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The Error Analysis of the Weak Galerkin Finite Element Method for Two-Dimensional Incompressible Immiscible Displacement in Porous Media

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Abstract

A nonlinear system of two couple partial differential equations models for two dimensional incompressible immiscible displacement fluid in a porous medium is discussed. A sequential implicit time-stepping procedure is defined, in which the pressure and Darcy velocity of the mixture are approximated by a weak Galerkin mixed finite element method and the saturation is approximated by a weak Galerkin finite element method. Error estimate in (L^2 and H^1) - norms, the stability and energy conservation are proved. Finally, a numerical experiment are presented to demonstrate the theoretical results.

Keywords: incompressible, immiscible displacement, porous media, mixed finite element method, weak Galerkin finite element method, stability, error estimate.

1 Introduction

The discontinuity of capillary pressure fields at interfaces that separate subdomains with varied rock characteristics complicates the numerical simulation of two-phase immiscible incompressible flows over heterogeneous porous media. When both phases are present on both sides of the interface, the saturation is forced to be discontinuous at the interface to ensure capillary pressure continuity; when one of the phases is absent, the capillary pressure can also be discontinuous due to the difference in entry pressures of the different rocks. Due to significant (up to several orders) permeability changes and rapid variations in capillary forces, the global pressure and saturation might exhibit strong discontinuities in such applications [1]. The development of numerical methods for two-phase flows in heterogeneous porous media that are compatible with the non-linearity and degeneracy of the pressure and saturation equations, as well as dealing satisfactorily with the discontinuity of the saturation and global pressure, is a relevant and challenging problem.

Numerical simulations and analysis for the system (3.1,3.2 and 3.3) were investigated extensively in the last several decades, applied mixed finite element method of the pressure and the Darcy velocity and upwind Galerkin finite element method of the saturation equation for immiscible displacement of one incompressible fluid in porous media[2]. Evolved discontinuous Galerkin methods for the numerical computation of incompressible two-phase flow in porous media[3]. Used a finite volume method with the modified implicit pressure explicit saturation approach to model the 3D incompressible and immiscible two-phase flow in porous media[4]. Designed and investigated the sequential discontinuous Galerkin finite element method to approximate two-phase immiscible incompressible flows in heterogeneous porous media with discontinuous capillary pressures [5]. Introduced a finite element level set method for simulation of immiscible fluid flows [6]. Considered the system of equations governing an incompressible immiscible two-phase flow within the heterogeneous porous medium made of two different rock types [7], considered the immiscible incompressible two-phase flow in a porous medium composed of the two different rocks [8]. Presented the sequential discontinuous Galerkin method for two-phase immiscible incompressible flows in heterogeneous porous media[1], described the competitive motion of $(N+1)$ incompressible immiscible phases within a porous medium [9]. Developed a high-order hybridisable discontinuous Galerkin formulation of the solution the immiscible and incompressible problems [10], considered a new efficient implicit pressure explicit saturation scheme for the simulation of incompressible and immiscible two-phase flow in heterogeneous porous media [11], used the multiphase Lattice

Boltzmann method for the numerical simulation of immiscible fluid displacement through a 2D porous medium [12].

The weak Galerkin method is a new finite element framework for the solution partial differential equations. Wang and Ye [13] introduced the weak Galerkin finite element method by using weakly defined gradient operator over discontinuous functions with heterogeneous properties and the solution of a model second order elliptic equations. The use of weak gradients, weak divergent and their approximations results called discrete weak gradients and discrete weak divergent which to play important roles in numerical methods for partial differential equations. Mu et al [14] conducted is to conducted a computational for the weak Galerkin method for various model second elliptic problem. Mu et al [15] introduced and analyzed the weak Galerkin mixed finite element method of the solution the biharmonic equation. Li and Wang [16] proposed developed weak Galerkin method of the solution parabolic equations. Zhu et al [17] combined weak Galerkin finite element method and characteristics method of treat the convection-diffusion problems. Gao et al [18] presented numerical weak Galerkin finite element schemes for Sobolev problem. Li et al [19] introduced and analyzed the weak Galerkin finite element method for the numerically solution the coupling of fluid flow with porous media flow. Li et al [19] introduced and analyzed weak Galerkin finite element method for numerically of the solution the coupled Stokes-Darcy problem. Liu et al [20] presented a lowest-order weak Galerkin finite element method of the solution the Darcy equation on general convex polygonal meshes. Kashkool and Hussein [21] used the weak Galerkin finite element method for the solution of two-Dimensional Burgers' equations. Zhu and Xie [22] presented and analyzed weak Galerkin method for the quasi-linear elliptic problem of non-monotone type. Hussein and Kashkool [23] presented the continuous time and discrete time weak Galerkin finite element method of the solution nonlinear two-dimensional coupled Burgers' problem.

In this paper, we will adopt the weak Galerkin finite element [13] and weak Galerkin mixed finite element [15] methods of the solution (3.1, 3.2 and 3.4). However, a direct application of the algorithms will lead to a non-linear algebraic system, in which the velocity and the saturation are coupled, since the diffusion-dispersion tensor in transport equation depends on velocity in flow equation. We set up weak Galerkin mixed finite element method for the pressure and the velocity for dual equations (3.1) and (3.2), and weak Galerkin finite element method for the saturation equation.

The paper is organized as follows.

In Sect. 2, we present some assumptions on the problem, the weak Galerkin finite element space, basic notations and some useful projections. In Sect. 3, properties for the weak Galerkin finite element method are presented. In Sect. 4, we are proved energy convection mass of weak Galerkin finite (3.4) and the stability for each of velocity, pressure and saturation, respectively. In Sect. 5, we are proved lemmas for the error equations and lemmas and theorems for error estimate for L^2-norm and H^1 -norm for each velocity, pressure, saturation and both, respectively. In Sect. 6, numerical results are given to demonstrate the accuracy of the proposed method. Finally, the conclusions in Sect. 7.

2 The Preliminaries Definitions

A weak Galerkin finite element methods were first introduced in [13],a weakly defined gradient operator over discontinuous functions is used. The concept of weak gradient, weak divergent and its approximations result in discrete weak gradients and discrete weak divergent, which will play an important role in the weak Galerkin finite element methods for porous media.

Let τ_h be a shape regular and body-fitted partition of Ω (see [24]). Denote by $P_k(T)$ the set of polynomials on T with degree on more than k, and $P_k(e)$ the set of polynomials on boundary of $T(e)$ with degree no $e \in \mathcal{E}$ with degree on more than k, where $\mathcal{E}$ by a set of all edges in τ_h, in particular.

On this partition of τ_h, we are introduced the following Sobolev space

$$H^m(\tau_h) := \{v \in L^2(\Omega) : v|_K \in H^1(K); \forall K \in \tau_h\}$$

for any integer $m \leq 0$.

For any weak function $\omega = \{\omega^0, \omega^b\}$ on a polygon K with boundary ∂K satisfying $\omega^0 \in L^2(K), \omega^b \in H^{\frac{1}{2}}(\partial K)$, we define its weak gradient [13] denote by $\nabla_d \omega$, in the dual space of $H(div, K)$ and weak divergent [25] denote by $\nabla_d \cdot \omega$ in the dual space of $H^1(K)$, respectively

$$(\nabla_d \omega, q) := -(\omega^0 \nabla \cdot q)_K + \langle \omega^b q \cdot n \rangle_{\partial K}, \forall q \in H(div, K)$$

$$(\nabla_d \cdot \omega, \psi) := -(\omega^0, \nabla \psi)_K + \langle \omega^b \cdot n, \psi \rangle_{\partial K}, \forall \psi \in H^1(K)$$

where n is the outward normal direction to ∂K. Basing on a weak function, the weak Galerkin finite element space is defined by

$$\mathbb{S}_h := \{\omega = \{\omega^0, \omega^b\}: \omega^0 \in P_k(T), \omega^b \in P_k(e), \forall\ e \in \mathcal{E}\ \forall\ T \in \tau_h\},$$

furthermore

$$\mathbb{S}_h^0 = \{\{\omega^0, \omega^b\} \in V_h, \omega^b|_{\partial T \cap \partial \Omega} = 0, \forall\ T \in \tau_h\}.$$

And introduce two finite element space which are necessary for formulating the numerical schemes. The first corresponds to the scalar (or pressure) variable and the second one corresponds to vector-valued functions, defined as

$$\mathbb{G}_h = \{w \in L^2(\Omega): w|_T \in P_{K+1}(T)\},$$

$$\mathbb{Q}_h = \{u = \{u^0, u^b\}: u^0|_T \in [P_K(T)]^d, u^b|_e = u^b n_e, u^b \in P_K(e), e \in \mathcal{E}\}.$$

where $k \geq 0$ is a non-negative integer. The following L^2 projections are introduced

$$Q_h: L^2(T) \to P_k(T); \forall T \in \tau_h;$$

$$R_h: [L^2(T)]^2 \to [P_{k-1}(T)]^2; \forall T \in \tau_h.$$

3 The Weak Galerkin Finite Element Methods

In many engineering applications, incompressible immiscible flow in porous media can be described by immiscible displacement system, let Ω be a bounded domain (oil field) in the plan R^2 with smooth boundary Γ and $T > 0$. The classical system in two-dimensional space are given as follows [2]

$$\nabla \cdot v = q(x,t), (x,t) \in \Omega \times (0,T] \tag{3.1}$$

$$v = -\alpha(s) \cdot \nabla p, (x,t) \in \Omega \times (0,T] \tag{3.2}$$

$$\phi \frac{\partial s}{\partial t} + \nabla \cdot (f(s)v) - \nabla \cdot (a(s)\nabla s) = g(x,t,s), (x,t) \in \Omega \times (0,T], \tag{3.3}$$

where

$$g(x,t,s) = \begin{cases} q_+(x,t)\tilde{s}(x,t) & , \quad at\ injection\ wells \\ -q_-(x,t)\ \ s(x,t) & , \quad at\ production\ wells \\ 0 & , \quad otherwise \end{cases}$$

where ϕ is the porosity of the medium of the rock ($\phi \in (0,1)$ in the domain Ω),s, v and p are the saturation, Darcy's velocity and pressure, respectively, $f(s)$ is the fractional flow function, $g(x,t,s)$

is acceleration caused by gravity, $\alpha(s)=\frac{k}{\mu}$ is a smooth function, $k=k(x)$ is the permeability of the porous rock, $\mu=\mu(s)$ is the viscosity of the fluid , q is the source and sink terms.
If $s=s_{oil}$, then $\tilde{s}=0$ and we can denote $g(x,t,s)=-q_{-}(x,t)\ s(x,t)$, where $-q_{-}(x,t)\geq 0$, so the above equation (3.3) becomes

$$\phi\frac{\partial s}{\partial t}+\nabla\cdot(f(s)v)-\nabla\cdot(a(s)\nabla s)+q_{-}(x,t)\ s(x,t)=0,(x,t)\in\Omega\times(0,T]. \quad (3.4)$$

The boundary conditions are

$$\frac{\partial p}{\partial n}=0,\frac{\partial s}{\partial n}=0,(x,t)\in\Gamma\times(0,T], \quad (3.5)$$

and the initial condition

$$s(x,0)=s^{0}(x),\ x\in\Omega. \quad (3.6)$$

The variational formulation for the above system. The pair $H_0(div,\Omega)\times L_0^2(\Omega)\times H^1(\Omega)$ forms the finite element approximation space for the unknown. Find $v\in H_0(div,\Omega)$, $p\in L_0^2(\Omega)$ and $s\in H^1(\Omega)$ such that

$$(\nabla\cdot v,w)=(q,w);\quad\forall\, w\in L_0^2(\Omega), \quad (3.7)$$

$$(v,u)=(\alpha(s)p,\nabla\cdot u);\quad\forall\, u\in H_0(div,\Omega) \quad (3.8)$$

$$\left(\phi\frac{\partial s}{\partial t},\varphi\right)+(a(s)\nabla\cdot s,\nabla\cdot\varphi)+(\nabla\cdot(f(s)v,\varphi)+(q_{-}s,\varphi)=0;\forall\,\varphi\in H_0^1(\Omega), \quad (3.9)$$

The weak Galerkin mixed finite element method for the Darcy velocity and pressure. Find $v_h=\{v_h^0,v_h^b\}\in\mathbb{Q}_h$ and $p_h=\{p_h^0,p_h^b\}\in\mathbb{G}_h$ such that

$$(\nabla_{d,r}\cdot v_h,w^0)=(q,w^0);\quad\forall\, w=\{w^0,w^b\}\in\mathbb{G}_h, \quad (3.10)$$

$$(v_h^0,u^0)=(\alpha(s_h^0)p_h^0,\nabla_{d,r}\cdot u);\quad\forall\, u=\{u^0,u^b\}\in\mathbb{Q}_h \quad (3.11)$$

The weak Galerkin finite element method for the saturation. Find $s_h=\{s_h^0,s_h^b\}\in\mathbb{S}_h^0(j,l)$ such that

$$\left(\phi\frac{\partial s_h^0}{\partial t},\varphi^0\right)+(a(s_h^0)\nabla_{d,r}\cdot s_h,\nabla_{d,r}\cdot\varphi)+(\nabla_{d,r}\cdot(f(s_h)v_h),\varphi^0)+$$
$$(q_{-}s_h^0,\varphi^0)=0;\forall\,\varphi=\{\varphi^0,\varphi^b\}\in\mathbb{S}_h^0(j,l), \quad (3.12)$$

$$s_h(x,0) = s_h^0(x).$$

Although the weak Galerkin scheme (3.12) is defined for arbitrary indices j, l, and r, the method can be shown to produce good numerical approximations for the solution of the original partial differential equation only with a certain combination of their values. For one thing, there are at least two prominent properties that the discrete gradient operator $\nabla_{d,r}$ should possess in order for the weak Galerkin finite element method to work well. These two properties are:

- A1: For any $\varphi \in \mathbb{S}(j,l)$, if $\nabla_{d,r}\varphi = 0$ on T , then one must have $\varphi \equiv constant$ on T . In other words, $\varphi^0 = \varphi^b = constant$ on T ;
- A2: Let $s \in H^m(\Omega)(m \geq 1)$ be a smooth function on Ω, and $Q_h^* s$ be a certain projection of s on the finite element space $\mathbb{S}_h(j,l)$. Then, the discrete weak gradient of $Q_h^* s$ should be a good approximation of $\nabla_d \cdot s$.

