Applications of Discrete Entropic Model to Queueing Theory with Infinite and Finite Space Capacity

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Abstract

This is well acknowledged authenticity that a collection of discrete entropic models in the probability spaces are unbelievably accessible in the literature but still there is unavoidability to produce supplementary parametric models to enlarge their application areas in a assortment of disciplines. This paper makes a comprehensive study of an innovative discrete entropic model along with its applications to the discipline of queueing theory with infinite capacity for the deliberations of the learning of variations of uncertainty. Our findings reveal the pattern of behavior of discrete entropy model in the *steady and non-steady state queueing process* with infinite channel capacity. Additionally, we have well designed the understanding of *maximum entropy principle* for the development of optimization principle in the discipline of queueing theory with finite channel capacity for the comprehension of distribution under consideration.

Keywords: Probability distribution, Queueing theory, Degenerate distributions, Entropy, Continuity, Symmetry, Concavity, Steady state, Non-steady state.

INTRODUCTION

The memorable authenticity about the discrete information models including entropic models intend to demonstrate their significance for expedient significance of information dispensing in broad-spectrum progression in statistical configuration pedestal on entropic model well established by Shannon [11]. This discrete entropic model influences some precious and obvious requirements and furthermore can be bestowed with outfitted association in crucial pragmatic optimization problems concerned with a diversity of disciplines. The convention that the entropic models have astonishingly congenial properties led the investigators working on entropic models to originate abundant new-fangled mathematical entropic models. Shannon [11] established the significant perception of entropy by means of subsequent manifestation:

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \tag{1.1}$$

It is well celebrated actuality that to facilitate sensible applications of discrete information models in the control of probabilistic coding theory, a broad-spectrum approach has been endowed with in a structure pedestal on entropic model pioneered by Shannon [11]. This quantitative entropic model persuades several advantageous self-evident prerequisites and furthermore it can be capable of allocation with an outfitted consequence in imperative convenient optimization problems of numerous disciplines. This is the distinguished authenticity that information models are significant for convenient relevance of information dispensing a broad-spectrum advancement in statistical structure. This entropy persuades some advantageous self-evident requirements and additionally can be dispensed outfitted connotation in imperative realistic problems.

In recent times, Zhang and Shi [14] deliberated astonishing commentary about Shannon's entropy that its structure has indispensable characteristic but its unboundedness over the universal category of distributions on an alphabet puts off its impending effectiveness from being entirely comprehended. To conquer this shortcoming, Zhang [13] projected a generalized feature of Shannon's entropy and made comprehensive study of its asymptotic properties. Additionally, the author verified that these properties require no suppositions on the original distribution.

After the establishment of Shannon [11] entropy, abundant entropic models were discussed and derived by assorted researchers. Some of the imperative explorations were made by Renyi's [10] additive and Havrada and Charvat's [3] non-additive entropy. To enhance the text of the discrete entropic models, Parkash and Kakkar [5, 6] delivered investigation and consequently organized protracted efforts for the exploration of abundant entropic models for the discrete probability spaces from application point of surveillance and consequently augmented the literature of discrete entropy models by means of twisting the subsequent innovative entropic models:

$$ {}_{\beta}S(P)=\frac{\sum_{i=1}^{n} p_i \beta^{\log_D p_i}-1}{1-\beta},\ \beta>1 \tag{1.2}$$

$$ S_{\beta}(P)=\frac{\beta^{\sum_{i=1}^{n} p_i \ln p_i}-1}{1-\beta},\ \beta>1 \tag{1.3}$$

Recently, Parkash and Mukesh [9] promoted the passage of information entropic models after creating another discrete entropy model for the furtherance of their research communications.

On the other hand, the perception of weighted models cannot be disregarded because of their extraordinarily productive responsibility. Employing significance to events and to provide the applications of their weighted entropic models to coding theory, Parkash and Kumar [7] completed protracted efforts for their investigations and consequently enriched the literature of weighted entropic models. Many other developments related with the study of information theoretic entropies for the discrete probability distributions have been made by Kapur [4], Elgawad, Barakat, Xiong and Alyami [1], Sholehkerdar, Tavakoli and Liu [12], Gao and Deng [2] etc.

