Coefficient of Variation Based Goodness–of–Fit Tests for the Two–Parameter Weibull Distribution

Ahmad Zghoul , Ahmed Alyaseen

Department of Mathematics, College of Science, The University of Jordan, Amman, 11942, Jordan

Ahmedyaseen886@gmail.com

Article Info
Page Number: 2233-2247
Publication Issue:
Vol. 71 No. 4 (2022)

Article History
Article Received: 25 March 2022
Revised: 30 April 2022
Accepted: 15 June 2022
Publication: 19 August 2022

Abstract

This article addresses Two-Parameter Weibull goodness-of-fit tests. First, tests based on the coefficient of variation of the log-transformed Weibull distribution are introduced. Then, the powers of the introduced tests are compared to empirical distribution-based tests, normalized spacing-based tests, and other types of tests. The results show that the simulated powers of the introduced tests compete well with the powers of previous tests.

Keywords: Weibull distribution, goodness-of-fit, empirical distribution-based tests, spacing-based tests, coefficient of variation-based tests.

1 **Introduction**

Because of its attractive mathematical features and numerous applications in various fields, Weibull distribution has attracted the attention of statisticians and practitioners from different disciplines. Besides Weibull distribution is the most widely used lifetime distribution; it has been exploited to model data from various fields such as economics, physics, medical sciences, ecology, biology, and climatology. This article aims to compare different known test procedures and present new goodness-of-fit tests to test the conformity of sample data to a two-parameter Weibull model.

Over the last forty years, many research articles tackled the problem of goodness-of-fit for Weibull distribution. Usually, these tests are based on some characteristics of the distribution. For example, Mann et al. (1973) developed a test statistic for two-parameter Weibull distribution based on normalized spacing. Many other tests are based on spacing and leaps, such as Mann and Fertig (1975a), Littell and others (1979), Tiku and Singh (1981), Lockhart and others (1986), and Gibson and Higgins (2000) tests.

The Shapiro-Wilk normality test, which is based on a ratio of two estimates of the population variance, was modified by Shapiro and Brain (1987) and then further modified by Öztürk and Korukoglu (1988) to test for Weibull. Furthermore, Coles (1989) proposed a Weibull goodness-of-fit test based on the sample correlation between the order statistics of a sample and their corresponding expectations.

Well-known test statistics, such as Kolmogorov-Smirnov (D), Cramer-von Mises (W^2), and Anderson Darling (A^2), have been modified to be applied for the Weibull distribution. For example, Bush and others (1983) modified W^2 and A^2 statistics and computed their corresponding critical values, Khamis (1997) modified the D statistic, and Liao and Shimokawa (1999) introduced a test combining D, $W^{2,}$ and A^2 tests.

Many authors have presented directed tests of Weibull distribution against a specified alternative distribution. For example, Hager and others (1971), Engelhardt and Bain (1975), and Gupta and Kundu (2003) proposed tests to test the Weibull against exponential distribution. Moreover, tests handling the gamma distribution as an alternative have been proposed, among others, by Bain and Engelhardt (1980b), Kappenman (1982), and Chen (1987). Furthermore, Fearn and Nebenzahl (1991), Balasooriya and Abeysinghe (1994), Dumonceaux et al. (1973b), and Kappenman (1982, 1988) suggested tests for the Weibull against the lognormal alternative.

In Section 2 of this article, we propose goodness-of-fit tests for two-parameter Weibull distribution based on the coefficient of variation (C.V.) of the logarithmic transformation of the Weibull variable. The distributions and asymptotic distributions of the proposed tests are investigated in Section 3. As with most goodness-of-fit test statistics, the distributions of the proposed tests do not have closed forms. Therefore, simulations are carried out in Section 3 to compute selected percentiles for the proposed statistics. In Section 4, selective quantiles of each of Kolmogorov-Smirnov, Cramer-von Mises, Anderson-Darling, Mann-Scheurer-Fertig, Liao-Shimokawa, Smith-Bain, and Shapiro-Brain statistics are simulated for samples of sizes 10, 20, and 50. Finally, the suggested tests are compared to the abovementioned tests in terms of their powers.

2 Test Statistics Based on the coefficient of variation

Let t G $= \left\{G(\mathrm{x}, \underline{\theta}), x \in (0,\infty), \underline{\theta} = (\beta,\tau) \in (0,\infty)^2\right\}$ be the family of two-parameter

Weibull distribution with respective distribution and density functions

$$G\left(x;\tau,\beta\right) = 1 - \exp\{-\left(\tfrac{x}{\beta}\right)^{\tau}\}, \tag{2.1}$$

$$g\left(x;\tau,\theta\right) = \frac{\tau}{\beta}\left(\frac{x}{\beta}\right)^{\tau-1} \exp\{-\left(\tfrac{x}{\beta}\right)^{\tau}\}, x > 0; \beta > 0, \tau > 0. \tag{2.2}$$

Assume that $X_1,\ldots,X_n$ is a random sample with distribution function $\mathrm{F}(x,\underline{\theta})$, where $\underline{\theta}$ is a vector of parameters. To test $H_0 : F(\underline{X},\underline{\theta}) \in$ G versus $H_1 : F(\underline{X},\underline{\theta}) \notin$ G, we develop test statistics that are based on the coefficient of variation of $X^* = log(X)$.