To verify property A2, let $s \in H^1(T)$ be a smooth function on T . Denote by $Q_h^* s = \{Q^* s^0, Q^* s^b\}$ the L^2 projection onto $P_j(T^0) \times P_l(\partial T)$. In other words, on each element T , the function $Q^* s^0$ is defined as the L^2 projection of s in $P_j(T)$ and on T , $Q^* s_b$ is the L^2 projection on $P_l(\partial T)$ and on ∂T. Furthermore, let R_h^* be the local L^2 projection onto $V(T,l)$. According to the definition of $\nabla_{d,r}$, the discrete weak gradient function $\nabla_{d,r} Q_h^* s$ is given by the following equation:

$$\int_T \nabla_{d,r} Q_h^* s \cdot v \, dT = -\int_T (Q^* s^0) \nabla \cdot v \, dT + \int_{\partial T} (Q^* s^b) v \cdot n dz, \quad \forall v \in V(K,r). \quad (3.13)$$

Since Q_0^* and Q_b^* are L^2-projection operators, then the right-hand side of (3.13) is given by

$$-\int_T (Q^* \varphi^0) \nabla \cdot v dT + \int_{\partial T} (Q^* s^b) v \cdot n dz = -\int_T s \nabla \cdot v dT + \int_{\partial T} s \, v \cdot n \, dz$$
$$= -\int_T (\nabla s) \cdot v dT$$
$$= \int_T (R_h^* \nabla s) \cdot v dT.$$

Thus, we have derived the following useful identity:

$$\nabla_{d,r}(Q_h^* s) = R_h^*(\nabla s), \quad \forall s \in H^1(T). \quad (3.14)$$

The above identity clearly indicates that $\nabla_{d,r}(Q_h^* s)$ is an excellent approximation of the classical gradient of s for any $s \in H^1(T)$. For simplicity of notation, we shall drop the subscript r in the discrete weak gradient operator $\nabla_{d,r}$ from now on. Readers should bear in mind that ∇_d refers to a discrete weak gradient operator defined. One may also define a projection Π_h^* such that $\Pi_h^* v \in$

$H(div,\Omega)$, and on each $T \in \mathcal{T}_h$, one has $\Pi_h^* v \in V(T, r = l)$ and the following identity

$$(\nabla \cdot v, \varphi^0)_T = (\nabla \cdot \Pi_h^* v, \varphi^0)_T, \quad \forall \varphi^0 \in P_j(T^0).$$

The following result is based on the above property of Π_h^*.

Lemma 3.1 *[13] For any $v \in H(div,\Omega)$, we have*

$$\sum_{T\in\mathcal{T}_h} (-\nabla \cdot v, \varphi^0)_T = \sum_{T\in\mathcal{T}_h} (\Pi^* v, \nabla_{d,r}\varphi)_T, \quad \forall \ \varphi = \{\varphi^0, \varphi^b\} \in \mathbb{S}_h^0(j,l). \quad (3.15)$$

4 The Energy Conservation and Stability of Weak Galerkin

The increase in interval energy in a small spatial region of the material, control volume, over the time period $[t - \Delta t, t + \Delta t]$ is given by

$$\int_{t-\Delta t}^{t+\Delta t} \int_K w_t dxdt = \int_K w(x, t + \Delta t)dx - \int_K w(x, t - \Delta t)dx$$

since equation (3.12), let

$$\phi \ \frac{\partial s}{\partial t} + \nabla \cdot \chi + q\_s = 0,$$

$$\chi = f(s)v - a(s)\nabla s,$$

suppose that a body obeys the heat equation and, in addition generates its own heat per unit volume at a known function q varying in space and time, the change in interval energy in K is accounted for by the flux of heat across the boundaries together with the source energy. By Fourier's law integral form of energy conservation

$$\int_{t-\Delta t}^{t+\Delta t} \int_K \phi \ \frac{\partial s^0}{\partial t} \ dxdt + \int_{t-\Delta t}^{t+\Delta t} \int_{\partial K} \chi \cdot n \ d\delta dt + \int_{t-\Delta t}^{t+\Delta t} \int_K q\_s \ dxdt = 0,$$

where the Green's formula was used.

We chose a test function $\varphi = \{\varphi^0, \varphi^b = 0\}$ so that $\varphi^0 = 1$ on K and $\varphi^0 = 0$ elsewhere. After integrating over the time period, we have

$$\int_{t-\Delta t}^{t+\Delta t} \int_K \phi \ \frac{\partial s^0}{\partial t} \ dxdt + \int_{t-\Delta t}^{t+\Delta t} \int_K a(s^0)\nabla_d \cdot s \ \nabla_d \cdot \varphi \ dxdt$$

$$+ \int_{t-\Delta t}^{t+\Delta t} \int_K f(s^0)v^0\nabla_d \cdot \varphi \ dxdt$$

$$+ \int_{t-\Delta t}^{t+\Delta t} \int_K q\_s^0 \ dxdt = 0,$$

using the definition of operator R_h^*, the divergence theorem and of the weak ($\nabla_{d,r} \cdot$), we arrive at

$$\int_K a(s_h^0)\nabla_{d,r} \cdot s_h \nabla_{d,r} \cdot \varphi \ dx = \int_K R_h^*(a(s^0)\nabla_{d,r} \cdot s)\nabla_{d,r} \cdot \varphi dx$$

$$= -\int_K \nabla_{d,r} \cdot R_h^*(a(s^0)\nabla_{d,r} \cdot s) \ dx$$

$$= -\int_{\partial K} R_h^*(a(s^0)\nabla_{d,r} \cdot s) \cdot n \, d\delta$$

$$\int_K f(s_h^0)v_h^0\nabla_{d,r} \cdot \varphi \, dx = \int_K R_h^*(f(s^0)v_h^0)\nabla_{d,r} \cdot \varphi \, dx$$
$$= -\int_K \nabla_{d,r} \cdot R_h^*(f(s^0)v_h^0)dx$$
$$= -\int_{\partial K} R_h^*(f(s^0)v_h^0) \cdot n \, d\delta.$$

Now, we have

$$\int_{t-\Delta t}^{t+\Delta t} \int_K \phi \frac{\partial s_h^0}{\partial t} dxdt - \int_{t-\Delta t}^{t+\Delta t} \int_{\partial K} R_h^*(a(s^0)\nabla_{d,r} \cdot s) \cdot n \, d\delta dt$$
$$- \int_{t-\Delta t}^{t+\Delta t} \int_{\partial K} R_h^*(f(s^0)v_h^0) \cdot n \, d\delta dt$$
$$+ \int_{t-\Delta t}^{t+\Delta t} \int_K q_- s_h^0 \, dxdt = 0,$$

$$\int_{t-\Delta t}^{t+\Delta t} \int_K \phi \frac{\partial s_h^0}{\partial t} dxdt - \int_{t-\Delta t}^{t+\Delta t} \int_{\partial K} R_h^*(a(s^0)\nabla_{d,r} \cdot s + f(s^0)v_h^0) \cdot n \, d\delta dt$$
$$+ \int_{t-\Delta t}^{t+\Delta t} \int_K q_- s_h^0 \, dxdt = 0,$$

which provides a numerical flux given for saturation by $\chi_h \cdot n = -R_h^*(a(s^0)\nabla_{d,r} \cdot s + f(s^0)v_h^0) \cdot n$. The numerical flux $\chi_h \cdot n + q_- s_h^0$ is continuous a cross the edge of each element K, which can be verified by a selection of the test function $\varphi = \{\varphi^0, \varphi^b\}$ so that $\varphi^0 \equiv 0$ and φ^b arbitrary.

Now the arguments for stability and bound on the approximation error are useful for the analysis of the discrete formulations.

Lemma 4.1 *(The stability for the pressure)*

Let a solution $p_h \in \mathbb{G}_h$ and C_1 is a constant and independent h, then

$$\| p_h \|^2 \leq C_1 \| q \|^2. \tag{4.1}$$

Proof. From equations (3.10 and 3.11) , and put $w = p_h$, we get

$$(\alpha(s_h^0)\nabla_{d,r} p_h, \nabla_{d,r} \cdot p_h) = (q, p_h^0),$$

by using Young's inequality and Cauchy inequality, we have

$$|(\alpha(s_h^0)\nabla_{d,r} \cdot p_h, \nabla_{d,r} \cdot p_h)| \leq \| (\alpha(s_h^0) \| \| \nabla_{d,r} \cdot p_h \|^2 \leq C \| (\alpha(s_h^0) \| \| p_h \|^2,$$

$$|(q, p_h^0)| \leq \frac{1}{4} \parallel q \parallel^2 + \parallel p_h^0 \parallel^2,$$

then

$$C \parallel (\alpha(s_h^0) \parallel \parallel p_h \parallel^2 \leq \frac{1}{4} \parallel q \parallel^2 + \parallel p_h^0 \parallel^2$$

$$\parallel p_h \parallel^2 \leq C_1 \parallel q \parallel^2,$$

where $C_1 = 1/min\{4C \parallel (\alpha(s_h) \parallel, -4 \parallel p_h^0 \parallel^2\}$.

Lemma 4.2 *(The stability for the velocity)*

Let a solution$v_h \in \mathbb{Q}_h$ and C_2 is a constant and independent h, then

$$\parallel v_h \parallel^2 \leq C_2\{\parallel v_h^0 \parallel^2 + \parallel p_h^0 \parallel^2\}. \tag{4.2}$$

Proof. Put $u = v_h$ in equation (3.10), we have

$$(v_h^0, v_h^0) = (\alpha(s_h^0) p_h^0, \nabla_{d,r} \cdot v_h),$$

then,

$$|(v_h^0, v_h^0)| \leq \alpha_1 \parallel v_h^0 \parallel^2,$$

$$|(\alpha(s_h^0) p_h^0, \nabla_{d,r} v_h)| \leq 2 \parallel \alpha(s_h^0) \parallel^2 \parallel p_h^0 \parallel^2 + \frac{1}{2} \parallel \nabla_{d,r} v_h \parallel^2$$

$$\leq 2 \parallel \alpha(s_h^0) \parallel^2 \parallel p_h^0 \parallel^2 + \frac{C}{2} \parallel v_h \parallel^2,$$

then

$$2 \parallel \alpha(s_h^0) \parallel^2 \parallel p_h^0 \parallel^2 + \frac{C}{2} \parallel v_h \parallel^2 \leq \alpha_1 \parallel v_h^0 \parallel^2,$$

$$\frac{C}{2} \parallel v_h \parallel^2 \leq \alpha_1 \parallel v_h^0 \parallel^2 - 2 \parallel \alpha(s_h^0) \parallel^2 \parallel p_h^0 \parallel^2,$$

then

$$\parallel v_h \parallel^2 \leq C_2\{\parallel v_h^0 \parallel^2 + \parallel p_h^0 \parallel^2\}.$$

where $C_2 = \max\{2\alpha_1/C, -4 \parallel \alpha(s_h^0) \parallel^2/C\}$

Theorem 4.1 *(The stability of velocity and the pressure)*

Let the dual solutions $v_h = \{v_h^0, v_h^b\} \in \mathbb{Q}_h$, $p_h = \{p_h^0, p_h^b\} \in \mathbb{G}_h$, and β is a constant and independent h. Then

$$\| p_h \|^2 + \| v_h \|^2 \leq \beta\{\| q \|^2 + \| v_h^0 \|^2\}. \tag{4.3}$$

Proof. From equations (3.10 and 3.11) and replacement test function with $u \in \mathbb{Q}_h$, such that $v_h \in \mathbb{Q}_h$, we have

$$(\alpha(s_h^0)\nabla_{d,r} p_h, \nabla_{d,r} \cdot u) = (q, u^0); \;\; u \in \mathbb{Q}_h,$$

put $u = v_h$ in the above equation, we have

$$(\alpha(s_h^0)\nabla_{d,r} p_h, \nabla_{d,r} \cdot v_h) = (q, v_h^0),$$

then,

$$|(\alpha(s_h^0)\nabla_{d,r} p_h, \nabla_{d,r} \cdot v_h)| \leq \frac{\alpha_2 \| \alpha(s_h^0) \|^2}{2} \| \nabla_{d,r} p_h \|^2 + 2\alpha_2 \| \nabla_{d,r} v_h \|^2$$

$$\leq C \| p_h \|^2 + C \| v_h \|^2,$$

$$|(q, v_h^0)| \leq \frac{1}{4} \| q \|^2 + \| v_h^0 \|^2,$$

then

$$C \| p_h \|^2 + C \| v_h \|^2 \leq \frac{1}{4} \| q \|^2 + \| v_h^0 \|^2,$$

we get,

$$\| p_h \|^2 + \| v_h \|^2 \leq \beta\{\| q \|^2 + \| v_h^0 \|^2\}.$$

where $\beta = 1/C \;\; max\{1/4, 1\}$.