To enhance the literature of discrete weighted entropic models, Parkash, Kumar, Mukesh and Kakkar [19] well thought-out prolonged efforts for the exploration of plentiful weighted parametric entropic models for the discrete probability spaces from application point of view in the field of coding theory. Through their cooperative efforts, the authors delivered numerous observations and consequently enhanced the literature of such models by means of twisting the two subsequent quantitative outward show:

$$ H_{\alpha,\beta}(P;W)=\frac{1}{\beta-\alpha}\log\left[\frac{\sum_{i=1}^{n} w_i p_i^{\alpha}}{\sum_{i=1}^{n} w_i p_i^{\beta}}\right],\ \alpha<1,\beta>1 \text{ or } \alpha>1,\beta<1 \tag{1.4}$$

One of the application areas of entropy measures in Operations Research finds compatibility with queueing theory under infinite channel capacity. While providing applications in a simple birth-death process, we assume $p_n(t)$ to be the probability of n persons at some time t and n_0 be the persons at some time $t = 0$, then if we describe the *probability generating function* by the subsequent appearance:

$$ \phi(s,t)=\sum_{n=o}^{\infty} p_n(t)s^n\ , \tag{1.5}$$

then, we acquire subsequent outcome:

$$\phi(s,t)=\left[\frac{(\lambda-\mu)s+\mu(x-1)}{(\lambda-\lambda x)s+(\lambda x-1)}\right]^{n_0},\ \lambda\neq\mu \tag{1.6}$$

$$=\left[\frac{\lambda t-(\lambda t-1)s}{1-\lambda t-\lambda ts}\right]^{n_0},\ \lambda=\mu \tag{1.7}$$

where $x=\exp(\lambda-\mu)t$ (1.8)

By making expansion of the function $\phi(s,t)$, one can discover $p_n(t)$. With the well acknowledged notations of the system, we presuppose λ and μ to be arrival and service rates respectively in the steady state, then we are well recognized with the subsequent form:

$$p_n=(1-\rho)\rho^n,\ n=0,1,2.3,....;\rho=\frac{\lambda}{\mu} \tag{1.9}$$

In the sequel, we have wrought out an innovative entropic model for the discrete probability distributions and completed the comprehensive learning of its interesting properties. Additionally, by employing this model, we have collected the knowledge of variations of uncertainty in the diverse states of queueing system with infinite channel capacity. More additionally, we have premeditated the responsiveness of *maximum entropy principle* for the development of optimization principle in the discipline of queueing theory with finite channel capacity for the knowledge of distribution under consideration.

2. A NEW-FANGLED ENTROPIC MODEL IN PROBABILITY SPACES

Here, we have produced a novel parametric entropic models in probability spaces and studied their crucial properties for their authenticity. The necessity for the development of these models arises due to their applicability to underline their applications in the disciplines of Statistics, Operations Research and Coding theory.

I. Firstly, we establish the subsequent quantitative discrete parametric entropic model specified by

$$H_\alpha(P)=-\sum_{i=1}^{n}p_i\log p_i-\frac{1}{\alpha}\sum_{i=1}^{n}\{1+\alpha p_i\}\log\{1+\alpha p_i\}+\frac{(1+\alpha)}{\alpha}\log(1+\alpha);\alpha\neq0,\alpha>0 \tag{2.1}$$

We observe that $\lim\limits_{\alpha\to0}H_\alpha(P)=-\sum_{i=1}^{n}p_i\log p_i$.

Hence, we examine that the appearance of $H_\alpha(P)$ is a generalized entropic model.

To authenticate that (2.1) is a convincing model, we study its subsequent fundamental properties:

(i) We have $H_\alpha(P) \geq 0$

For n degenerate distributions, we have $H_\alpha(P)=0$. Since, entropy gives minimum value for degenerate distributions and the minimum value is 0, we must have $H_\alpha(P) \geq 0$.

(ii) $H_\alpha(P)$ is symmetric.

(iii) $H_\alpha(P)$ is continuous.