let $Y_i = log(X_i/\theta)$, $i = 1,2,...,n$. It can be shown that under the null hypothesis, each of Y_i, i=1,..,n, has the Extreme-Value distribution with scale parameter $(1/\tau)$ and location zero,

hence $E[Y_i] = -\frac{\gamma}{\tau}$, and for $i = 1,¼, n$, $Var[Y_i] = Var[\log(X_i/\theta)] = Var(\mathrm{X}_i^*) = \frac{\pi^2}{6\tau^2}$

,

where γ is the Euler constant, $\gamma \approx 0.577216$. Thus, the coefficient of variation of Y_i is

$$\frac{E(Y_i)}{\sqrt{Var(Y_i)}} = \frac{-(\gamma/\tau)}{\sqrt{6\pi^2/\tau^2}} = \frac{-\gamma}{\sqrt{6}\pi}.$$

Obviously, $\frac{-\bar{Y}_n}{\gamma}$ is an unbiased estimator of $(1/\tau)$ and $\frac{6S_n^2}{\pi^2}$ is an unbiased estimator of $\frac{1}{\tau^2}$, where $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$, $\bar{X}_n^* = \frac{1}{n}\sum_{i=1}^{n} log(X_i)$ and $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i^* - \bar{X}_n^*)^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2$.

By the laws of large numbers, $\bar{Y}_n \to -\gamma\tau^{-1}$ and $S_n^2 \to \frac{\pi^2}{6}\tau^{-2}$, hence $\frac{\pi^2}{6\gamma^2}\frac{\bar{Y}_n^2}{S_n^2} \to 1$.

Based on the statistics $\bar{Y}_n^2$ and S_n^2, several test statistics may be suggested. The two-sided tests $T_{1,n}$ and $T_{2,n}$ defined below, are just two possibilities $T_{1,n} = \sqrt{n}\left|\frac{\bar{Y}_n}{S_n} - \frac{\sqrt{6}\gamma}{\pi}\right|$ and

$$T_{2,n} = \sqrt{n}\left|\frac{\pi^2}{6\gamma^2}\frac{\bar{Y}_n^2}{S_n^2} - 1\right|$$

Each of $T_{1,n}$ and $T_{2,n}$ measures deviations of the sample coefficient of variation from its asymptotic limit.

In most real life applications, the parameters β and τ are unknown and must be estimated from the sample. As a matter of fact, the distributions of $T_{1,n}$ and $T_{2,n}$, and consequently their powers, depend on the method of estimation. The maximum likelihood (ML) and a hybrid method of estimation will be used to estimate the unknown parameters.

The two likelihood equations are

$$-\frac{n\tau}{\beta} + \frac{\tau}{\beta}\sum_{i=1}^{n}\left(\frac{X_i}{\beta}\right)^{\tau} = 0$$

$$\frac{n}{\tau} - n\log\beta + \sum_{i=1}^{n}\log(X_i) - \sum_{i=1}^{n}\left(\frac{X_i}{\beta}\right)^{\tau}\log\left(\frac{X_i}{\beta}\right)^{\tau} = 0$$

Explicit solution is not possible, so some numerical methods must be used to compute estimates of β and τ based on a given sample.

Using $Var(\log(X_i)) = \frac{\pi^2}{6\tau^2}$, the moment estimator of τ is

$\hat{\tau} = \dfrac{\sqrt{6}\pi}{S_n}$. Equation (1) can be rewritten as $\beta = \left(\dfrac{1}{n}\sum_{i=1}^{n} X_i^{\tau}\right)^{1/\tau}$. Plugging in the above moment estimator of τ, we get $\hat{\beta} = \left(\dfrac{1}{n}\sum_{i=1}^{n} X_i^{\hat{\tau}}\right)^{1/\hat{\tau}}$. In the sequel, we will refer to $\hat{\tau}$ and $\hat{\beta}$ as hybrid estimators of τ and β, respectively.

1 Asymptotic Distributions of $\mathbf{T}_{1,n}$ and $\mathbf{T}_{2,n}$ when θ is known

Let $\mathbf{X} = (X_1,\ldots,X_n)'$ be a vector of iid random variables each distributed Weibull(τ,θ), where θ is assumed to be known, and let $W_n = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} Y_i \\ \frac{1}{n}\sum_{i=1}^{n} Y_i^2 \end{pmatrix}$ where $Y_i = \mathrm{Log}(X_i/\theta)$; i=1, 2, …, n, we then have

$$E[W_n] = \begin{pmatrix} E(Y_1) \\ E(Y_1^2) \end{pmatrix} \text{ and } \mathrm{Cov}[\sqrt{n}W_n] = \begin{pmatrix} \mathrm{Var}(Y_1) & \mathrm{Cov}(Y_1, Y_1^2) \\ \mathrm{Cov}(Y_1, Y_1^2) & \mathrm{Var}(Y_1^2) \end{pmatrix}. \qquad (3.6)$$

If $\mu_r' = E(Y_1^r), r = 1, 2, \ldots,$ are the rth non-central moment of Y_1, then it can be shown that

$$\mu_1' = -\frac{\gamma}{\tau},\ \mu_2' = \frac{\gamma^2}{\tau^2} + \frac{\pi^2}{6\tau^2},\ \mu_3' = -\left(\frac{2\xi(3)}{\tau^3} + \frac{\pi^2}{2\tau^3} + \frac{\gamma^3}{\tau^3}\right), \text{and}$$

$$\mu_4' = \gamma^4 + \gamma^2\pi^2 + \frac{3\pi^4}{20} + \frac{\pi^2}{2\tau^3} + 8\gamma\xi(3)$$

where $\xi(3)$ is the Apéry's constant approximately equals to 1.2020569.

