamma N^\alpha + \lambda 2^{N-1}\eta) \end{pmatrix}}{\left(2^{N-1}\eta N^\alpha\left(\left(\gamma + \sum_{m=2}^{N+1} \frac{\gamma}{2^{m-2}\eta}\right) + \sum_{m=1}^{N} \frac{\lambda}{m^\alpha} + \tau\right)\left(\left(\gamma + \sum_{m=2}^{N} \frac{\gamma}{2^{m-2}\eta}\right) + \sum_{m=1}^{N-1} \frac{\lambda}{m^\alpha} + \tau\right)\right)} \tag{3}$$

Let

$$B(N)=\frac{(c+r)\lambda\left(2^{N-1}\eta\left(\gamma+\sum_{m=2}^{N}\frac{\gamma}{2^{m-2}\eta}\right)-N^{\alpha}\gamma\sum_{m=1}^{N-1}\frac{1}{m^{\alpha}}+\tau 2^{N-1}\eta\right)}{(R+r\tau)(\gamma N^{\alpha}+\lambda 2^{N-1}\eta)} \tag{4}$$

Now, we prove the following Lemma.

Lemma 4.1.

$$C(N+1)>C(N)\Leftrightarrow B(N)>1$$

$$C(N+1)=C(N\ )\Leftrightarrow B(N)=1$$

$$C(N+1)<C(N)\ \Leftrightarrow B(N)<1$$

Proof.

From Equation (3), the indication of $C(N+1)-C(N)$is plainly equivalent to the indication of its numerator. Since the denominator of $C(N+1)-C(N)$ is always positive.

Lemma 4.2.$B(N)$is non-diminishing in N.

Proof. From equation (4),

Let

$$\zeta(N)=\frac{(c+r)\lambda}{(R+r\tau)(\gamma N^{\alpha}+\lambda 2^{N-1}\eta)(\gamma(N+1)^{\alpha}+\lambda 2^{N}\eta)}$$

Now,

$B(N+1)-B(N)$

$$=C(N)\left(\begin{array}{c}(\gamma N^{\alpha}+\lambda 2^{N-1}\eta)\left(2^{N}\eta\left(\gamma+\displaystyle\sum_{m=2}^{N+1}\frac{\gamma}{2^{m-2}\eta}\right)-(N+1)^{\alpha}\gamma\displaystyle\sum_{m=1}^{N}\frac{1}{m^{\alpha}}+\tau 2^{N}\eta\right)-\\(\gamma(N+1)^{\alpha}+\lambda 2^{N}\eta)\left(2^{N-1}\eta\left(\gamma+\displaystyle\sum_{m=2}^{N}\frac{\gamma}{2^{m-2}\eta}\right)-N^{\alpha}\gamma\displaystyle\sum_{m=1}^{N-1}\frac{1}{m^{\alpha}}+\tau 2^{N-1}\eta\right)\end{array}\right)$$

$$= \zeta(N)\left(\begin{array}{c} \left(\gamma^2\eta + \lambda\eta\gamma\ \sum_{m=1}^{N-1}\frac{1}{m^\alpha}\right)(N^\alpha 2^N - (N+1)^\alpha 2^{N-1}) + \\ \left(\eta\gamma\sum_{m=2}^{N}\frac{\gamma}{2^{m-2}\eta} + \eta\gamma\tau\right)\left((N^\alpha 2^N - (N+1)^\alpha 2^{N-1})\right) + \\ \gamma^2(2N^\alpha - (N+1)^\alpha) + \lambda\eta\gamma(2^N - 2^{N-1}(N+1)^\alpha N^\alpha) \end{array}\right)$$

$$= \zeta(N)\left(\begin{array}{c} (N^\alpha 2^N - (N+1)^\alpha 2^{N-1})\left(\left(\gamma^2\eta + \lambda\eta\gamma\ \sum_{m=1}^{N-1}\frac{1}{m^\alpha}\right) + \left(\eta\gamma\sum_{m=2}^{N}\frac{\gamma}{2^{m-2}\eta} + \eta\gamma\tau\right)\right) + \\ \left(\gamma^2(2N^\alpha - (N+1)^\alpha) + \lambda\eta\gamma(2^N - 2^{N-1}(N+1)^\alpha N^\alpha)\right) \end{array}\right)$$

After Simplifications, we get

$$B(N+1) - B(N)$$

$$= \frac{(c+r)\lambda\left(\begin{array}{c} (N^\alpha 2^N - (N+1)^\alpha 2^{N-1})\left(\left(\gamma^2\eta + \lambda\eta\gamma\ \sum_{m=1}^{N-1}\frac{1}{m^\alpha}\right) + \left(\eta\gamma\sum_{m=2}^{N}\frac{\gamma}{2^{m-2}\eta} + \eta\gamma\tau\right)\right) + \\ \left(\gamma^2(2N^\alpha - (N+1)^\alpha) + \lambda\eta\gamma(2^N - 2^{N-1}(N+1)^\alpha N^\alpha)\right) \end{array}\right)}{(R + r\tau)(\gamma N^\alpha + \lambda 2^{N-1}\eta)(\gamma(N+1)^\alpha + \lambda 2^N\eta)}$$

$B(N)$is non-diminishing in N for $\alpha < 0$ and $\eta > 0$

Finally, we prove the main theorem of this section.