(iv) Concavity: To demonstrate the concavity, we carry on subsequently:

We have $\frac{\partial H_\alpha(P)}{\partial p_i} = -\left[2+\log p_i + \log\{1+\alpha p_i\}\right]$

Also $\frac{\partial^2 H_\alpha(P)}{\partial p_i^2} = -\frac{1}{p_i} - \frac{\alpha}{1+\alpha p_i} < 0$ which verifies concavity of $H_\alpha(P)$.

Moreover, with the assistance of numerical data exposed in the subsequent Table-2.1 for $n=2$ and $\alpha=2$, we have presented the entropic model $H_\alpha(P)$ against p as revealed in Figure-2.1.

Table-2.1: $H_\alpha(P)$ against p for $n=2$ and $\alpha=2$

p	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$H_\alpha(P)$	0.0000000	0.2756456	0.3884567	0.4545678	0.4893489	0.5000000	0.4895636	0.4546767	0.3887878	0.2714587	0.0000000

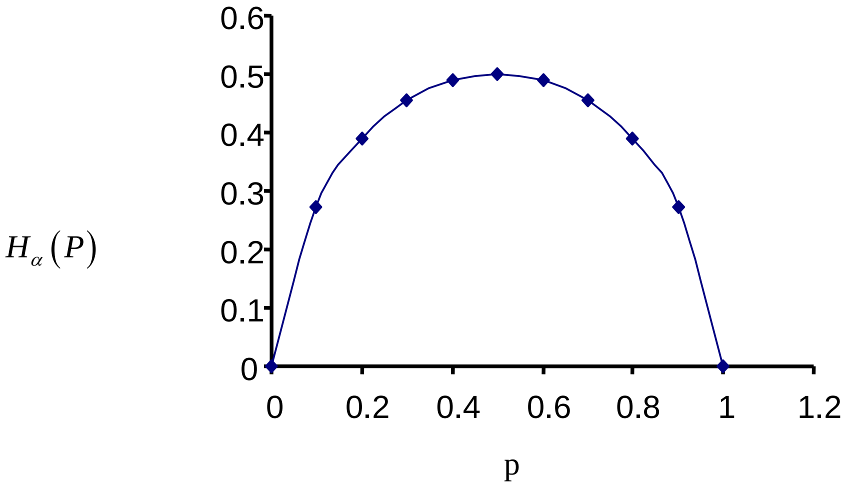

Figure-2.1: Concavity of $H_\alpha(P)$ with respect to P

The Figure-2.1 undoubtedly demonstrates the concavity of (2.1). Consequently, we claim that $H_\alpha(P)$ is an acceptable entropy model.

(v) Maximization: We make exploitation of Lagrange's technique to deliberate maximization the entropic model (2.1) under the discussion of the usual and accustomed constraint.

In this case, we think about the equivalent Lagrange's function appearing in the subsequent structure:

$$L \equiv H_\alpha(P) - \lambda\left(\sum_{i=1}^{n} p_i - 1\right) \tag{2.2}$$

Differentiating (2.2) with respect to p_i and connecting its derivatives to nil, we acquire the correspondence which is promising only if $p_1 = p_2 = ... = p_n$.

Further, by means of the property $\sum_{i=1}^{n} p_i = 1$, we search out that $p_i = \frac{1}{n}, \forall i$.

As a consequence, we monitor that $\left[H_\alpha(P)\right]_{\max}$ takes place at the uniform distribution and this consequence is essentially attractive one.

(vi) If we symbolize $\left[H_\alpha(P)\right]_{\max}$ by $f(n)$ then, we acquire

$$f'(n) = \frac{2}{n} + \frac{1}{\alpha}\log\left\{1 + \frac{\alpha}{n}\right\} > 0$$

which provide the evidence that $f(n)$ increases with n, which all over again an striking consequence for the reason that the maximum value of any entropy measure forever increase.

In the next section, we have deliberated the applications of the discrete entropic model shaped above in the direction of its relevance in *queueing theory* with infinite channel capacity.