Thus, $E(Y_1) = \dfrac{-\gamma}{\tau}$, $E(Y_1^2) = \dfrac{\gamma^2}{\tau^2} + \dfrac{\pi^2}{6\tau^2}$, $\mathrm{Var}(Y_1) = \dfrac{\pi^2}{6\tau^2}$,

$$\mathrm{Var}(Y_1^2) = \mu_4' - \mu_2'^2 = \frac{11\pi^4 + 720\xi(3)\gamma + 60\pi^2\gamma^2}{90\tau^4}, \text{ and}$$

$$\mathrm{Cov}(Y_1, Y_1^2) = \mu_3' - \mu_1'\mu_2' = -\frac{6\xi(3) + \pi^2\gamma}{3\tau^3}.$$

Replacing γ and $\xi(3)$ by their approximate values, we have

$$E(W_n) = \begin{pmatrix} \dfrac{-0.577216}{\tau} \\ \dfrac{1.978112}{\tau^2} \end{pmatrix} \text{ and } \mathrm{Cov}[\sqrt{n}W_n] \approx \begin{pmatrix} \dfrac{1.644934}{\tau^2} & \dfrac{-4.303077}{\tau^3} \\ \dfrac{-4.303077}{\tau^3} & \dfrac{19.648547}{\tau^4} \end{pmatrix}.$$

By the multivariate version of the central limit theorem, we have, as $n \to \infty$,

$$\sqrt{n}\begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} Y_i - \frac{-0.577216}{\tau} \\ \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \frac{1.978112}{\tau^2} \end{pmatrix} \to N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1.644934}{\tau^2} & \frac{-4.303077}{\tau^3} \\ \frac{-4.303077}{\tau^3} & \frac{19.648547}{\tau^4} \end{pmatrix}\right). \qquad (3.7)$$

Let $g(u,v) = \frac{cu^2}{v-u^2}$, where $c = \frac{\pi^2(n\text{-}1)}{6\gamma^2 n}$. Then, we have $g(\bar{Y}, \frac{1}{n}\sum_{i=1}^{n} Y_i^2) = \frac{c\bar{Y}^2}{\frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \bar{Y}^2}$,

therefore

$$\left.\frac{\partial g(u,v)}{\partial u}\right|_{(\mu_1', \mu_2')} = \frac{2c\mu_1'\mu_2'}{\sigma^4} \text{ and } \left.\frac{\partial g(u,v)}{\partial v}\right|_{(\mu_1', \mu_2')} = \frac{-c\mu_1'^2}{\sigma^4}. \qquad (3.8)$$

Upon simplifying, we obtain $\left.\left(\frac{\partial g}{\partial u} \quad \frac{\partial g}{\partial v}\right)\right|_{(\mu_1', \mu_2')} = \left(-\frac{4.166720(n-1)\tau}{n} \quad -\frac{0.607927(n-1)\tau^2}{n}\right)$

. (3.9)

Thus, by the delta method, we have

$\sqrt{n}\left(\frac{c\bar{Y}_n^2}{S_{y,n}^2} - 1\right) \to N(0, 14.020305)$. Hence

$$P(T_{1,n} \le t) \to 2\Phi\left(\frac{t}{\sqrt{14.020305}}\right) - 1 \text{ and } P(T_{2,n} \le t) \to \Phi\left(\frac{t}{\sqrt{14.020305}}\right), \qquad (3.10)$$

where $\Phi(z)$ signifies the standard normal distribution function.

In practice, the parameters, , are unknown. In this case, we modify the tests $T_{1,n}$ and $T_{2,n}$ to be

$$T_{1,n} = \left|\frac{\pi^2}{6\gamma^2}\frac{\bar{Y}_n^2}{S_{Y,n}^2} - 1\right| \text{ and } T_{2,n} = \frac{\pi^2}{6\gamma^2}\frac{\bar{Y}_n^2}{S_{Y,n}^2} - 1,$$

where $Y_i = (X_i / \hat{\theta}), i = 1, \ldots, n$.

The distributions of these statistics may not have closed forms, further it is expected that the convergence of the asymptotic distributions to be slower than that in the case of known parameters. This lets simulated quantiles to be our best choice.