Lemma 4.3 *(The stability for Saturation)*

Let a solution $s_h \in \mathbb{S}_h^0$ and α_4 is a constant and independent h, then

$$\| s_h^0(t) \|^2 \leq \| s_h^0(0) \|^2 \, exp(-\alpha_1 t) + \alpha_4 exp(-\alpha_1 t) \int_0^t exp(\alpha_1 z)\{\| s_h(z) \|^2 + \| v_h(z) \|^2\} dz. \quad (4.4)$$

Proof. Rewrite (3.12), and put $\varphi = s_h$, we get

$$\left(\phi \frac{\partial s_h}{\partial t}, s_h^0\right) + (a(s_h^0)\nabla_{d,r} \cdot s_h, \nabla_{d,r} \cdot s_h) + (v_h^0 \nabla_{d,r} \cdot f(s_h), s_h^0)$$
$$+(f(s_h^0)\nabla_{d,r} \cdot v_h, s_h^0) + (q_- s_h^0, s_h^0) = 0, \quad (4.5)$$

we have,

$$\left(\phi \frac{\partial s_h^0}{\partial t}, s_h^0\right) \leq \frac{1}{2} \| \phi \| \frac{\partial}{\partial t} \| s_h^0(t) \|^2$$

$$|(a(s_h^0)\nabla_{d,r} \cdot s_h, \nabla_{d,r} \cdot s_h)| \leq \frac{1}{4} \| a(s_h^0) \|^2 + \| \nabla_{d,r} s_h \|^2 \leq C \| s_h^0 \|^2 + C \| s_h \|^2,$$

$$|(v_h^0 \nabla_{d,r} \cdot f(s_h), s_h^0)| \leq \frac{1}{4} \| v_h^0 \|^2 \| \nabla_{d,r} f(s_h) \|^2 + \| s_h^0 \|^2 \leq C \| s_h \|^2 + \| s_h^0 \|^2,$$

$$|(f(s_h^0)\nabla_{d,r} \cdot v_h, s_h^0)| \leq \frac{1}{4} \| (f(s_h^0) \|^2 \| \nabla_{d,r} v_h \|^2 + \| s_h^0 \|^2 \leq \frac{1}{16} \| f(s_h^0 \|^2$$
$$+ \frac{1}{4} \| \nabla_{d,r} v_h \|^2 + \| s_h^0 \|^2 \leq C \| s_h^0 \|^2 + C \| v_h \|^2,$$

$$|(q_- s_h^0, s_h^0)| \leq \lambda \| s_h^0 \|^2,$$

became equation (4.5), the follows

$$\| \phi \| \frac{1}{2} \frac{\partial}{\partial t} \| s_h^0(t) \|^2 + C \| s_h^0 \|^2 + C \| s_h \|^2 + C \| s_h \|^2 + \| s_h^0 \|^2$$
$$+C \| s_h^0 \|^2 + C \| v_h \|^2 + \lambda \| s_h^0 \|^2 \leq 0,$$

$$\frac{\partial}{\partial t} \| s_h^0(t) \|^2 + \alpha_1 \| s_h^0 \|^2 \leq \alpha_2 \| s_h \|^2 + \alpha_3 \| v_h \|^2,$$

where $\alpha_1 = \frac{2}{\|\phi\|}(2C + 1 + \lambda)$, $\alpha_2 = \frac{-4C}{\|\phi\|}$ and $\alpha_3 = \frac{-2C}{\|\phi\|}$. Multiply both sides of the above inequality by the integral factor $exp(\alpha_1 z)$ and then integrate from 0 to t, we get

$$\| s_h^0(t) \|^2 \leq \| s_h^0(0) \|^2 \ exp(-\alpha_1 t) + \alpha_4 exp(-\alpha_1 t) \int_0^t \ exp(\alpha_1 z)\{\| s_h(z) \|^2$$
$$+\| v_h(z) \|^2\}dz,$$

where $\alpha_4 = max\{\alpha_2, \alpha_3\}$.

5 The Error Analysis

In this section we are derived error estimate for the weak Galerkin finite element methods in (3.11)-(3.12). Let us begin with the derivations of the error equations for weak Galerkin approximation (v_h, p_h, s_h) and the L^2 projection of the exact solution (v, p, s) in the weak finite element space $\mathbb{Q}_h \times \mathbb{G}_h \times \mathbb{S}_h^0$, will prove error estimate for L^2 norm and H^1 norm for each variables.

The first let for each element $T \in \mathcal{T}_h$, denoted by Q_h^0 and Q'^0_h the L^2 projection operator onto $P_{k-1}(T)$ and $[P_k(T)]^d$, respectively. For each edge $e \in \mathcal{E}_h$, denote by Q_h^b and Q'^b_h the L^2 projection onto $P_{k-1}(e)$ and $[P_k(e)]^d$, respectively. We shall combine Q_h^0 and Q_h^b by writing $Q_h = \{Q_h^0, Q_h^b\}$ and Q'^0_h and Q'^b_h by writing $Q'_h = \{Q'^0_h, Q'^b_h\}$. Then we can define two projection onto the finite element space $\mathbb{Q}_h$ and $\mathbb{G}_h$ such that on each element T,

$$Q_h v = \{Q_h v^0, Q_h v^b\}, Q'_h p = \{Q'_h p^0, Q'_h p^b\}.$$

In the current application, we shall employ the following decomposition: Let

$$v - v_h = (v - Q_h v) + (Q_h v - v_h),$$
$$p - p_h = (p - Q'_h p) + (Q'_h p - p_h),$$
$$s - s_h = (s - Q_h^* s) + (Q_h^* s - s_h).$$

For simplicity, we introduce the following notation:

$$e_v = \{e_v^0, e_v^b\} = Q_h v - v_h, e_p = \{e_p^0, e_p^b\} = Q'_h p - p_h, e_s = \{e_s^0, e_s^b\} = Q_h^* s - s_h.$$

An error equations are given by the following lemmas 5.1, 5.2 and 5.3

Lemma 5.1 *Let v_h be the solution in dual equations (3.10) and (3.11) and e_v is the error estimate for the velocity, then*

$$(e_v^0, u^0) = (Q_h v^0 - \Pi_h v^0, u^0), \tag{5.1}$$

Proof. Rewrite equation (3.11), then

$$\sum_{T\in\mathcal{T}_h} (v_h^0, u^0)_T = (\alpha(s_h^0)p_h^0, \nabla_{d,r} u),$$

by using the projection Π_h for the above equation , we arrive at

$$\sum_{T\in\mathcal{T}_h} (\Pi_h v^0, u^0) = (\alpha(s_h^0)p_h^0, \nabla_{d,r} u).$$

Adding and subtracting the term $(Q_h v^0, u^0)$ on the right hand side of the above equation and using (3.14) we obtain,

$$(\Pi_h v^0 - Q_h v^0, u^0) + (Q_h v^0, u^0) = (\alpha(s_h^0)p_h^0, \nabla_{d,r} u).$$

since $-\alpha(s)\cdot\nabla p = v$, we get

$$(\Pi_h v^0 - Q_h v^0, u^0) + (Q_h v^0, u^0) = (v_h^0, u^0),$$

then the error equation is

$$(e_v^0, u^0) = (Q_h v^0 - \Pi_h v^0, u^0),$$

where $e_v^0 = Q_h v^0 - v_h^0$.

Lemma 5.2 *Let p_h be the solution of equation (3.11) and e_p is the error estimate for the pressure, then*

$$(\nabla_{d,r} e_p, \nabla_{d,r}\cdot w) = \left(\nabla_{d,r}(Q'_h p - \Pi'_h p), \nabla_{d,r}\cdot w\right). \tag{5.2}$$

Proof. From dual equations (3.10) and (3.11), we have

$$\sum_{T\in\mathcal{T}_h} (\alpha(s_h^0)\nabla_{d,r} p_h, \nabla_{d,r}\cdot w)_T = (q, w^0),$$

$$\sum_{T\in\mathcal{T}_h} (\alpha(s_h^0)\Pi'_h \nabla_{d,r} p, \nabla_{d,r}\cdot w) = (q, w^0),$$

Adding and subtracting the term $(\alpha(s_h^0)Q'_h\nabla_{d,r} p, \nabla_{d,r}\cdot w)$ on the left hand side of the above equation and using (3.14) we obtain,

$$\left(\alpha(s_h^0)\left(\Pi'_h\nabla_{d,r} p - Q'_h\nabla_{d,r} p\right), \nabla_{d,r}\cdot w\right) + (\alpha(s_h^0)Q'_h\nabla_{d,r} p, \nabla_{d,r}\cdot w)$$
$$= (\alpha(s_h^0)\nabla_{d,r} p_h, \nabla_{d,r}\cdot w),$$

the following that

$$(\alpha(s_h^0)\nabla_{d,r} e_p, \nabla_{d,r}\cdot w) = \left(\alpha(s_h^0)\nabla_{d,r}(Q'_h p - \Pi'_h p), \nabla_{d,r}\cdot w\right),$$

then, the error equation is

$$(\nabla_{d,r} e_p, \nabla_{d,r} \cdot w) = \left(\nabla_{d,r}(Q'_h p - \Pi'_h p), \nabla_{d,r} \cdot w\right),$$

where $e_p = p_h - Q'_h p$.

Lemma 5.3 *Let s_h be the solution of equation (3.12) and e_s is the error estimate for the saturation, then*

$$a((c_h - Q_h^* c), \varphi) = (D(u_h^0)\nabla_{d,r}(\Pi_h^* c - Q_h^* c), \nabla_{d,r}\varphi) \tag{5.3}$$
$$+(u_h^0 \cdot \nabla_{d,r}(\Pi_h^* c - Q_h^* c), \varphi^0) - (f(\Pi_h^* c^0 - Q_h^* c^0), \varphi^0).$$

Proof. By testing (3.12) against φ^0 and using (3.15), we get

$$0 = \left(\phi \frac{\partial s_h^0}{\partial t}, \varphi^0\right) + \sum_{T\in\mathcal{T}_h} [(a(s_h^0)\nabla_{d,r} \cdot s_h, \nabla_{d,r} \cdot \varphi)_T + (\nabla_{d,r} \cdot (f(s_h)v_h), \varphi^0)_T +$$
$$(q_- s_h^0, \varphi^0)_T],$$

$$0 = \left(\phi \frac{\partial c_h^0}{\partial t}, \varphi^0\right) + \sum_{T\in\mathcal{T}_h} [(a(\Pi_h^* s^0)\Pi_h^* \nabla_{d,r} s, \nabla_{d,r}\varphi)_T$$
$$+(\nabla_{d,r} \cdot (f(\Pi_h^* s)v_h), \varphi^0)_T + (q_- \Pi_h^* s^0, \varphi^0)_T].$$

Adding and subtracting the term $A(Q_h^* s, \varphi) \equiv (a(Q_h^* s^0)\nabla_{d,r} Q_h^* s, \nabla_{d,r}\varphi) + (\nabla_{d,r} \cdot (f(Q_h^* s)v_h), \varphi^0) +$ $(q_- Q_h^* s^0, \varphi^0)$ on the right hand side of the above equation and then using (3.14) we obtain

$$0 = \left(\phi \frac{\partial c_h^0}{\partial t}, \varphi^0\right) + (a(Q_h^* s^0)\nabla_{d,r} Q_h^* s, \nabla_{d,r}\varphi) + (\nabla_{d,r} \cdot (f(Q_h^* s)v_h), \varphi^0)$$
$$+(q_- Q_h^* s^0, \varphi^0)$$
$$+(a(\Pi_h^* s^0)\nabla_{d,r}(\Pi_h^* s - Q_h^* s), \nabla_{d,r}\varphi) + (\nabla_{d,r} \cdot ((f(\Pi_h^* s) - Q_h^* s)v_h), \varphi^0)$$
$$+(q_-(\Pi_h^* s^0 - Q_h^* s^0), \varphi^0) - (a(Q_h^* s^0)\nabla_{d,r}(\Pi_h^* s - Q_h^* s), \nabla_{d,r}\varphi),$$

which can be rewritten as

$$A(s_h, \varphi) = A(Q_h^* s, \varphi) + (a(\Pi_h^* s^0)\nabla_{d,r}(\Pi_h^* s - Q_h^* s), \nabla_{d,r}\varphi)$$
$$+(\nabla_{d,r} \cdot ((f(\Pi_h^* s) - Q_h^* s)v_h), \varphi^0)$$
$$+(q_-(\Pi_h^* s^0 - Q_h^* s^0), \varphi^0) - (a(Q_h^* s^0)\nabla_{d,r}(\Pi_h^* s - Q_h^* s), \nabla_{d,r}\varphi),$$

then, the error equation is

$$A((s_h - Q_h^* s), \varphi) = (a(\Pi_h^* s^0)\nabla_{d,r}(\Pi_h^* s - Q_h^* s), \nabla_{d,r}\varphi)$$
$$+(\nabla_{d,r} \cdot ((f(\Pi_h^* s) - Q_h^* s)v_h), \varphi^0) + (q_-(\Pi_h^* s^0 - Q_h^* s^0), \varphi^0) \quad (5.4)$$
$$-(a(Q_h^* s^0)\nabla_{d,r}(\Pi_h^* s - Q_h^* s), \nabla_{d,r}\varphi).$$

5.1 L^2 The Error Analysis for Velocity, Pressure and Saturation

Lemma 5.4 *(L^2 Error Analysis for Velocity)*

Let $v_h \in \mathbb{Q}_h$ be a solution to the dual equations (3.10 and 3.11). Then

$$\| e_v \|^2 \leq C_1 h[\| v_h^0 \|^2 + \| v_h \|^2 + \| e_p^0 \|^2 + \| e^0 \|^2]. \quad (5.5)$$

where C_1 is a constant and independent h.