3. APPLICATIONS OF DISCRETE PARAMETRIC ENTROPIC MODEL IN QUEUEING THEORY

Here, we have completed the learning of discrepancy of entropy in the different states of queueing process and for this rationale, we have well thought-out the subsequent cases:

Case-I: Variations of entropic model in the steady state with infinite capacity

For the steady state queueing process, we rewrite the above model (2.1) as

$$S^{\alpha}\left(\lambda,\mu\right)=-\sum_{n=o}^{\infty}p_n\log p_n-\frac{1}{\alpha}\sum_{n=0}^{\infty}\left\{1+\alpha p_n\right\}\log\left[1+\alpha p_n\right]+\frac{(1+\alpha)}{\alpha}\log(1+\alpha) \tag{3.1}$$

Substituting the standard result of queueing theory, that is, $p_n=(1-\rho)\rho^n$, we acquire the subsequent appearance:

$$S^{\alpha}\left(\lambda,\mu\right)=-\sum_{n=o}^{\infty}(1-\rho)\rho^n\left\{\log((1-\rho)+\log\rho^n\right\}-\frac{1}{\alpha}\sum_{n=0}^{\infty}\left\{1+\alpha(1-\rho)\rho^n\right\}\log\left[1+\alpha(1-\rho)\rho^n\right]+\frac{(1+\alpha)}{\alpha}\log(1+\alpha)$$

$$\begin{aligned}&=-\frac{1}{(1-\rho)}\left[(1-\rho)\log(1-\rho)+\rho\log\rho\right]\\&\quad-\frac{1}{\alpha}\sum_{n=0}^{\infty}\left\{1+\alpha(1-\rho)\rho^n\right\}\log\left[1+\alpha(1-\rho)\rho^n\right]+\frac{(1+\alpha)}{\alpha}\log(1+\alpha)\end{aligned} \tag{3.2}$$

To provide the solution of the problem under deliberation, we reflect on the IInd and IIrd terms of equation (3.2) and taking limit as $\alpha\to 0$, the above equation gives the subsequent consequence:

$$Lt_{\alpha\to 0}-\frac{1}{\alpha}\sum_{n=0}^{\infty}\left\{1+\alpha(1-\rho)\rho^n\right\}\log\left[1+\alpha(1-\rho)\rho^n\right]+Lt_{\alpha\to 0}\frac{(1+\alpha)}{\alpha}\log(1+\alpha)$$

$$=-\left[(1-\rho)+(1-\rho)\rho+(1-\rho)\rho^2+(1-\rho)\rho^3+\ldots\right]+1$$

$$=-(1-\rho)\left[1+\rho+\rho^2+\rho^3+\ldots\right]+1=0$$

As a consequence, equation (3.2) becomes

$$Lt_{\alpha\to 0}S^{\alpha}\left(\lambda,\mu\right)=-\frac{1}{(1-\rho)}\left[(1-\rho)\log(1-\rho)+\rho\log\rho\right] \tag{3.3}$$

Differentiating equation (3.3) w.r.t. ρ, we search out the subsequent result:

$$Lt_{\alpha\to 0}\frac{\partial}{\partial\rho}S^{\alpha}\left(\lambda,\mu\right)=-\frac{\log\rho}{(1-\rho)^2}>0$$

This surveillance implies that the in steady-state, uncertainty prevailed gets enhanced from 0 to ∞ as ρ gets amplified from 0 to unity. Accordingly, in this case, we perceive that uncertainty gets enlarged with the increase in utilization factor.

Case-II: Entropic Variations in non-steady state queueing process with infinite capacity

Here, we primarily develop the results by considering Kapur's [4] observation that

$\sum_{n=0}^{\infty} p_n(t) = \frac{\lambda t}{1+\lambda t}\left(1-\frac{\lambda t-1}{\lambda t}s\right)\sum_{n=0}^{\infty}\left(\frac{\lambda t}{1+\lambda t}\right)^n s^n$ so that $p_n(t) = \frac{(\lambda t)^{n-1}}{(1+\lambda t)^{n+1}}, \; n \geq 1.$

Also $p_0(t) = \frac{\lambda t}{1+\lambda t}$

Consequently, we acquire the subsequent mathematical appearance:

$$p_n(t) = \begin{cases} \frac{(\lambda t)^{n-1}}{(1+\lambda t)^{n+1}}, \; n \geq 1 \\ \frac{\lambda t}{1+\lambda t}, n = 0 \end{cases} \tag{3.4}$$

Now, we provide the wide-ranging learning of the diverse variations by taking into consideration the probabilistic entropic model (3.1).