To simulate quantiles, we first simulate a random sample of specific sample size n from a Weibull distribution with some shape and scale parameters. Then, the maximum likelihood method is applied to estimate both of the parameters from the simulated sample. We transform the sample values $(x_1, \ldots, x_n)$ into $(y_1, \ldots, y_n)$, where $y_i = \log(x_i / \hat{\theta}), i = 1, \ldots, n$, and $\hat{\theta}$ is the MLE of θ. After that, we compute the values of $T_{1,n}$ and $T_{2,n}$ from the y sample. This procedure is repeated 20,000 times, for each of which $T_{1,n}$ and $T_{2,n}$ are computed, then all computed values are sorted and specified quantiles are located. It is noticed that the Y's are θ free and each of $T_{1,n}$ and $T_{2,n}$ is τ free. Thus, the tests are invariant of the choices of θ and τ. Simulated quantiles of $T_{1,n}$ and $T_{2,n}$, for the unknown parameters case, are displayed in Tables A1 and A2, respectively.

2 Power Comparisons

The simulated powers of $T_{1,n}$ and $T_{2,n}$ will be compared to tests based on EDFs, spacing, and on ratios of estimators of τ. First, we conduct a comparison among the following three EDFs based tests:

1. Kolmogorov-Smirnov D test, where $D = max\left[\max_j\left(\frac{j}{n} - F(z_{(j)}) \right), \max_j\left(F(z_{(j)}) - \frac{j-1}{n} \right) \right]$, (4.1)

where F is the distribution function of Weibull(1,1) variate and $z_{(j)}, j = 1,2,...,n$ is the jth ordered standardized sample value; that is $Z_{(j)} = \left(\frac{X_{(j)}}{\hat{\theta}} \right)^{\hat{\tau}}$.

The null hypothesis is rejected when D is large.

2. Cramer-von Mises W^2 test with computational form

$$W^2 = \frac{1}{12n} + \left(\sum_{j=1}^{n} \frac{2j-1}{2n} - F(z_{(j)}) \right)^2, \quad (4.2)$$

The hypothesis is rejected for large values of W^2.

3. Anderson-Darling A^2 test with computational form

$$A^2 = -n - \frac{1}{n} \sum_{j=1}^{n} [(2j-1)log(F(z_{(j)}) + (2n-2j+1)log(1-F(z_{(j)}))] . \quad (4.3)$$

The hypothesis is rejected when A^2 is large.

The simulated critical points of these tests, and other tests to be discussed later, with nominal values α = 0.1, 0.05, and 0.01 and sample sizes n=10, 20, 30, 50, and 100 are displayed in Table 2. These values are obtained based on 20,000 simulated samples. Also, the simulated powers of the above three tests are computed when testing the two unknown parameters Weibull distribution against wide spectrum of Alternative distributions. For comparison purposes, only α =0.05 is considered. The power is computed as the proportion of rejections in 10,000 samples from alternative distribution.

Table 2. Critical values of considered tests' statistics based on ML estimates.

n	**α**	D	W^2	A^2	MSF	SB	LS	$T_{1,n}$	$T_{2,n}$		SHBR	
10	0.1	0.241	0.100	0.622	2.504	0.145	1.151	0.229	-.26	0.198	-	1.47
20		0.174	0.100	0.623	1.823	0.097	1.013	0.204	-.22	0.192	1.15	1.58
30		0.144	0.102	0.628	1.605	0.074	0.956	0.183	-.20	0.177	-	1.64

50 100		0.113 0.081	0.102 0.102	0.634 0.629	1.422 1.266	0.053 0.033	0.893 0.824	0.156 0.121	-.17 -.13	0.150 0.116	1.31 - 1.37 - 1.44 - 1.48	1.61 1.57
10 20 30 50 100	0.05	0.263 0.189 0.156 0.123 0.088	0.120 0.120 0.123 0.124 0.122	0.730 0.726 0.751 0.759 0.753	3.299 2.129 1.813 1.558 1.353	0.180 0.120 0.094 0.068 0.042	1.253 1.094 1.036 0.970 0.896	0.282 0.246 0.219 0.187 0.145	-.31 -.27 -.25 -.21 -.16	0.234 0.222 0.205 0.173 0.134	- 1.33 - 1.52 - 1.61 - 1.68 - 1.75	1.81 1.96 1.97 1.92 1.92
10 20 30 50 100	0.01	0.300 0.219 0.180 0.144 0.102	0.169 0.171 0.170 0.177 0.179	0.992 1.004 1.013 1.051 1.055	5.714 2.916 2.312 1.870 1.526	0.256 0.181 0.142 0.110 0.074	1.539 1.313 1.210 1.140 1.059	0.389 0.344 0.313 0.267 0.207	- .410 - .375 - .347 - .291 - .232	0.321 0.282 0.253 0.215 0.169	- 1.62 - 1.93 - 2.01 - 2.11 - 2.27	2.65 2.75 2.62 2.66 2.54

Table 3. Powers of D, W^2, and A^2 for n=20 and 50 at α=0.05.

	n=20			n=50		
Alternative	D	W^2	A^2	D	W^2	A^2
W (1,1)	0.049	0.051	0.052	0.050	0.047	0.048
HN(2)	0.063	0.066	0.078	0.097	0.104	0.125
LN(0,1)	0.153	0.206	0.221	0.357	0.482	0.549
G(0.5,1)	0.066	0.074	0.085	0.094	0.103	0.126
G(4,1)	0.073	0.086	0.087	0.116	0.146	0.165
U(0,1)	0.233	0.308	0.386	0.549	0.696	0.811
Pa(2,1)	0.871	0.935	0.953	1.000	1.000	1.000
B(2,1)	0.235	0.311	0.385	0.561	0.706	0.823
N[1,1]	0.0913	0.1029	0.1227	0.152	0.175	0.221
Logistic[1,1]	0.0627	0.0711	0.0785	0.079	0.084	0.101