Proof. Put $v_h = \Pi_h v$ for dual equations (3.10)and (3.11), we have

$$(\Pi_h \nabla_{d,r} \cdot v, w_0) = (q, w_0); \quad \forall w \in \mathbb{G}_h, \quad (5.6)$$
$$(\Pi_h v^0, u^0) = (\alpha(s_h^0)p_h^0, \nabla_{d,r} u); \quad \forall u \in \mathbb{Q}_h \quad (5.7)$$

subtracting equation (3.10) from (5.6) and equation (3.11) from (5.7), then

$$(\Pi_h v^0 - v_h^0, u^0) = ((\Pi_h \nabla_{d,r} \cdot v - \nabla_{d,r} \cdot v_h), w^0).$$

Adding and subtract the two terms $(Q_h v^0, u^0)$ and $(\nabla_{d,r} \cdot Q_h v, w^0)$, put $u = e_v = (Q_h v - v_h)$ and $w = e_p = (Q_h p - p_h)$, we have

$$((\Pi_h v^0 - Q_h v^0 + Q_h v^0 - v_h^0), e_v^0)$$
$$= \left((\Pi_h \nabla_{d,r} \cdot v - \nabla_{d,r} \cdot Q_h v + \nabla_{d,r} \cdot Q_h v - \nabla_{d,r} \cdot v_h), e_p^0\right),$$

$$((\Pi_h v^0 - Q_h v^0), e_v^0) + ((Q_h v^0 - v_h^0), e_v^0) = \left(\nabla_{d,r} \cdot (\Pi_h v - Q_h v), e_p^0\right)$$
$$+\left(\nabla_{d,r} \cdot (Q_h v - v_h), e_p^0\right), \quad (5.8)$$

so,

$$J_{11} + J_{12} = J_{13} + J_{14}$$

by Cauchy-Schwarz inequality and Young's inequality, we get

$$|J_{11}| = \left|\left((\Pi_h v^0 - Q_h v^0), e_v^0\right)\right| \leq \frac{1}{4} \| (\Pi_h v^0 - Q_h v^0) \|^2 + \| e_v^0 \|^2 \leq Ch^2 \| v_h^0 \|^2 + \| e_v^0 \|^2,$$

$$|J_{12}| = |((Q_h v^0 - v_h^0), e_v^0)| \leq C \parallel e_v^0 \parallel^2,$$

$$|J_{13}| = |\left(\nabla_{d,r} \cdot (\Pi_h v - Q_h v), e_p^0\right)| \leq \frac{1}{4} \parallel \nabla_{d,r} \cdot (\Pi_h v - Q_h v) \parallel^2 + \parallel e_p^0 \parallel^2$$
$$\leq Ch \parallel v_h \parallel^2 + \parallel e_p^0 \parallel^2,$$

$$|J_{14}| = |\left(\nabla_{d,r} \cdot (Q_h v - v_h), e_p^0\right)| \leq \frac{1}{4} \parallel \nabla_{d,r} \cdot e_v \parallel^2 + \parallel e_p^0 \parallel^2 \leq C \parallel e \parallel^2 + \parallel e_p^0 \parallel^2,$$

then equation (5.8) becomes

$$Ch \parallel v_h \parallel^2 + \parallel e_p^0 \parallel^2 + C \parallel e_v \parallel^2 + \parallel e_p^0 \parallel^2 \leq Ch^2 \parallel v_h^0 \parallel^2 + \parallel e_v^0 \parallel^2 + C \parallel e_v^0 \parallel^2,$$

then

$$\parallel e_v \parallel^2 \leq C_1 h[\parallel v_h^0 \parallel^2 + \parallel v_h \parallel^2 + \parallel e_p^0 \parallel^2 + \parallel e_v^0 \parallel^2],$$

where $C_1 = max\{h, -1, -2/Ch, 1/Ch, 1/h\}$.

Lemma 5.5 *(L^2 Error Analysis for Pressure)*

Let $p_h \in \mathbb{G}_h$ be a solution to the dual equations (3.10 and 3.11). Then

$$\parallel e_p \parallel^2 \leq C'_1 h[\parallel p_h \parallel^2 + \parallel e_v^0 \parallel^2], \tag{5.9}$$

where C'_1 is a constant and independent h.

Proof. Rewrite equation (3.11), we get

$$(v_h^0, u^0) = (\alpha(s_h^0)\nabla_{d,r} p_h, u^0); \quad \forall u \in \mathbb{Q}_h, \tag{5.10}$$

put $p_h = \Pi'_h p$ for equation (5.10), we have

$$(v_h^0, u^0) = (\alpha(s_h^0)\Pi'_h \nabla_{d,r} p, u^0). \tag{5.11}$$

Subtracting equation (5.10) from (5.11), then

$$(\alpha(s_h^0)(\Pi'_h \nabla_{d,r} p - \nabla_{d,r} p_h), u^0) = 0.$$

Adding and subtract the term $(\alpha(s_h^0)\nabla_{d,r} Q'_h p, u^0)$,

$$(\alpha(s_h^0)(\Pi'_h \nabla_{d,r} p - \nabla_{d,r} Q'_h p), u^0) + (\alpha(s_h^0)(\nabla_{d,r} p_h - \nabla_{d,r} Q'_h p), u^0) = 0, \tag{5.12}$$

put $u = e_v = (Q_h v - v_h)$, we get

$$(\alpha(s_h^0)(\Pi'_h \nabla_{d,r} p - \nabla_{d,r} Q'_h p), e_v^0) + (\alpha(s_h^0)(\nabla_{d,r} p_h - \nabla_{d,r} Q'_h p), e_v^0) = 0, \tag{5.13}$$

so,

$$J_{31} + J_{32} = 0,$$

by Cauchy-Schwarz inequality and Young's inequality, we get

$$|J_{31}| = |(\alpha(s_h^0)(\Pi'_h \nabla_{d,r} p - \nabla_{d,r} Q'_h p), e_v^0)| \leq \frac{1}{4} \| \alpha(s_h^0) \|^2 \| e_v^0 \|^2$$

$$+ \| (\Pi'_h \nabla_{d,r} p - \nabla_{d,r} Q'_h p) \|^2 \leq \frac{1}{4} \| \alpha(s_h^0) \|^2 \| e_v^0 \|^2 + C'h \| p_h \|^2,$$

$$|J_{32}| = |(\alpha(s_h^0)(\nabla_{d,r} Q'_h p - \nabla_{d,r} p_h), e_v^0)| \leq \frac{1}{4} \| \alpha(s_h^0) \|^2 \| e_v^0 \|^2$$

$$+ \| (\nabla_{d,r} Q'_h p - \nabla_{d,r} p_h) \|^2 \leq \frac{1}{4} \| \alpha(s_h^0) \|^2 \| e_v^0 \|^2 + C' \| e_p \|,$$

become equation (5.13) as the follows

$$\frac{1}{4} \| \alpha(s_h^0) \|^2 \| e_v^0 \|^2 + C' \| e_p \|^2 + \frac{1}{4} \| \alpha(s_h^0) \|^2 \| e_v^0 \|^2 + C'h \| p_h \|^2 \leq 0,$$

then,

$$\| e_p \|^2 \leq C'_1 h [\| p_h \|^2 + \| e_v^0 \|^2],$$

where $C'_1 = max\{\frac{-\|\alpha(s_h^0)\|^2}{2h}, 1\}$.

Theorem 5.1 *(L^2 Error Analysis for Velocity and Pressure)*

Let $v_h \in \mathbb{Q}_h$ and $p_h \in \mathbb{G}_h$ be the solutions to the dual equations (3.10 and 3.11). Then

$$\| e_v \|^2 + \| e_p \|^2 \leq C''h[\| v_h \|^2 + \| u_h^0 \|^2 + \| p_h \|^2 + \| e_v^0 \|^2 + \| e_p^0 \|^2]. \quad (5.14)$$

where C'' is a constant and independent h.

Proof. From equations (3.10)and (5.10), put $v_h = \Pi_h v$ and $p_h = \Pi'_h p$ for equations we have

$$(\Pi_h \nabla_{d,r} \cdot v, w^0) = (q, w^0); \quad \forall w \in \mathbb{G}_h, \quad (5.15)$$

$$(\Pi_h v^0, u^0) = (\alpha(s_h^0) \Pi'_h \nabla_{d,r} p, u^0); \quad \forall u \in \mathbb{Q}_h \quad (5.16)$$

subtracting equation (3.10) from (5.15) and equation (5.10) from (5.16), then

$$(\Pi_h v^0 - v_h^0, u^0) + (\alpha(s_h^0)(\nabla_{d,r} p_h - \Pi'_h \nabla_{d,r} p), u^0)$$

$$= ((\Pi_h \nabla_{d,r} \cdot v - \nabla_{d,r} \cdot v_h), w^0).$$

Adding and subtracting the three terms $(\nabla_{d,r} \cdot Q_h v, w^0)$, $(\alpha(s_h^0) \nabla_{d,r} Q'_h p, u^0)$ and $(Q_h v^0, u^0)$, and put $u = e_v$ and $w = e_p$,we have

$$\left(\nabla_{d,r}\cdot(\Pi_h v - Q_h v), e_p^0\right) + \left(\nabla_{d,r}\cdot(Q_h v - v_h), e_p^0\right) = ((\Pi_h v^0 - Q_h v^0), e_v^0)$$
$$+((Q_h v^0 - v_h^0), e_v^0) + (\alpha(s_h^0)\nabla_{d,r}(Q'_h p - \Pi'_h p), e_v^0)$$
$$+(\alpha(s_h^0)\nabla_{d,r}(Q'_h p - p_h), e_v^0), \quad (5.17)$$

suppose equation (5.17) as

$$J_{13} + J_{14} = J_{11} + J_{12} - J_{31} - J_{32},$$

from lemmas (5.4) and (5.5), we get

$$Ch \parallel v_h \parallel^2 + \parallel e_p^0 \parallel^2 + C \parallel e_v \parallel^2 + \parallel e_p^0 \parallel^2 \leq Ch^2 \parallel v_h^0 \parallel^2 + \parallel e_v^0 \parallel^2 + C \parallel e_v^0 \parallel^2$$
$$+\frac{1}{4}\parallel \alpha(s_h^0)\parallel^2 \parallel e_v^0 \parallel^2 + C' \parallel e_p \parallel^2 + \frac{1}{4}\parallel \alpha(s_h^0)\parallel^2 \parallel e_v^0 \parallel^2 + C'h \parallel p_h \parallel^2,$$

then,

$$\parallel e_v \parallel^2 + \parallel e_p \parallel^2 \leq C''h[\parallel v_h \parallel^2 + \parallel v_h^0 \parallel^2 + \parallel p_h \parallel^2 + \parallel e_v^0 \parallel^2 + \parallel e_p^0 \parallel^2],$$

where $C'' = max\{-C, Ch,, C'/h, 1/Ch, C/h, \frac{\|\alpha(s_h^0)\|^2}{2h}, -2/h\}/min\{C, -C'\}$

Lemma 5.6 *(L^2 Error Analysis for Saturation).*
Let $Q_h^* s$ be the L^2 projection of s_h on the corresponding finite element space such that s_h be a solution (3.12), and C and β are the constants. Then

$$\parallel e_s^0(t) \parallel^2 \leq exp(-\beta t) \parallel e_s^0(0) \parallel^2 + \beta_1 h \int_0^t exp((z-t)\beta)[\parallel s_h(z) \parallel^2$$
$$+\parallel s_h^0(z) \parallel^2 + \parallel e_s(z) \parallel^2]\, dz. \quad (5.18)$$

provided that the meshsize h is sufficiently small.