Consider the first term of (3.1) as

$$-\sum_{n=o}^{\infty} p_n(t) \log p_n(t) = \frac{2\left[\{1+\lambda t\}\log\{1+\lambda t\} - \lambda t \log \lambda t\right]}{\{1+\lambda t\}}$$

Accordingly, equation (3.1) acquires the precise manifestation, the second term of which in the limiting case as $\alpha \to 0$, provides the subsequent mathematical communication:

$$Lt_{\alpha \to 0} \frac{1}{\alpha} \sum_{n=0}^{\infty} \{1+\alpha p_n\} \log\left[1+\alpha p_n(t)\right] = 1$$

Similarly, captivating limit as $\alpha \to 0$, the third term of the same manifestation confers with the subsequent noticeable illustration:

$$Lt_{\alpha \to 0} \frac{(1+\alpha)}{\alpha} \log(1+\alpha) = 1$$

Thus, in the limiting case, equation (3.1) bestows with the subsequent emergence:

$$Lt_{\alpha \to 0} S^{\alpha}(\lambda, \mu) = \frac{2\left[\{1+\lambda t\}\log\{1+\lambda t\} - \lambda t \log \lambda t\right]}{\{1+\lambda t\}} \tag{3.5}$$

Now

$$\frac{d}{d(\lambda t)} S^{\alpha}(\lambda, t) = -\frac{2 \log \lambda t}{((1-\lambda t)^2} \begin{cases} > 0 \quad if \; \lambda t < 1 \\ < 0 \quad if \; \lambda t > 1 \end{cases}$$

which implies that the uncertainty increases if λt <1 and decreases if $\lambda t \geq 1$.

Furthermore from (3.5), the highest uncertainty happens when λt =1 and in this case,

Max $S^{\alpha}(\lambda, t) = 2\log 2$

Additionally, at the time when $t = 0$, uncertainty is 0 and at the time when $t \to \infty$, we acquire subsequent manifestation:

$$\underset{t\to\infty}{Lt}\ S^{\alpha}(\lambda, t) \ = 0$$

Consequently, we scrutinize that uncertainty begins absolutely with worth 0 at time $t = 0$ and finishes absolutely with worth 0 as time $t \to \infty$, and sandwiched between this time, it accomplishes the greatest value at λt =1,that is, at t = $\frac{1}{\lambda}$.

In the succeeding segment, we have well planned the awareness of *maximum entropy principle* for the development of optimization principle in the discipline of queueing theory with finite channel capacity for the knowledge of distribution under consideration.

4. DEVELOPMENT OF OPTIMIZATIONAL PRINCIPLE IN THE FIELD OF QUEUEING THEORY

In this fragment, we make available the relevance of *MaxEnt principle* for estimation of discrete probability distribution when simply fractional information concerning the specified distribution is accessible. To elaborate structure to this *MaxEnt principle*, we think about the subsequent model:

Model: Optimizational principle by the employment of entropic model with finite space capacity and known mean size of the system

To accomplish our intention, we exploit the discrete parametric entropic model (2.1) previously commenced in section-2. Consequently, our optimizational problem gets converted into the subsequent mathematical appearance:

Maximize (2.1) under the subsequent set of constraints:

$$\sum_{i=1}^{n} p_i = 1 \tag{4.1}$$

and

$$\sum_{i=1}^{n} ip_i = m \tag{4.2}$$

To act in accordance with the optimizational problem, we think about the equivalent Lagrange's function appearing in the subsequent mathematical structure:

$$L = -\sum_{i=1}^{n} p_i \log p_i - \frac{1}{\alpha}\sum_{i=1}^{n}\{1+\alpha p_i\}\log[1+\alpha p_i] + \frac{(1+\alpha)}{\alpha}\log(1+\alpha) - \lambda\left(\sum_{i=1}^{n} p_i - 1\right) - \mu\left(\sum_{i=1}^{n} ip_i - m\right)$$