T[1]	0.2717	0.3573	0.3684	0.630	0.738	0.764
T[4]	0.0562	0.0640	0.0676	0.071	0.080	0.090
C[1,1]	0.3832	0.4726	0.4759	0.781	0.859	0.866
IG[1,1]	0.1839	0.2453	0.2611	0.435	0.590	0.674
F[3,2]	0.3024	0.3930	0.4111	0.682	0.797	0.836
Inv.Chi[2]	0.5217	0.6655	0.6904	0.931	0.975	0.988
GD[1,1]	0.0965	0.1132	0.1314	0.178	0.217	0.257
Inv. Gam[2,1]	0.3958	0.5201	0.5490	0.835	0.926	0.957
Levy[0.5,1]	0.8666	0.9386	0.9535	1.000	1.000	1.000

Testing Weibull against wide spectrum of alternatives, We can conclude from Table 3. that Anderson-Darling test considerably outperforms Kolmogorov-Smirnov and slightly overtakes Cramer-von Mises tests. Subsequently, out of the above three tests, we will consider only A^2 to be compared to our proposed tests and to other tests under consideration. Besides A^2, four more tests are considered:

4. Mann, Scheure, and Fertig (1973), to be labeled MSF, test. This test is based on normalized spacing of the sample. The test statistic form is

$$MSF = \frac{r\sum_{j=r+1}^{n}\frac{(z_{(j)}-z_{(j-1)})}{l_j-l_{j-1}}}{(n-r)\sum_{j=1}^{r}\frac{(z_{(j)}-z_{(j-1)})}{l_j-l_{j-1}}}, \quad (4.4)$$

where r is the floor of $(n/2)$, and

$$l_j = log\left(-log\left(1-\frac{(j-0.5)}{(n+0.25)}\right)\right) \quad (4.5).$$

The test is one-sided and the hypothesis is rejected when *MSF* is large.

5. Smith and Bain (1976), to be labeled S.B., test. This test is based on the sample correlation between the sample order statistics ant its expectation.

$$SB = 1-\frac{\left(\sum_{i=1}^{n}\left(y_{(i)}-\bar{y}\right)\left(l_{(i)}-\bar{l}\right)\right)^2}{\sum_{i=1}^{n}\left(y_{(i)}-\bar{y}\right)^2\sum_{i=1}^{n}\left(l_{(i)}-\bar{l}\right)^2}, \quad (4.6)$$

Where $l_{(i)}$ is as in (4.5).

The test is one-sided and the hypothesis is rejected when *S.B.* is large.

6. Shapiro and Brain (1987), to be labeled SBr. This test is based on the ratio of two estimates of scale parameter $b=(1/\tau)$ of the Log-Weibull distribution. We consider a modification of the test statistic by Öztürk and Korukoglu (1988) which has the form

$$W^* = \frac{(\hat{b}/\hat{\sigma}) - 1.0 - 0.13/\sqrt{n} + 1.18/n}{0.49/\sqrt{n} - 0.36n} \quad , \tag{4.7}$$

Where $\hat{\sigma}$ is the standard deviation of the sample, and $\hat{b}$ is D'Agostino (1971a) estimate of $(1/\tau)$ given by

$$\hat{b} = \frac{1}{n}\left[0.6079\sum_{i=1}^{n}\omega_{n+i}Y_{i:n} - 0.2570\sum_{i=1}^{n}\omega_i Y_{i:n}\right], \tag{4.8}$$

Where

$\omega_i = ln\left(\frac{n+1}{n+1-i}\right); i = 1,...,n-1;\ \omega_n = n - \sum_{i=1}^{n-1}\omega_i; \omega_{n+i} = \omega_i(1 + ln\,\omega_i) - 1; i = 1,...,n-1;$ and

$\omega_{2n} = 0.4228n - \sum_{i=1}^{n-1}\omega_{n+i}$.

The test is two-sided and the hypothesis is rejected for small and large values of W^*.

7. Liao and Shimokawa (1999), to be labeled L.S., test. This test combines D, W^2, and A^2 tests. The test statistic has the form

$$LS = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{max[(j/n) - F(z_{(j)}),(F(z_{(j)}) - (j-1)/n)]}{\sqrt{(F(z_{(j)}))(1 - F(z_{(j)}))}} \quad . \tag{4.9}$$

The test is one-sided and the hypothesis is rejected for large values of *L.S.*

The simulated critical points, computed under ML estimation, for designated nominal values and sample sizes are displayed in Table 2. It is worth noting that the tests 4, 5, and 6 are parameter free. The powers of the abovementioned four tests along with that of A^2 and the proposed tests $T_{1,n}$ and $T_{2,n}$ for n=20 and $\alpha = 0.05$ are depicted in Table 4. Simulated powers for other sample sizes n=10, 50, and 100 are displayed in Tables A3-A5 in Appendix A.