Proof. Put $s_h = \Pi_h^* s$ in equation (3.12), we get

$$\left(\phi \frac{\partial \Pi_h^* s^0}{\partial t}, \varphi^0\right) + (a(\Pi_h^* s^0)\Pi_h^* \nabla_{d,r}\cdot s, \nabla_{d,r}\cdot \varphi) + (\nabla_{d,r}\cdot (f(\Pi_h^* s)v_h), \varphi^0)$$
$$+(q_- \Pi_h^* s^0, \varphi^0) = 0, \quad \varphi \in \mathbb{S}_h^0(j,l), \quad (5.19)$$

subtracting equation (3.12) from (5.19) and using the fact $\left(\phi \frac{\partial s_h^0}{\partial t}, \varphi^0\right) = \left(\phi \frac{\partial Q_h^* s^0}{\partial t}, \varphi^0\right)$, then

$$\left(\phi \frac{\partial}{\partial t}(Q_h^* s^0 - s_h^0), \varphi^0\right) + (a(\Pi_h^* s^0)\Pi_h^* \nabla_{d,r}\cdot s, \nabla_{d,r}\cdot \varphi)$$
$$-(a(s_h^0)\nabla_{d,r}\cdot s_h, \nabla_{d,r}\cdot \varphi) + (q_- \Pi_h^* s^0, \varphi^0) - (\nabla_{d,r}\cdot (f(s_h)v_h), \varphi^0)$$
$$-(q_- s_h^0, \varphi^0) + (\nabla_{d,r}\cdot (f(\Pi_h^* s)v_h), \varphi^0) = 0,$$

Adding and subtracting for each, the terms $(a(Q_h^* s^0) Q_h^* \nabla_{d,r} \cdot s, \nabla_{d,r} \cdot \varphi)$, $(\nabla_{d,r} \cdot (f(Q_h^* s) v_h), \varphi^0)$ and $(q_- Q_h^* s^0, \varphi^0)$, and put $\varphi = e_s = (Q_h^* s - s_h)$,we have

$$\left(\phi \frac{\partial}{\partial t}(Q_h^* s^0 - s_h^0), e_s^0\right) + (a(\Pi_h^* s^0) \nabla_{d,r} \cdot (\Pi_h^* s - Q_h^* s), \nabla_{d,r} \cdot e_s)$$
$$+(q_-(Q_h^* s^0 - s_h^0), e_s^0) + ((a(\Pi_h^* s^0) - a(Q_h^* s^0)) \nabla_{d,r} \cdot Q_h^* s, \nabla_{d,r} \cdot e_s)$$
$$+((a(Q_h^* s^0) - a(s_h^0)) \nabla_{d,r} \cdot s_h, \nabla_{d,r} \cdot e_s) + \left(\nabla_{d,r} \cdot \left(v_h\left(f(\Pi_h^* s) - f(Q_h^* s)\right), e_s^0\right) +\right.$$
$$(\nabla_{d,r} \cdot (v_h(f(Q_h^* s) - f(s_h)), e_s^0) + (q_-(\Pi_h^* s^0 - Q_h^* s^0), e_s^0)$$
$$+(a(Q_h^* s^0) \nabla_{d,r} \cdot (Q_h^* s - s_h), \nabla_{d,r} \cdot e_s) = 0, \tag{5.20}$$

so,

$$\left(\phi \frac{\partial}{\partial t}(Q_h^* s^0 - s_h^0), e_s^0\right) + J_{51} + J_{52} + J_{53} + J_{54} + J_{55} + J_{56} + J_{57} + J_{58} = 0. \tag{5.21}$$

To estimate $J_{51} - J_{58}$ by Cauchy-Schwarz inequality and Young's inequality, we get

$$|J_{51}| = |(a(\Pi_h^* s^0) \nabla_{d,r} \cdot (\Pi_h^* s - Q_h^* s), \nabla_{d,r} \cdot e_s)| \leq \| \nabla_{d,r} (\Pi_h^* s - Q_h^* s) \|^2$$
$$+ \frac{\| a(\Pi_h^* s^0) \|^2}{4} \| \nabla_{d,r} e_s) \|^2 \leq Ch \| s_h \|^2 + C \| e_s \|^2,$$

$$|J_{52}| = |(q_-(Q_h^* s^0 - s_h^0), e_s^0)| \leq \| q_- \| \| e_s^0 \|^2 \leq C \| e_s^0 \|^2,$$

$$|J_{53}| = |((a(\Pi_h^* s^0) - a(Q_h^* s^0)) \nabla_{d,r} \cdot Q_h^* s, \nabla_{d,r} \cdot e_s)|$$
$$\leq \| (a(\Pi_h^* s^0) - a(Q_h^* s^0)) \|^2 \| \nabla_{d,r} \cdot Q_h^* s \|^2 + \frac{1}{4} \| \nabla_{d,r} e_s \|^2$$
$$\leq \alpha \ \| (a(\Pi_h^* s^0) - a(Q_h^* s^0)) \|^2 + \frac{\alpha}{4} \| \nabla_{d,r} \cdot Q_h^* s \|^2 + \frac{1}{4} \| \nabla_{d,r} e_s \|^2$$
$$\leq C \| (\Pi_h^* s^0 - Q_h^* s^0) \|^2 + C \| s_h \|^2 + C \| \nabla_{d,r} e_s \|^2 \leq Ch \| s_h^0 \|^2$$
$$+C \| s_h \|^2 + C \| e_s \|^2,$$

$$|J_{54}| = |((a(Q_h^* s^0) - a(s_h^0)) \nabla_{d,r} \cdot s_h, \nabla_{d,r} \cdot e_s)|$$
$$\leq \| ((a(Q_h^* s^0) - a(s_h^0)) \|^2 \| \nabla_{d,r} \cdot s_h \|^2 + \frac{1}{4} \| \nabla_{d,r} e_s \|^2$$
$$\leq \alpha \ \| ((a(Q_h^* s^0) - a(s_h^0)) \|^2 + \frac{\alpha}{4} \| \nabla_{d,r} \cdot s_h \|^2 + \frac{1}{4} \| \nabla_{d,r} e_s \|^2$$
$$\leq C \| (Q_h^* s^0 - s_h^0) \|^2 + C \| s_h \|^2 + C \| \nabla_{d,r} e_s \|^2 \leq C \| e_s^0 \|^2$$
$$+C \| s_h \|^2 + C \| e_s \|^2,$$

$$|J_{55}| = |(\nabla_{d,r} \cdot (v_h(f(\Pi_h^* s) - f(Q_h^* s)), e_s^0)| \leq \frac{1}{4} \| \nabla_{d,r} \cdot (v_h\big(f(\Pi_h^* s) - f(Q_h^* s)\big) \|^2 + \| e_s^0 \|^2 \leq \frac{\alpha}{4} \| v_h \|^2 \| (f(\Pi_h^* s) - f(Q_h^* s)) \|^2 + \| e_s^0 \|^2$$

$$\leq \frac{\alpha_1}{4} \| v_h \|^2 \| (\Pi_h^* s - Q_h^* s) \|^2 + \| e_s^0 \|^2 \leq Ch \| s_h \|^2 + \| e_s^0 \|^2,$$

$$|J_{56}| = |(\nabla_{d,r} \cdot (v_h(f(Q_h^* s) - f(s_h)), e_s^0)| \leq \frac{1}{4} \| \nabla_{d,r} \cdot (v_h\big(f(Q_h^* s) - f(s_h)\big) \|^2 + \| e_s^0 \|^2 \leq \frac{\alpha}{4} \| v_h \|^2 \| (f(Q_h^* s) - f(s_h)) \|^2 + \| e_s^0 \|^2$$

$$\leq \frac{\alpha_1}{4} \| v_h \|^2 \| (Q_h^* s - s_h) \|^2 + \| e_s^0 \|^2 \leq Ch \| e_s \|^2 + \| e_s^0 \|^2,$$

$$|J_{57}| = |(q_-(\Pi_h^* s^0 - Q_h^* s^0), e_s^0)| \leq \frac{\| q_- \|^2}{4} \| (\Pi_h^* s^0 - Q_h^* s^0) \|^2 + \| e_s^0 \|^2$$

$$\leq Ch^2 \| s_h^0 \|^2 + \| e_s^0 \|^2,$$

$$|J_{58}| = |(a(Q_h^* s^0)\nabla_{d,r} \cdot (Q_h^* s - s_h), \nabla_{d,r} \cdot e_s)| = |(a(Q_h^* s^0)\nabla_{d,r} \cdot e_s, \nabla_{d,r} \cdot e_s)|$$

$$\leq \| a(Q_h^* s^0) \| \| \nabla_{d,r} \cdot e_s \|^2 \leq C \| e_s \|^2,$$

substituting $J_{51} - J_{58}$ in the equation (5.21), we get

$$\left(\phi \frac{\partial}{\partial t} (Q_h^* s^0 - s_h^0), e_s^0\right) + Ch \| s_h \|^2 + C \| e_s \|^2 + C \| e_s^0 \|^2 + Ch \| s_h^0 \|^2$$

$$+ C \| s_h \|^2 + C \| e_s \|^2 + C \| e_s^0 \|^2 + C \| s_h \|^2 + C \| e_s \|^2 + Ch \| s_h \|^2$$

$$+ \| e_s^0 \|^2 + Ch \| e_s \|^2 + \| e_s^0 \|^2 + Ch^2 \| s_h^0 \|^2 + \| e_s^0 \|^2 + C \| e_s \|^2 \leq 0,$$

so,

$$\| \phi \| \frac{1}{2} \frac{\partial}{\partial t} \| e_s^0 \|^2 + Ch \| s_h \|^2 + C \| e_s \|^2 + C \| e_s^0 \|^2 + Ch \| s_h^0 \|^2 + C \| s_h \|^2$$

$$+ C \| e_s \|^2 + C \| e_s^0 \|^2 + C \| s_h \|^2 + C \| e_s \|^2 + Ch \| s_h \|^2 + \| e_s^0 \|^2$$

$$+ Ch \| e_s \|^2 + \| e_s^0 \|^2 + Ch^2 \| s_h^0 \|^2 + \| e_s^0 \|^2 + C \| e_s \|^2 \leq 0,$$

$$\frac{\partial}{\partial t} \| e_s^0 \|^2 + \beta \| e_s^0 \|^2 \leq \beta_1 h [\| s_h \|^2 + \| s_h^0 \|^2 + \| e_s \|^2].$$

where $\beta = min\{2C, 3\}$ and $\beta_1 = max\{-2C, -2C/h, -C, -Ch, -4C/h, -1/h\}$.

Multiply both sides of the above inequality by the integral factor $exp(\beta z)$ and then integrate from 0 to t, we have

$\| e_s^0(t) \|^2 \leq exp(-\beta t) \| e_s^0(0) \|^2 + \beta_1 h \ exp(-\beta t) \int_0^t exp(\beta z)[\| s_h(z) \|^2 + \| s_h^0(z) \|^2 + \| e_s(z) \|^2] \, dz.$

Theorem 5.2 *Let $v_h \in \mathbb{Q}_h$, $p_h \in \mathbb{G}_h$ and $s_h \in \mathbb{S}_h^0$ are the solutions for the equations (3.10) , (3.11) and (3.12), respectively. And $Q_h v$, $Q'_h p$ and $Q_h^* s$ are the L^2 projection operators for all v_h, p_h and s_h on the corresponding finite element spaces . Then*

$$\| e_v \|^2 + \| e_p \|^2 + \| e_s^0(t) \|^2 \leq exp(-\beta t) \| e_s^0(0) \|^2 + \beta_2 h[\| v_h \|^2 + \| v_h^0 \|^2 + \| p_h \|^2 + \| e_v^0 \|^2 + \| e_p^0 \|^2 + \int_0^t exp(\beta(z-t))[\| s_h(z) \|^2 + \| s_h^0(z) \|^2 + \| e_s(z) \|^2] \, dz]. \quad (5.22)$$

where β_2 and β are the constants and independent h.

Proof. From the equations (5.14) and (5.18), we get proof.

5.2 H^1 Error Analysis for Velocity, Pressure and Saturation

Lemma 5.7 *(H^1 Error Analysis for Velocity).*

Let $v_h \in \mathbb{Q}_h$ be a solution to the dual equations (3.10 and 3.11). Then

$$\| \nabla_{d,r} \cdot e_v(t) \|^2 + \| e_v(t) \|^2 \leq C_2 h \int_0^t [\| (v_h)_z(z) \|^2 + \| e_p^0(z) \|^2 + \| (v_h^0)_\delta(z) \|^2 + \| e_v^0(z) \|^2 + \| (e_v^0)_z(z) \|^2] \, dz + \| \nabla_{d,r} \cdot e_v(0) \|^2 + \| e_v(0) \|^2. \quad (5.23)$$

where C_2 is a constant and independent h.