Now, in the limiting case as $\alpha \to 0$, $\frac{\partial L}{\partial p_i} = 0$ gives

$$p_i = e^{-\lambda}e^{-i\mu} = ab^i \tag{4.3}$$

where $a = e^{-(2+\lambda)}$, $b = e^{-\mu}$ and a, b are to be determined from the subsequent equations:

$$a\sum_{i=1}^{n} b^i = 1 \text{ and } a\sum_{i=1}^{n} ib^i = m \tag{4.4}$$

Now,

$$a\sum_{i=1}^{n} b^i = a\left[b + b^2 + b^3 + b^4 + ... + b^n\right] \quad = ab\left\{\frac{1-b^n}{1-b}\right\}$$

Also $a\sum_{i=1}^{n} ib^i = a\left[b + 2b^2 + 3b^3 + 4b^4 + ... + nb^n\right]$

$$= ab\left[\frac{1-b^n}{(1-b)^2} - \frac{nb^n}{1-b}\right]$$

Employing these values equations (4.4), we acquire the subsequent communication

$$ab\frac{1-b^n}{1-b} = 1 \text{ and } \frac{1}{1-b} - \frac{nb^n}{1-b^n} = m \tag{4.5}$$

With the known values of m and n, the equations (4.5) give the values of a and b and hence equation (4.3) determines the required set of probabilities. Thus, we observe that the maximizing entropy probability distribution is ***geometric distribution***.

The above process has been exemplified with the facilitation of subsequent numerical illustration:

Numerical Illustration:

Upon maximization of the discrete entropic model (2.1) with the employment of (4.1) and (4.2) particularly, for $n = 9$ and $m = 1.5$, the equations (4.5) and (4.3) provide the subsequent communications:

$p_1 = 0.6012$ $p_2 = 0.1475,$ $p_3 = 0.0827$, $p_4 = 0.0534,$ $p_5 = 0.03412,$ $p_6 = 0.0312,$ $p_7 = 0.0225,$ $p_8 = 0.0121,$ $p_9 = 0.0102$

Obviously, we have the subsequent probabilistic appearance:

$$\sum_{i=1}^{9} p_i = 1.0020$$

This modus operandi is reiterated for dissimilar values of m when we capture $n = 9$. For the period of the modus operandi implemented, we experimented that in certain cases, we contracted a small amount of negative probabilities, not advantageous at all. To undertake the circumstances, we overlooked these probabilities and once again originated the problem for outstanding probabilities and resolved it for its explanation. The consequences of our working out are exposed in the subsequent Table-4.1.

Table-4.1

m	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
1.5	0.6012	0.1475	0.0827	0.0534	0.0412	0.0312	0.0225	0.0121	0.0102
2.5	0.4565	0.1853	0.1272	0.0882	0.0662	0.0518	0.0425	0.0387	0.0371
3.5	0.2334	0.1835	0.1283	0.0989	0.0867	0.0767	0.0694	0.5565	0.0565
4.5	0.1375	0.1264	0.1075	0.0989	0.0978	0.0896	0.0886	0.0879	0.0784
5.5	0.0000	0.0000	0.2202	0.1862	0.1632	0.1432	0.1138	0.1034	0.0913
6.5	0.0000	0.0000	0.0000	0.0000	0.3063	0.2634	0.1723	0.1422	0.1218
7.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.4012	0.3078	0.2167	0.1005
8.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.7023	0.2130	0.1003
9.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5000	0.5000

Concluding Remarks: With the development of entropy model, we have deliberated the study of variations of uncertainty and observed that the uncertainty always increases in the *steady state* despite the fact that in the *non-steady state*, uncertainty in the beginning increases and receives maximum value whereas with time, it gets decreased and receives its minima. The consequences of our findings bring about intact compatibility with reality and consequently our conclusions are fascinating. Furthermore, we have well premeditated the responsiveness of *maximum entropy principle* for the enlargement of optimization principle in the discipline of queueing theory for the understanding of distribution under contemplation.

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