Table 4. Powers of A^2, L.S., S.B., MSF, SBr, $T_{1,n}$, and $T_{2,n}$ for n=20 and α=0.05 .

Alternative	A^2	LS	SB	MSF	SBr	$T_{1,n}$	$T_{2,n}$
W (1,1)	0.052	0.050	0.051	0.050	0.051	0.051	0.049
HN(2)	0.078	0.099	0.064	0.020	0.088	0.092	0.074
LN(0,1)	0.221	0.168	0.251	0.306	0.288	0.229	0.295
G(0.5,1)	0.085	0.105	0.067	0.020	0.092	0.091	0.071
G(4,1)	0.087	0.058	0.088	0.133	0.109	0.074	0.105
U(0,1)	0.386	0.419	0.144	0.003	0.467	0.210	0.170
Pa(2,1)	0.953	0.902	0.990	0.952	0.942	0.964	0.980

B(2,1)	0.385	0.420	0.147	0.004	0.471	0.212	0.168
N[1,1]	0.1227	0.1555	0.088	0.016	0.136	0.1245	0.0974
Logistic[1,1]	0.0785	0.0974	0.067	0.026	0.081	0.0917	0.0753
T[1]	0.3684	0.3306	0.332	0.354	0.432	0.3247	0.3738
T[4]	0.0676	0.0749	0.068	0.052	0.078	0.0774	0.0752
C[1,1]	0.4759	0.4396	0.392	0.385	0.500	0.3737	0.4154
IG[1,1]	0.2611	0.1819	0.338	0.387	0.336	0.2902	0.3747
F[3,2]	0.4111	0.3497	0.428	0.465	0.498	0.4153	0.4831
Inv.Chi[2]	0.6904	0.6046	0.776	0.755	0.742	0.7484	0.8091
GD[1,1]	0.1314	0.1658	0.093	0.015	0.150	0.1303	0.1061
Inv. Gam[2,1]	0.5490	0.4649	0.641	0.638	0.640	0.6070	0.6841
Levy[0.5,1]	0.9535	0.9044	0.991	0.960	0.942	0.9623	0.9763

4. Comparing the powers of the one-sided test $T_{1,n}$ to that of the two-sided test $T_{2,n}$ we see from Table 4. that the two tests compete for many alternatives. However, the two-sided $T_{2,n}$ test outperforms the one sided $T_{1,n}$ test , considerably, when testing against L.N. (1,1), IGaus(1,1), IChis(2,1), and IGam(2,1). So, choosing between $T_{1,n}$ and $T_{2,n}$ tests, it might be reasonable to choose $T_{2,n}$.

From Table 4., it can be noticed that $T_{2,n}$ outperforms all considered tests when testing against F(3,2), INVCHI[2], and INVGAM[2,1], and outperformed by at least one test when testing against UNIF[0,1], BETA[2,1], FNOR(1,1), and FCAU[1,1], and performs almost the same as other tests for the rest of alternatives. Inspecting

Table 4. and Table A3, It is noticed that, for small and moderate sample sizes, the powers of all considered tests, when testing against HNOR[2], GAM[0.5,1], GAM[4,1], FNOR(1,1), FLOG[0,1], T[4], and GUM[1,1], barely exceed the nominal value. This motivates developing directed tests addressing these alternatives.

From the above discussion, we conclude that the proposed one-sided $T_{2,n}$ test competes well with prominent tests such as Anderson-Darling, Smith-Bain and Shapiro-Brain tests. Also, we notice that none of the tests considered in this article is uniformly most powerful for all alternatives. Moreover, all considered tests show low powers against gamma and folded symmetric alternatives. However, some tests show, comparably, higher powers than others. Finally, computed powers of all considered tests improve as the sample size gets larger, which indicates consistency of these tests.

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Appendix A.

Table A1. Selected simulated quantiles of $\mathrm{T}_{1,n}$ when both parameters are unknown

0.5	**0.75**	**0.9**	**.95**	**.975**	**.995**
0.0917	0.1576	0.2292	0.2819	0.3289	0.4179
0.0835	0.1422	0.2037	0.2459	0.2873	0.3838
0.0758	0.1289	0.1827	0.2190	0.2554	0.3401
0.0699	0.1176	0.1685	0.2014	0.2367	0.3099
0.0637	0.1085	0.1556	0.1870	0.2185	0.2987
0.0604	0.1019	0.1450	0.1753	0.2015	0.2806
0.0576	0.0973	0.1394	0.1672	0.1927	0.2666
0.0521	0.0886	0.1254	0.1519	0.1784	0.2427
0.0504	0.0850	0.1209	0.1445	0.1705	0.2385

Table A2. Selected simulated quantiles of $T_{2,n}$ when both parameters are unknown

0.001	0.025	0.05	0.90	0.95	0.975	0.995
-0.4096	-0.3140	-0.2602	0.1556	0.1976	0.2344	0.3206
-0.3751	-0.2720	-0.2215	0.1542	0.1919	0.2220	0.2818
-0.3472	-0.2507	-0.1987	0.1433	0.1770	0.2050	0.2534
-0.3287	-0.2304	-0.1807	0.1304	0.1597	0.1842	0.2329
-0.2913	-0.2086	-0.1647	0.1223	0.1499	0.1732	0.2153