Proof. Put $e_v = (e_v)_t$ and $e_p = (e_p)_t$ in equation (5.8) , we have

$$((\Pi_h v^0 - Q_h v^0), (e_v^0)_t) + ((Q_h v^0 - v_h^0), (e_v^0)_t)$$
$$= \left(\nabla_{d,r} \cdot (\Pi_h v - Q_h v), (e_p^0)_t\right) + \left(\nabla_{d,r} \cdot (Q_h v - v_h), (e_p^0)_t\right), \quad (5.24)$$

so,

$$J_{21} + J_{22} = J_{23} + J_{24},$$

by Cauchy-Schwarz inequality and Young's inequality, we get

$$|J_{21}| = |((\Pi_h v^0 - Q_h v^0), (e_v^0)_t)| \leq |\frac{\partial}{\partial t}((\Pi_h v^0 - Q_h v^0), e_v^0)|$$
$$-|\left(\frac{\partial}{\partial t}(\Pi_h v^0 - Q_h v^0), e_v^0\right)| \leq \frac{\partial}{\partial t}|((\Pi_h v^0 - Q_h v^0), e_v^0)|$$
$$+\frac{1}{4} \| (\Pi_h v^0 - Q_h v^0)_t \|^2 + \| e_v^0 \|^2 \leq Ch^2 \| (v_h^0)_t \|^2 + \| e_v^0 \|^2,$$

$$|J_{22}| = |((Q_h v^0 - v_h^0), (e_v^0)_t)| \leq |\frac{\partial}{\partial t}\left((Q_h v^0 - v_h^0), e_v^0\right)| - \left|\left(\frac{\partial}{\partial t}(Q_h v^0 - v_h^0), e_v^0\right)\right| \leq |\frac{\partial}{\partial t}((Q_h v^0 - v_h^0), e_v^0)| + C \| (e_v^0)_t \|^2 \leq C \| (e_v^0)_t \|^2,$$

$$|J_{23}| = |\left(\nabla_{d,r} \cdot (\Pi_h v - Q_h v), (e_p^0)_t\right)| \leq |\frac{\partial}{\partial t}\left(\nabla_{d,r} \cdot (\Pi_h v - Q_h v), e_p^0\right)|$$
$$-|\left(\frac{\partial}{\partial t}\nabla_{d,r} \cdot (\Pi_h v - Q_h v), e_p^0\right)| \leq |\frac{\partial}{\partial t}\left(\nabla_{d,r} \cdot (\Pi_h v - Q_h v), e_p^0\right)|$$
$$+\frac{1}{4} \| \nabla_{d,r} \cdot ((\Pi_h v - v_h) - (Q_h v - v_h))_t \|^2 + \| e_p^0 \|^2$$
$$\leq |\frac{\partial}{\partial t}\left(\nabla_{d,r} \cdot (\Pi_h v - Q_h v), e_p^0\right)| + Ch \| (v_h)_t \|^2 + Ch \| (e_v)_t \|^2 + \| \varrho^0 \|^2 \leq Ch \| (v_h)_t \|^2 + Ch \| (e_v)_t \|^2 + \| e_p^0 \|^2,$$

$$|J_{24}| = |\left(\nabla_{d,r} \cdot (Q_h v - v_h), (e_p^0)_t\right)| \leq |\frac{\partial}{\partial t}\left(\nabla_{d,r} \cdot (Q_h v - v_h), e_p^0\right)|$$
$$-|\left(\frac{\partial}{\partial t}\nabla_{d,r} \cdot (Q_h v - v_h), e_p^0\right)| \leq |\frac{\partial}{\partial t}\left(\nabla_{d,r} \cdot (Q_h v - v_h), e_p^0\right)|$$
$$+\frac{1}{4} \| \nabla_{d,r} \cdot (e_v)_t \|^2 + \| e_p^0 \|^2 \leq \frac{1}{4} \| \nabla_{d,r} \cdot (e_v)_t \|^2 + \| e_p^0 \|^2,$$

then equation (5.24) becomes

$$Ch \| (v_h)_t \|^2 + Ch \| (e_v)_t \|^2 + \| e_p^0 \|^2 + \frac{1}{4} \| \nabla_{d,r} \cdot (e_v)_t \|^2 + \| e_p^0 \|^2$$
$$\leq Ch^2 \| (v_h^0)_t \|^2 + \| e_v^0 \|^2 + C \| (e_v^0)_t \|^2,$$

then

$$\frac{\partial}{\partial t}[\| \nabla_{d,r} \cdot e_v \|^2 + \| e_v \|^2] \leq C_2 h[\| (v_h)_t \|^2 + \| e_p^0 \|^2 + \| (v_h^0)_t \|^2 + \| e_v^0 \|^2 + \| (e_v^0)_t \|^2],$$

where $C_2 = max\{-C, -2/h, Ch, 1/h, C/h\}/min\{1/4, Ch\}$, and integrate with respect to z from 0 to t, we have

$$\| \nabla_{d,r} \cdot e_v(t) \|^2 + \| e_v(t) \|^2 \leq C_2 h^2 \int_0^t [\| (v_h)_z(z) \|^2 + \| e_p^0(z) \|^2 + \| (v_h^0)_z(z) \|^2$$
$$+ \| e_v^0(\delta) \|^2 + \| (e_v^0)_z(z) \|^2]\, dz + \| \nabla_{d,r} \cdot e_v(0) \|^2 + \| e_v(0) \|^2.$$

Lemma 5.8 *(H^1 Error Analysis for Pressure).*

Let $p_h \in \mathbb{G}_h$ be a solution to the equation (3.11). Then

$$\| \nabla_{d,r} e_p(t) \|^2 + \| e_p(t) \|^2 \leq \| \nabla_{d,r} e_p(0) \|^2 + \| e_p(0) \|^2$$
$$+ C'_2 h \int_0^t [\| (p_h)_z(z) \|^2 + \| e_v^0(z) \|^2]\, dz. \quad (5.25)$$

where C'_2 is a constant and independent h.

Proof. Put $u = (e_v)_t$ in equation (5.12) , we have

$$(\alpha(s_h^0)(\Pi'_h \nabla_{d,r} p - \nabla_{d,r} Q'_h p), (e_v^0)_t) + (\alpha(s_h^0)(\nabla_{d,r} p_h - \nabla_{d,r} Q'_h p), (e_v^0)_t) = 0$$
$$(\alpha(s_h^0)\nabla_{d,r}(\Pi'_h p - Q'_h p), (e_v^0)_t) + (\alpha(s_h^0)\nabla_{d,r}(p_h - Q'_h p), (e_v^0)_t) = 0, \quad (5.26)$$

so,

$$J_{41} + J_{42} = 0,$$

by Cauchy-Schwarz inequality and Young's inequality, we get

$$|J_{41}| = |(\alpha(s_h^0)\nabla_{d,r}(\Pi'_h p - Q'_h p), (e_v^0)_t)| \leq \left| \frac{\partial}{\partial t}\left(\alpha(s_h^0)\nabla_{d,r}\left(\Pi'_h p - Q'_h p\right), e_v^0\right) \right| -$$
$$\left| \left(\frac{\partial}{\partial t}\alpha(s_h^0)\nabla_{d,r}(\Pi'_h p - Q'_h p), e_v^0\right) \right| \leq \left| \frac{\partial}{\partial t}(\alpha(s_h^0)\nabla_{d,r}(\Pi'_h p - Q'_h p), e_v^0) \right|$$
$$+ \frac{\|\alpha(s_h^0)\|^2}{4} \| e_v^0 \|^2 + \| \nabla_{d,r}(\Pi'_h p - p_h - (Q'_h p - p_h))_t \|^2$$
$$\leq \left| \frac{\partial}{\partial t}(\alpha(s_h^0)\nabla_{d,r}(\Pi'_h p - Q'_h p), e_v^0) \right| + \frac{\|\alpha(s_h^0)\|^2}{4} \| e^0 \|^2 + C'h \| (p_h)_t \|^2$$
$$+ C'h \| \nabla_{d,r}(e_p)_t \|^2 \leq C'h \| (p_h)_t \|^2 + C'h \| \nabla_{d,r}(e_p)_t \|^2 + \frac{\|\alpha(s_h^0)\|^2}{4} \| e_v^0 \|^2,$$

$$|J_{42}| = |(\alpha(s_h^0)(\nabla_{d,r}(Q'_h p - p_h), (e_v^0)_t)| \leq \left| \frac{\partial}{\partial t}\left(\alpha(s_h^0)\left(\nabla_{d,r}\left(Q'_h p - p_h\right), e_v^0\right) \right| -$$
$$\left| \left(\frac{\partial}{\partial t}\alpha(s_h^0)(\nabla_{d,r}(Q'_h p - p_h), e_v^0\right) \right| \leq \left| \frac{\partial}{\partial t}(\alpha(s_h^0)(\nabla_{d,r}(Q'_h p - p_h), e_v^0) \right|$$
$$+ \frac{\|\alpha(s_h^0)\|^2}{4} \| e_v^0 \|^2 + \| (\nabla_{d,r}(Q'_h p - p_h)_t) \|^2 \leq \frac{\|\alpha(s_h^0)\|^2}{4} \| e_v^0 \|^2 + C' \| (e_p)_t \|^2,$$

substituting J_{41} and J_{42} in the equation (5.26), we get

$$\frac{\|\alpha(s_h^0)\|^2}{4} \| e_v^0 \|^2 + C' \| (e_p)_t \|^2 + \frac{\|\alpha(s_h^0)\|^2}{4} \| e_v^0 \|^2 + C'h \| (p_h)_t \|^2$$
$$+C'h \| \nabla_{d,r}(e_p)_t \|^2 \leq 0$$

$$\frac{\partial}{\partial t}[\| \nabla_{d,r} e_p \|^2 + \| e_p \|^2] \leq C'_2 h[\| (p_h)_t \|^2 + \| e_v^0 \|^2],$$

where $C'_2 = max\{-C', -\frac{\|\alpha(s_h^0)\|^2}{2h}\}/min\{C', C'h\}$, and integrate with respect to z from 0 to t, we get

$$\| \nabla_{d,r} e_p(t) \|^2 + \| e_p(t) \|^2 \leq \| \nabla_{d,r} e_p(0) \|^2 + \| e_p(0) \|^2$$
$$+C'_2 h^2 \int_0^t [\| (p_h)_z(z) \|^2 + \| e_v^0(z) \|^2] \, dz.$$

Theorem 5.3 *(H^1 Error Analysis for Velocity and Pressure).*
Let $v_h \in \mathbb{Q}_h$ and $p_h \in \mathbb{G}_h$ are the solutions to the dual equations (3.10, 3.11). Then

$$\| \nabla_{d,r} \cdot e_v(t) \|^2 + \| e_v(t) \|^2 + \| \nabla_{d,r} e_p(t) \|^2 + \| e_p(t) \|^2 \leq C''_1 h \int_0^t [\|$$
$$(v_h)_z(z) \|^2 + \| (v_h^0)_z(z) \|^2 + \| (p_h)_z(z) \|^2 + \| e_v^0(z) \|^2 + \| (e_v^0)_z(z) \|^2$$
$$+\| e_p^0(z) \|^2] dz + \| \nabla_{d,r} \cdot e_v(0) \|^2 + \| e_v(0) \|^2$$
$$+\| \nabla_{d,r} e_p(0) \|^2 + \| e_p(0) \|^2. \tag{5.27}$$

where C''_1 is a constant and independent h.