Theorem 4.1. The optimal replacement strategy N^* is chosen by

$$N^* = \min\{N/B(N) \geq 1\}$$

Moreover, N^* is unique if and only if$B(N^*) > 1$

Proof. We note that $C(N)$ vhas a minimum value when$C(N+1) \geq C(N)$. Thus

$$N^* = \min\{N\ /\ C(N) \geq C(N)\}$$

$= \min\{N\ /\ B(N) \geq 1\}$(By lemma (4.1)).

Since $B(N)$ is non- diminishing in N,there exist an integer N^* such that

$B(N) \geq 1 \Leftrightarrow N \geq N^*$ And $B(N) < 1 \Leftrightarrow N < N^*$

Thus N^* is unique if and only if $B(N^*) > 1$

5 Numeric Illustration

In this segment, we give a guide to represent the hypothetical result.

Let the boundary values be $\alpha = -0.98$, $\eta = 1.5, R = 5500$, $\lambda = 3$, $\gamma = 50, c = 15, r = 45$ and $\tau = 10$.Substituting the above values into (2) and (4) ,the mathematical outcomes are introduced in Table 1 and Table 2 and the comparing figures are plotted in Figure 1 and Figure 2 separately.

N	$C(N)$	N	$C(N)$
1	54.16666667	6	5.94965438
2	18.63321800	7	6.50660318
3	9.53402094	8	7.20364793
4	6.54754439	9	7.92208448
5	**5.78302555**	10	8.60721304

Table1: The values of $C(N)$ for various values of N.

N	$B(N)$	N	$BN)$
1	0.04995759772	6	1.160150181
2	0.2242781753	7	1.219427129
3	0.5261412216	8	1.248640434
4	0.8323471117	9	1.263009858
5	**1.043157389**	10	1.270124784

Table2: The values of $B(N)$ for various values of N.

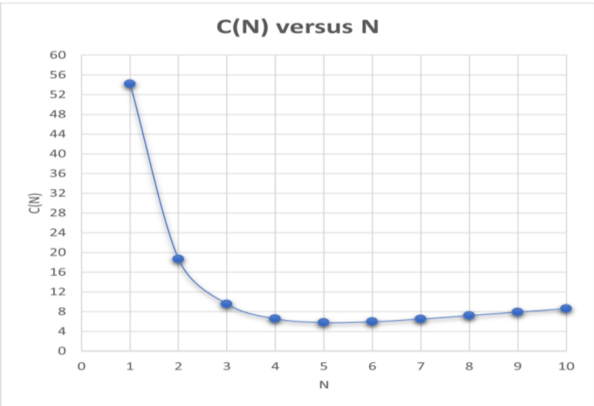

Figure1 : $C(N)$ verses N

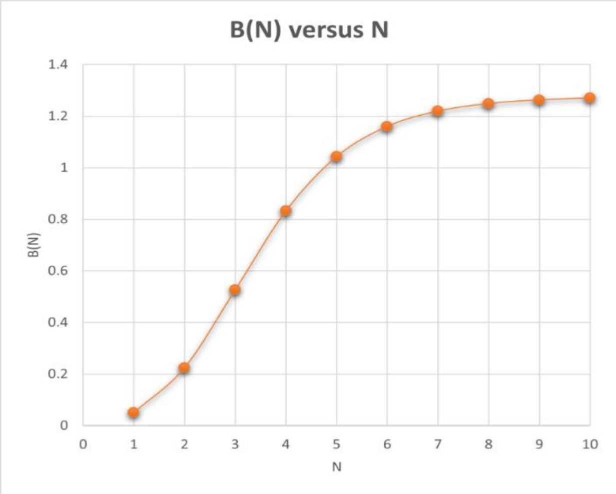

Figure 2: $B(N)$versus N

Clearly $C(5) = 5.78302555$ is the minimum of the mean cost. On the other hand $B(N) = 1.043157389 > 1$and $\min\{N / B(N)\} = 5$. Therefore, the unique optimal replacement strategy for the further developing system is$N^* = 5$, the system ought to be replaced at the time of the 5^{th} failure.

6 Conclusion

By examining a maintenance model for a decaying system with increasing running times, an explicit expression for the long run mean cost for every unit time under N is derived using partial sum process and successive restore times following the alpha -series process. Analytically, the optimal replacement strategy N^* for minimizing long run mean cost for every unit time is found. Naturally, it is an overall information that the more seasoned the further developing system is, the better the system is. This implies that we shall repair the system when it fails without replacement. A mathematical model is given to outline the system created.

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