ISSN: 2094-0343
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-0.2819	-0.1945	-0.1577	0.1151	0.1397	0.1623	0.1995
-0.2623	-0.1864	-0.1474	0.1074	0.1322	0.1547	0.1934
-0.2556	-0.1717	-0.1384	0.1023	0.1259	0.1464	0.1825
-0.2401	-0.1673	-0.1321	0.0983	0.1201	0.1402	0.1744
-0.2321	-0.1599	-0.1265	0.0939	0.1155	0.1337	0.1690

Table A3. Simulated powers of A^2, L.S., S.B., MSF, SBr, $T_{1,n}$, and $T_{2,n}$ for n=10 and α=0.05

Alternative	D	W^2	A^2	LS	SB	MSF	SBr	$T_{1,n}$	$T_{2,n}$
W (1,1)	0.051	0.047	0.046	0.049	0.051	0.053	0.052	0.048	0.049
HN(2)	0.063	0.058	0.066	0.088	0.047	0.032	0.061	0.080	0.064
LN(0,1)	0.096	0.108	0.098	0.060	0.150	0.154	0.143	0.068	0.127
G(0.5,1)	0.059	0.056	0.063	0.086	0.046	0.031	0.069	0.076	0.062
G(4,1)	0.061	0.061	0.056	0.034	0.079	0.087	0.070	0.061	0.065
U(0,1)	0.135	0.154	0.195	0.241	0.075	0.014	0.203	0.145	0.110
Pa(2,1)	0.481	0.610	0.603	0.463	0.786	0.703	0.642	0.526	0.673
B(2,1)	0.136	0.150	0.187	0.233	0.076	0.014	0.204	0.148	0.113
N[1,1]	0.068	0.069	0.081	0.113	0.053	0.026	0.084	0.092	0.073
Logistic[1,1]	0.057	0.061	0.065	0.083	0.050	0.036	0.065	0.072	0.062
T[1]	0.144	0.167	0.158	0.121	0.203	0.169	0.212	0.132	0.200
T[4]	0.058	0.060	0.060	0.067	0.061	0.051	0.067	0.064	0.066
C[1,1]	0.193	0.224	0.216	0.177	0.242	0.184	0.269	0.183	0.248
IG[1,1]	0.106	0.121	0.113	0.063	0.180	0.186	0.148	0.076	0.146
F[3,2]	0.144	0.177	0.167	0.118	0.239	0.216	0.240	0.136	0.216
Inv.Chi[2]	0.262	0.326	0.316	0.227	0.444	0.398	0.387	0.279	0.408
GD[1,1]	0.080	0.079	0.094	0.123	0.055	0.026	0.098	0.102	0.082
Inv. Gam[2,1]	0.198	0.245	0.232	0.154	0.332	0.304	0.310	0.194	0.308
Levy[0.5,1]	0.483	0.612	0.607	0.471	0.782	0.714	0.639	0.535	0.681

Table A4. Powers of A^2, L.S., S.B., MSF, SBr, $T_{1,n}$, and $T_{2,n}$ for n=50 and α=0.05

Alternative	D	W^2	A^2	LS	SB	MSF	SBr	$T_{1,n}$	$T_{2,n}$
W (1,1)	0.050	0.047	0.048	0.052	0.051	0.050	0.050	0.051	0.051
HN(2)	0.097	0.104	0.125	0.147	0.097	0.009	0.158	0.120	0.094
LN(0,1)	0.357	0.482	0.549	0.506	0.528	0.664	0.699	0.628	0.697
G(0.5,1)	0.094	0.103	0.126	0.149	0.096	0.010	0.159	0.122	0.095
G(4,1)	0.116	0.146	0.165	0.147	0.129	0.272	0.255	0.157	0.205
U(0,1)	0.549	0.696	0.811	0.822	0.304	0.001	0.914	0.375	0.311
Pa(2,1)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

B(2,1)	0.561	0.706	0.823	0.834	0.319	0.000	0.907	0.376	0.307
N[1,1]	0.152	0.175	0.221	0.263	0.139	0.005	0.299	0.191	0.147
Logistic[1,1]	0.079	0.084	0.101	0.126	0.091	0.015	0.105	0.106	0.085
T[1]	0.630	0.738	0.764	0.759	0.571	0.722	0.815	0.610	0.657
T[4]	0.071	0.080	0.090	0.106	0.080	0.058	0.100	0.078	0.077
C[1,1]	0.781	0.859	0.866	0.865	0.661	0.752	0.858	0.613	0.644
IG[1,1]	0.435	0.590	0.674	0.598	0.747	0.774	0.785	0.809	0.864
F[3,2]	0.682	0.797	0.836	0.818	0.763	0.864	0.899	0.830	0.865
Inv.Chi[2]	0.931	0.975	0.988	0.978	0.994	0.987	0.992	0.997	0.999
GD[1,1]	0.178	0.217	0.257	0.304	0.149	0.006	0.250	0.185	0.146
Inv. Gam[2,1]	0.835	0.926	0.957	0.934	0.968	0.965	0.975	0.986	0.992
Levy[0.5,1]	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table A5. Powers of A^2, L.S., S.B., MSF, SBr, $T_{1,n}$, and $T_{2,n}$ for n=100 and α=0.05