Proof. Put $e_v = (e_v)_t$ and $e_p = (e_p)_t$ in equation (5.17) , we have

$$\left(\nabla_{d,r} \cdot (\Pi_h v - Q_h v), (e_p^0)_t\right) + \left(\nabla_{d,r} \cdot (Q_h v - v_h), (e_p^0)_t\right)$$
$$= ((\Pi_h v^0 - Q_h v^0), (e_v^0)_t) + ((Q_h v^0 - v_h^0), (e_v^0)_t) \tag{5.28}$$
$$+(\alpha(s_h^0)\nabla_{d,r}(Q'_h p - \Pi'_h p), (e_v^0)_t) + (\alpha(s_h^0)\nabla_{d,r}(Q'_h p - p_h), (e_v^0)_t),$$

then, equation (5.29) becomes

$$J_{23} + J_{24} = J_{21} + J_{22} - J_{41} - J_{42},$$

by lemmas (5.7) and (5.8), we get

$$Ch \| (v_h)_t \|^2 + Ch \| (e_v)_t \|^2 + \| e_p^0 \|^2 + \frac{1}{4} \| \nabla_{d,r} \cdot (e_v)_t \|^2 + \| e_p^0 \|^2 \leq Ch^2 \| (v_h^0)_t \|^2$$

$$+\| e_v^0 \|^2 + C \| (e_v^0)_t \|^2 - \frac{\|\alpha(s_h^0)\|^2}{4} \| e_v^0 \|^2 - C' \| (e_p)_t \|^2$$
$$-\frac{\|\alpha(s_h^0)\|^2}{4} \| e_v^0 \|^2 - C'h \| (p_h)_t \|^2 - C'h \| \nabla_{d,r}(e_p)_t \|^2,$$

so,

$$\frac{\partial}{\partial t}[\| \nabla_{d,r} \cdot e_v \|^2 + \| e_v \|^2 + \| \nabla_{d,r} e_p \|^2 + \| e_p \|^2] \leq C''_1 h[\| (v_h)_t \|^2 + \| (v_h^0)_t \|^2$$
$$+\| (p_h)_t \|^2 + \| e_v^0 \|^2 + \| (e_v^0)_t \|^2 + \| e_p^0 \|^2],$$

where $C''_1 = max\{-C/h, Ch, -C', 1/h, -\frac{\|\alpha(s_h^0)\|^2}{2h}, C/h, -2/h\}/\{1/4, Ch, C', C'h\}$, and integrate with respect to z from 0 to t, we have

$$\| \nabla_{d,r} \cdot e_v(t) \|^2 + \| e_v(t) \|^2 + \| \nabla_{d,r} e_p(t) \|^2 + \| e_p(t) \|^2 \leq C''_1 h \int_0^t [\| (v_h)_z(z) \|^2 + \| (v_h^0)_z(z) \|^2 + \| (p_h)_z(z) \|^2 + \| e_v^0(z) \|^2 + \| (e_v^0)_z(z) \|^2$$
$$+\| e_p^0(z) \|^2]dz + \| \nabla_{d,r} \cdot e_v(0) \|^2 + \| e_v(0) \|^2$$
$$+\| \nabla_{d,r} e_p(0) \|^2 + \| e_p(0) \|^2.$$

Lemma 5.9 *(Error Analysis H^1 for Saturation).*

Let $s_h \in \mathbb{S}_h(j;l)$ be the weak Galerkin approximation of s arising from (3.12). Denote by $e_s = s_h - Q_h^* s$ the difference between the weak Galerkin approximation and the L^2 projection of the exaction solution $s = (s_1, s_2)$. Then there exists a constant $C"_3$ such that

$$\| \nabla_{d,r} \cdot e_s(t) \|^2 + \| e_s(t) \|^2 \leq C''_3 h \int_0^t [\| s_h^0(z) \|^2 + \| (s_h^0(z))_z \|^2 + \| s_h(z) \|^2 + \| (s_h(z))_z \|^2 + \| (e_s^0)_z(z) \|^2 + \| e_s^0(z) \|^2 + \| e_s(z) \|^2$$
$$+\| (v_h(z))_z \|^2]dz - \| \nabla_{d,r} \cdot e_s(0) \|^2 - \| e_s(0) \|^2. \qquad (5.29)$$

provided that the meshsize h is sufficiently small.

Proof. Put $e_s = (e_s)_t$ in equation (5.20), we get

$$\left(\phi \frac{\partial}{\partial t}(Q_h^* s^0 - s_h^0), (e_s^0)_t\right) + (\alpha(\Pi_h^* s^0)\nabla_{d,r} \cdot (\Pi_h^* s - Q_h^* s), \nabla_{d,r} \cdot (e_s)_t)$$

$$+(q_{-}(Q_h^* s^0 - s_h^0), (e_s^0)_t) + ((a(\Pi_h^* s^0) - a(Q_h^* s^0))\nabla_{d,r} \cdot Q_h^* s, \nabla_{d,r} \cdot (e_s)_t) + ((a(Q_h^* s^0) - a(s_h^0))\nabla_{d,r} \cdot s_h, \nabla_{d,r} \cdot (e_s)_t)$$

$$+(\nabla_{d,r} \cdot (v_h(f(\Pi_h^* s) - f(Q_h^* s)), (e_s^0)_t) + (\nabla_{d,r} \cdot (v_h(f(Q_h^* s) - f(s_h)), (e_s^0)_t) + (q_{-}(\Pi_h^* s^0 - Q_h^* s^0), (e_s^0)_t) + (a(Q_h^* s^0)\nabla_{d,r} \cdot (Q_h^* s - s_h), \nabla_{d,r} \cdot (e_s)_t) = 0,$$

so,

$$\left(\phi \frac{\partial}{\partial t}(Q_h^* s^0 - s_h^0), (e_s^0)_t\right) + J_{61} + J_{62} + J_{63} + J_{64} + J_{65} + J_{66} + J_{67} + J_{68} = 0,$$

then the above equation becomes

$$C \ \| \phi \| \| (e_s^0)_t \|^2 + J_{61} + J_{62} + J_{63} + J_{64} + J_{65} + J_{66} + J_{67} + J_{68} = 0. \tag{5.30}$$

To estimate $I_{61} - I_{68}$,by Cauchy-Schwarz inequality and Young's inequality, we get

$$|J_{61}| = |(a(\Pi_h^* s^0)\nabla_{d,r} \cdot (\Pi_h^* s - Q_h^* s), \nabla_{d,r} \cdot (e_s)_t)|$$
$$\leq |\frac{\partial}{\partial t} a(\Pi_h^* s^0)\nabla_{d,r} \cdot (\Pi_h^* s - Q_h^* s), \nabla_{d,r} e_s)|$$
$$-|(\frac{\partial}{\partial t} a(\Pi_h^* s^0)\nabla_{d,r} \cdot (\Pi_h^* s - Q_h^* s), \nabla_{d,r} e_s)|$$
$$\leq |\frac{\partial}{\partial t}(a(\Pi_h^* s^0)\nabla_{d,r} \cdot (\Pi_h^* s - Q_h^* s), \nabla_{d,r} e_s)|$$
$$+\frac{\|a(\Pi_h^* s^0)\|^2}{4} \| \nabla_{d,r} \cdot (\Pi_h^* s - s_h - (Q_h^* s - s_h))_t \|^2 + C \| e_s \|^2$$
$$\leq C \| a(\Pi_h^* s^0) \|^2 + C \| e_s \|^2 + \frac{\alpha_1}{4} \| \nabla_{d,r} \cdot (\Pi_h^* s - s_h) \|^2$$
$$+\frac{\alpha_1}{16} \| \nabla_{d,r} \cdot (Q_h^* s - s_h) \|^2 \leq C \| s_h^0 \|^2 + Ch \| s_h \|^2$$
$$+C \| \nabla_{d,r} \cdot (e_s)_t \|^2 + C \| e_s \|^2,$$

$$|J_{62}| = |(q_{-}(Q_h^* s^0 - s_h^0), (e_s^0)_t)| \leq |\frac{\partial}{\partial t} q_{-}(Q_h^* s^0 - s_h^0), e_s^0)|$$
$$-|(\frac{\partial}{\partial t}(q_{-}(Q_h^* s^0 - s_h^0), e_s^0)|$$
$$\leq |\frac{\partial}{\partial t} q_{-}(Q_h^* s^0 - s_h^0), e_s^0)| + \frac{\|q_{-}\|}{2} \| (e_s^0)_t \|^2 \leq C \ \| (e_s^0)_t \|^2,$$

$$|J_{63}| = |((a(\Pi_h^* s^0) - a(Q_h^* s^0))\nabla_{d,r} \cdot Q_h^* s, \nabla_{d,r} \cdot (e_s)_t)|$$
$$\leq \left|\frac{\partial}{\partial t}((a(\Pi_h^* s^0) - a(Q_h^* s^0))\nabla_{d,r} \cdot Q_h^* s, \nabla_{d,r} \cdot e_s)\right|$$
$$-\left|\left(\frac{\partial}{\partial t}((a(\Pi_h^* s^0) - a(Q_h^* s^0))\nabla_{d,r} \cdot Q_h^* s, \nabla_{d,r} \cdot e_s\right)\right|$$

$$\leq \| \nabla_{d,r} \cdot e_s \|^2 + \frac{\alpha_2}{4} \| ((a(\Pi_h^* s^0) - a(Q_h^* s^0))_t \|^2$$

$$+ \left| \frac{\partial}{\partial t}((a(\Pi_h^* s^0) - a(Q_h^* s^0))\nabla_{d,r} \cdot Q_h^* s, \nabla_{d,r} \cdot e_s) \right|$$

$$+ \frac{\alpha_2}{16} \| \nabla_{d,r} \cdot Q_h^* s_t \|^2 \leq C \| e_s \|^2 + C \| (s_h)_t \|^2 + Ch \| (s_h^0)_t \|^2,$$

$$|J_{64}| = |((a(Q_h^* s^0) - a(s_h^0))\nabla_{d,r} \cdot s_h, \nabla_{d,r} \cdot (e_s)_t)|$$

$$\leq |\frac{\partial}{\partial t}((a(Q_h^* s^0) - a(s_h^0))\nabla_{d,r} \cdot s_h, \nabla_{d,r} \cdot e_s)|$$

$$-|(\frac{\partial}{\partial t}(a(Q_h^* s^0) - a(s_h^0))\nabla_{d,r} \cdot s_h, \nabla_{d,r} \cdot e_s)|$$

$$\leq |\frac{\partial}{\partial t}((a(Q_h^* s^0) - a(s_h^0))\nabla_{d,r} \cdot s_h, \nabla_{d,r} \cdot e_s)|$$

$$+ \| \nabla_{d,r} \cdot e_s \|^2 + \frac{\alpha_2}{4} \| (\nabla_{d,r} \cdot s_h)_t \|^2 + \frac{\alpha_2}{16} \| (a(Q_h^* s^0) - a(s_h^0))_t \|^2$$

$$\leq C \| e_s \|^2 + C \| (s_h)_t \|^2 + C \| (e_s^0)_t \|^2,$$

$$|J_{65}| = |(\nabla_{d,r} \cdot (v_h(f(\Pi_h^* s) - f(Q_h^* s)), (e_s^0)_t)|$$

$$\leq |\frac{\partial}{\partial t}(\nabla_{d,r} \cdot (v_h(f(\Pi_h^* s) - f(Q_h^* s)), e_s^0)|$$

$$-|(\frac{\partial}{\partial t}\nabla_{d,r} \cdot (v_h(f(\Pi_h^* s) - f(Q_h^* s)), e_s^0)|$$

$$\leq |\frac{\partial}{\partial t}(\nabla_{d,r} \cdot (v_h(f(\Pi_h^* s) - f(Q_h^* s)), e_s^0)|$$

$$+ \frac{1}{4} \| \frac{\partial}{\partial t}(\nabla_{d,r} \cdot (v_h(f(\Pi_h^* s) - f(Q_h^* s)) \|^2 + \| e_s^0 \|^2$$

$$\leq \frac{\alpha_3}{4} \| \frac{\partial}{\partial t}((v_h(f(\Pi_h^* s) - f(Q_h^* s)) \|^2 + \| e_s^0 \|^2 \leq \frac{\alpha_3}{4} \| (v_h)_t \|^2$$

$$+ \frac{\alpha_3}{16} \| ((\Pi_h^* s - s_h) - (Q_h^* s - s_h))_t \|^2 + \| e_s^0 \|^2 \leq C \| (v_h)_t \|^2$$

$$+ Ch^2 \| (s_h)_t \|^2 + C \| (e_s)_t \|^2 + \| e_s^0 \|^2,$$

$$|J_{66}| = |(\nabla_{d,r} \cdot (v_h(f(Q_h^* s) - f(s_h)), (e_s^0)_t)|$$

$$\leq |\frac{\partial}{\partial t}(\nabla_{d,r} \cdot (v_h(f(Q_h^* s) - f(s_h)), e_s^0)|$$

$$-|(\frac{\partial}{\partial t}\nabla_{d,r} \cdot (v_h(f(Q_h^* s) - f(s_h)), e_s^0)|$$

$$\leq \frac{\partial}{\partial t}(\nabla_{d,r} \cdot (v_h(f(Q_h^* s) - f(s_h)), e_s^0)|$$

$+\frac{1}{4}\| \frac{\partial}{\partial t}(\nabla_{d,r}\cdot(v_h(f(Q_h^* s)-f(s_h))\|^2+\| e_s^0\|^2$

$\leq \frac{\alpha_3}{4}\| \frac{\partial}{\partial t}(Q_h^* s-s_h)\|^2+\frac{\alpha_3}{16}\| \frac{\partial}{\partial t}(v_h)\|^2$

$+\| e_s^0\|^2\leq C\| (e_s)_t\|^2+C\| (v_h)_t\|^2+\| e_s^0\|^2,$

$|J_{67}|=|(q_-(\Pi_h^* s^0-Q_h^* s^0),(e_s^0)_t)|\leq|\frac{\partial}{\partial t}(q_-(\Pi_h^* s^0-Q_h^* s^0),e_s^0)|$

$-|(\frac{\partial}{\partial t}q_-(\Pi_h^* s^0-Q_h^* s^0),e_s^0)|\leq|\frac{\partial}{\partial t}(q_-(\Pi_h^* s^0-Q_h^* s^0),e_s^0)|$

$+\frac{\|q_-\|^2}{4}\|(\Pi_h^* s^0-Q_h^* s^0)_t\|^2+\| e_s^0\|^2\leq Ch^2\|(s_h^0)_t\|^2+\| e_s^0\|^2,$

$|J_{68}|=|(a(Q_h^* s^0)\nabla_{d,r}\cdot(Q_h^* s-s_h),\nabla_{d,r}\cdot(e_s)_t)|$

$\leq|\frac{\partial}{\partial t}(a(Q_h^* s^0)\nabla_{d,r}\cdot(Q_h^* s-s_h),\nabla_{d,r}\cdot e_s)|$