Alternative	D	W^2	A^2	LS	SB	MSF	SBr	$T_{1,n}$	$T_{2,n}$
W (1,1)	0.053	0.049	0.051	0.045	0.059	0.059	0.048	0.049	0.049
HN(2)	0.146	0.165	0.203	0.208	0.130	0.004	0.291	0.178	0.140
LN(0,1)	0.666	0.804	0.870	0.825	0.846	0.914	0.943	0.933	0.959
G(0.5,1)	0.151	0.168	0.205	0.211	0.127	0.004	0.288	0.178	0.139
G(4,1)	0.221	0.277	0.321	0.281	0.216	0.462	0.464	0.332	0.415
U(0,1)	0.894	0.966	0.991	0.987	0.589	0.000	0.998	0.606	0.534
Pa(2,1)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
B(2,1)	0.888	0.963	0.993	0.987	0.589	0.000	0.997	0.611	0.539
N[1,1]	0.278	0.328	0.402	0.412	0.207	0.001	0.538	0.309	0.251
Logistic[1,1]	0.117	0.120	0.141	0.151	0.125	0.008	0.155	0.154	0.123
T[1]	0.906	0.951	0.962	0.960	0.818	0.940	0.974	0.828	0.855
T[4]	0.101	0.118	0.137	0.145	0.086	0.066	0.114	0.078	0.074
C[1,1]	0.978	0.990	0.991	0.980	0.884	0.951	0.984	0.807	0.834
IG[1,1]	0.791	0.915	0.959	0.924	0.982	0.969	0.979	0.995	0.998
F[3,2]	0.943	0.978	0.988	0.980	0.961	0.989	0.995	0.981	0.988
Inv.Chi[2]	0.999	1.000	1.000	0.990	1.000	1.000	1.000	1.000	1.000
GD[1,1]	0.333	0.403	0.458	0.474	0.229	0.001	0.353	0.296	0.246
Inv. Gam[2,1]	0.992	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Levy[0.5,1]	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

hybrid

Alternative	kol	cvm	A.D.	wats	lish	T1	T2
Weibull[1,1]	0.051	0.053	0.051	0.051	0.046	0.047	0.047
Weibull[3,4]	0.050	0.049	0.049	0.051	0.048	0.049	0.049

HalfNormal[10]	0.061	0.061	0.056	0.040	0.029	0.042	0.017
LogNormal[0,1]	0.305	0.338	0.353	0.388	0.393	0.372	0.413
Gamma[0.5,1]	0.058	0.057	0.052	0.037	0.028	0.040	0.017
Gamma[4,1]	0.119	0.130	0.138	0.158	0.159	0.144	0.170
Uniform[{0,1}]	0.165	0.198	0.191	0.118	0.079	0.087	0.009
Pareto[2,1]	0.963	0.976	0.983	0.980	0.982	0.971	0.979
Pareto[0.5,1]	0.964	0.976	0.982	0.980	0.982	0.973	0.980
Beta[2,1]	0.166	0.195	0.188	0.117	0.080	0.087	0.010
Beta[0.5,2.5]	0.082	0.088	0.080	0.048	0.030	0.048	0.007
InverseGamma[4,1]	0.558	0.615	0.630	0.659	0.661	0.643	0.682
InverseChiSquare[2]	0.768	0.811	0.825	0.838	0.840	0.827	0.854

Alternative	kol	cvm	AD	wats	lish	T1	T2
Weibul[1,1]	0.046	0.046	0.049	0.044	0.050	0.049	0.049
Weibul[3,4]	0.049	0.051	0.053	0.048	0.052	0.054	0.055
Exponentia[1]	0.046	0.047	0.049	0.043	0.048	0.053	0.052
HalfNorma[10]	0.065	0.066	0.079	0.042	0.099	0.100	0.089
LogNorma[0,1]	0.156	0.200	0.215	0.263	0.163	0.170	0.223
Gamma[0.5,1]	0.067	0.069	0.082	0.046	0.103	0.101	0.090
Gamma[4,1]	0.075	0.083	0.084	0.111	0.058	0.049	0.069
Uniform[{0,1}]	0.225	0.287	0.369	0.192	0.407	0.228	0.205
Pareto[2,1]	0.876	0.935	0.951	0.955	0.901	0.941	0.961
Pareto[0.5,1]	0.874	0.933	0.951	0.955	0.900	0.941	0.962
Beta[2,1]	0.232	0.291	0.374	0.191	0.416	0.237	0.214
Beta[0.5,2.5]	0.106	0.117	0.147	0.067	0.179	0.139	0.125
InverseGamma[2,1]	0.401	0.525	0.553	0.603	0.471	0.536	0.606
InverseGamma[4,1]	0.315	0.418	0.450	0.501	0.371	0.419	0.491
InverseGamma[8,1]	0.261	0.349	0.375	0.429	0.303	0.337	0.407
InverseChiSquare[2]	0.523	0.653	0.685	0.725	0.604	0.677	0.742
Levy[0,0.5]	0.663	0.787	0.814	0.840	0.741	0.808	0.857
Levy[0,2]	0.661	0.784	0.813	0.835	0.737	0.810	0.854