$-|(\frac{\partial}{\partial t}a(Q_h^* s^0)\nabla_{d,r}\cdot(Q_h^* s-s_h),\nabla_{d,r}\cdot e_s)|$

$\leq|\frac{\partial}{\partial t}(a(Q_h^* s^0)\nabla_{d,r}\cdot(Q_h^* s-s_h),\nabla_{d,r}\cdot e_s)|$

$+\frac{\alpha_4}{4}\|(a(Q_h^* s^0)_t\|^2+\frac{\alpha_4}{16}\|(\nabla_{d,r}\cdot(Q_h^* s-s_h))_t\|^2+\|\nabla_{d,r}\cdot e_s\|^2$

$\leq C\|(s_h^0)_t\|^2+C\|\nabla_{d,r}\cdot(e_s)_t\|^2+C\| e_s\|^2,$

substituting $J_{61}-J_{68}$ in equation (5.30), we get

$C\ \|\phi\|\|(e_s^0)_t\|^2+C\| s_h^0\|^2+Ch\| s_h\|^2+C\|\nabla_{d,r}\cdot(e_s)_t\|^2+C\| e_s\|^2+C\|$
$(e_s^0)_t\|^2$

$+C\| e_s\|^2+C\|(s_h)_t\|^2+Ch\|(s_h^0)_t\|^2+C\| e_s\|^2+C\|(s_h)_t\|^2$

$+C\|(e_s^0)_t\|^2+C\|(v_h)_t\|^2+Ch^2\|(s_h)_t\|^2+C\|(e_s)_t\|^2+\| e_s^0\|^2$

$+C\|(e_s)_t\|^2+C\|(v_h)_t\|^2+\| e_s^0\|^2+Ch^2\|(s_h^0)_t\|^2+\| e_s^0\|^2$

$+C\|(s_h^0)_t\|^2+C\|\nabla_{d,r}\cdot(e_s)_t\|^2+C\| e_s\|^2\leq 0,$

so,

$\frac{\partial}{\partial t}[\|\nabla_{d,r}\cdot e_s\|^2+\| e_s\|^2]\leq C''{}_3 h[\| s_h^0\|^2+\|(s_h^0)_t\|^2+\| s_h\|^2+\|(s_h)_t\|^2+\|$
$(e_s^0)_t\|^2+\| e_s^0\|^2+\| e_s\|^2+\|(v_h)_t\|^2]$

where $C''_3=max\{C/h,C,Ch,2C/h,3/h,4C/h,C\|\phi\|/h\}/(-2C)$, and integrate with respect to z

from 0 to t, we have

$$\| \nabla_{d,r} \cdot e_s(t) \|^2 + \| e_s(t) \|^2 \leq C''{}_3 h \int_0^t [\| s_h^0(z) \|^2 + \| (s_h^0(z))_z \|^2 + \| s_h(z) \|^2 + \| (s_h(z))_z \|^2 + \| (e_s^0)_z(z) \|^2 + \| e_s^0(z) \|^2 + \| e_s(z) \|^2 + \| (v_h(z))_z \|^2] dz - \| \nabla_{d,r} \cdot e_s(0) \|^2 - \| e_s(0) \|^2.$$

6 Numerical Result

For numerical validation of the weak Galerkin method presented in the previous section, we consider a two-dimensional test case of incompressible immiscible displacement.

The weak Galerkin method in the synthetic problem for a coupled system of pressure-saturation equation that admits an exact solution, and investigates the effects of the accuracy of total velocity reconstruction on the convergence order for a coupled system. The numerical simulations have been performed using developed by authors' MATLAB software package that implements the weak Galerkin method in two-dimensional geometry in a fast sparse matrix programming environment.

We are considered in $\Omega = (0,1) \times (0,1)$ for the benchmark problem given below [1] [26]:

$$\nabla \cdot (k(s) \nabla p) = 0, \tag{6.1}$$

$$v = k(s) \nabla p\,, \tag{6.2}$$

$$\frac{\partial}{\partial t} s + \nabla \cdot (-\epsilon \nabla s + v f(s)) = F\,, \tag{6.3}$$

with $k(s) = (0.5 - 0.2s)^{-1}$, $\epsilon = 0.01$, $f(s) = s$, where $F = 2\pi^2 \epsilon \sin(\pi(x + y - 2t))$, boundary and initial conditions correspond to the exact solution

$$p = \frac{0.2}{\pi} \cos(\pi(x + y - 2t)) + 0.5(x + y),$$

$$s = \sin(\pi(x + y - 2t)).$$

In Tables 1, 2 and 3 for the errors and convergence orders of first order weak Galerkin method, calculated on nested sequences of structured triangular meshes, are presented for pressure, total velocity and saturation at final time $t = 1$. In the simulation, a small uniform time step was used to eliminate time error pollution. We can observe that the convergence order is optimal for all three variables.

h	$\| v - v_h \|_{H^1}$	Order	$\| v - v_h \|_{L^2}$	Order	$\| v - v_h \|_{L^\infty}$	Order
1/2	1.7387e-01	0	2.1316e-01	0	2.0750e-01	0
1/4	1.0541e-01	7.2207e-01	4.3763e-02	2.2842e+00	5.4918e-02	1.9177e+00
1/8	4.5946e-02	1.1980e+00	1.0297e-02	2.0874e+00	1.4161e-02	1.9553e+00
1/16	2.1004e-02	1.1292e+00	2.5313e-03	2.0243e+00	3.5977e-03	1.9768e+00
1/32	9.9384e-03	1.0796e+00	6.3010e-04	2.0063e+00	9.0684e-04	1.9882e+00
1/64	4.8180e-03	1.0446e+00	1.5735e-04	2.0016e+00	2.2765e-04	1.9940e+00

Table 1: Errors for Velocity of weak Galerkin method with fixed $t = 1, \Delta t = 0.001$ and $\varepsilon = 0.01$.

h	$\| p - p_h \|_{H^1}$	Order	$\| p - p_h \|_{L^2}$	Order	$\| p - p_h \|_{L^\infty}$	Order
1/2	4.6756e-03	0	3.9482e-03	0	1.9560e-01	0
1/4	2.6169e-03	8.3730e-01	1.0132e-03	1.9622e+00	5.7881e-02	1.7567e+00
1/8	1.3364e-03	9.6948e-01	2.5313e-04	2.0010e+00	1.4568e-02	1.9903e+00
1/16	6.7069e-04	9.9466e-01	6.3244e-05	2.0009e+00	3.6525e-03	1.9959e+00
1/32	3.3559e-04	9.9895e-01	1.5808e-05	2.0002e+00	9.1401e-04	1.9986e+00
1/64	1.6782e-04	9.9976e-01	3.9519e-06	2.0001e+00	2.2857e-04	1.9996e+00

Table 2: Errors for Pressure of weak Galerkin method with fixed $t = 1, \Delta t = 0.001$ and $\varepsilon = 0.01$.

h	$\| s - s_h \|_{H^1}$	Order	$\| s - s_h \|_{L^2}$	Order	$\| s - s_h \|_{L^\infty}$	Order
1/2	5.0581e-02	0	6.2516e-02	0	3.7532e-03	0
1/4	2.5187e-02	1.0059e+00	1.9268e-02	1.6980e+00	9.3995e-04	1.9975e+00
1/8	1.2563e-02	1.0035e+00	5.2248e-03	1.8828e+00	2.3492e-04	2.0004e+00
1/16	6.2715e-03	1.0023e+00	1.3479e-03	1.9547e+00	5.8956e-05	1.9944e+00
1/32	3.1325e-03	1.0015e+00	3.4152e-04	1.9807e+00	1.4901e-05	1.9842e+00
1/64	1.5652e-03	1.0010e+00	8.5947e-05	1.9905e+00	3.8285e-06	1.9605e+00

Table 3: Errors for Saturation of weak Galerkin method with fixed $t = 1, \Delta t = 0.001$ and $\varepsilon = 0.01$

7 Conclusions and Future Work

We have developed a weak Galerkin method sequential solution technique for miscible fluid flows in porous media, in which we use the weak Galerkin mixed finite element method to solve the pressure and Darcy velocity equations and weak Galerkin method to solve the transport equation for saturation. Error convergence in H^1 and L^2 is proved for semi-discrete schemes by the error equations, and the stabilization and energy conservation for the weak Galerkin methods.

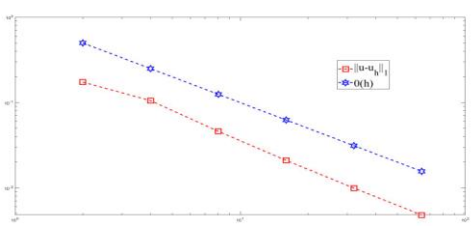
The numerical results demonstrate that the weak Galerkin simulator generates accurate and physically reasonable solutions. The use of the weak Galerkin mixed finite element method for the pressure equation yields accurate Darcy velocity fields that conserve mass, whereas the saturation equation by using the standard weak Galerkin method ,clearly in figures (4, 8 and 12), respectively.

Future work consists of the following steps:

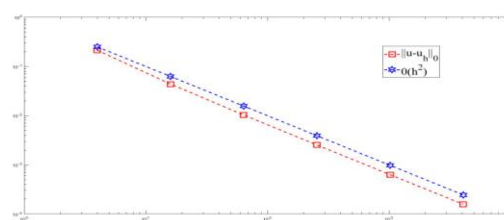
* Studying the weak Galerkin hybrid finite element methods for two-dimensional incompressible immiscible displacement fluid in a porous medium.

* Studying the Crank-Nicolson-weak Galerkin finite element scheme for two-dimensional incompressible immiscible displacement fluid in a porous medium.

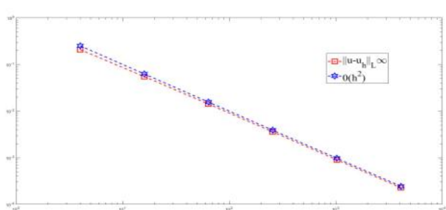
* Studying a simplified weak Galerkin finite element model for two-dimensional incompressible immiscible displacement fluid in a porous medium.



(a) Order error for H^1

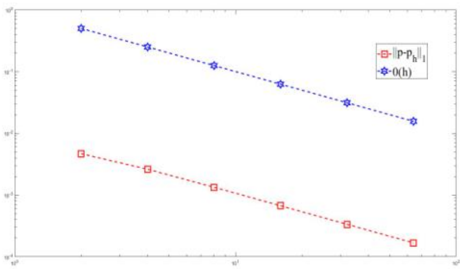


(b) Order error for L^2

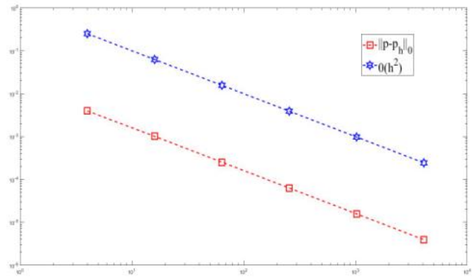


(c) Order error for L^∞

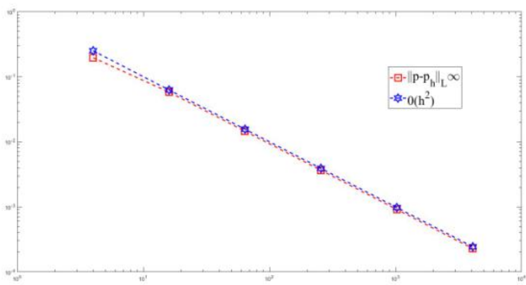
Figure 1: Order Error for Velocity with H^1-, L^2 −and $L^\infty -$ norms.



(a) Order error for H^1

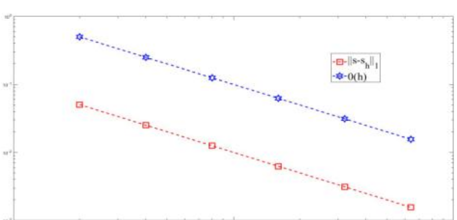


(b) Order error for L^2

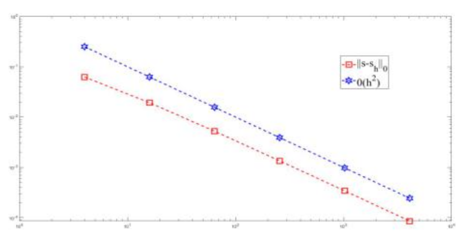


(c) Order error for L^∞

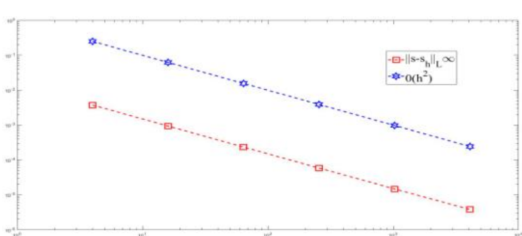
Figure 2: Order Error for Pressure with H^1-, L^2 −and $L^\infty -$ norms.



(a) Order error for H^1



(b) Order error for L^2



(c) Order error for L^∞

Figure 3: Order Error for Saturation with H^1-, L^2 −and $L^\infty -$ norms.

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