Type 2: Aggregation of Interval-Valued Pentagonal Fuzzy Neutrosophicsets and Its Application in Solving Multi-Attribute Decision Making Environment

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Article Info
Page Number: 3598 - 3621
Publication Issue:
Vol 71 No. 4 (2022)

Article History
***Article Received**: 25 March 2022*
***Revised**: 30 April 2022*
***Accepted**: 15 June 2022*
***Publication**: 19 August 2022*

Abstract

This paper proposes interval-valued pentagonal fuzzy neutrosophic set by combining pentagonal fuzzy sets and interval-valued neutrosophic sets to get more efficient results. Operational laws have been discussed and also the weighted arithmetic aggregation operator of interval-valued pentagonal neutrosophic sets have been established and also a theorem is proved and some of its properties were dealt with. Finally, an illustrative example is solved to validate the proposed weighted arithmetic aggregation operator based on its alternatives and attributes.

Keywords: Neutrosophic Sets, Pentagonal fuzzy sets, Interval-valued Neutrosophic Sets, Multi-attribute, Aggregation Operators.

1. Introduction

Zadeh in 1965 [1] established the idea of fuzziness to deal with hesitant theory, this has created a tremendous change in various fields like engineering, space research, medical diagnosis, robotics, statistical analysis etc.Intuitionistic fuzzy sets were presented by Atanassov [2], is stressed upon membership and non-membership functions.Added to this, Smarandache [3] in 1998 createdneutrosophic set which includes truth, indeterminacy and falsity membership function.This paved a new idea to sort out the mathematical models which dealt with vague and uncertain data in a very efficient way for real life models.Multi-criteria decision making (MCDM) problems were solved more effectively by Zadeh's [4] interval valued fuzzy set theory. Atanassov [5] introduced extrusive form by the combined effect of intuitionistic and interval valued fuzzy sets. Mendel et al (2002) [6] gave prominent insight into type 2 fuzzy sets. Chakraborty [7] gave an insight view regarding the score function and also the de-neutrosophic technique of fuzzy pentagonal neutrosophic numbers whereas Umamageswari et al [8] presented interval valued pentagonal fuzzy numbers.

The most crucial method for resolving multi-criteria decision-making, when qualities and alternatives are stated in terms of neutrosophic values, is the aggregation of neutrosophic sets. Ye [9] introduced weighted geometric and arithmetic average operators for neutrosophic sets.

In this paper, the objective includes:

- To propose interval-valued pentagonal fuzzy neutrosophic sets (IVPFNS), its arithmetic operators, score function, accuracy function.
- Propose an aggregation operator for interval-valued pentagonal fuzzy neutrosophic weighted arithmetic averaging(IVPFNWAA) operator.
- Prove some properties of the proposed operator of IVPFNWAA.
- Establishes a multi-criteria decision making based on IVPFNWAA.
- Giving a concrete illustration of the MCDM approach.

2. Preliminaries

Definition 2.1 [10]"A fuzzy number $\dot{A}$ on R is said to be a pentagonal fuzzy number which is represented as $(\breve{a}_1, \breve{a}_2, \breve{a}_3, \breve{a}_4, \breve{a}_5; \breve{W})$ if its membership function satisfies,

$$\mu_{\dot{A}}(x) = \begin{cases} \left(\frac{x - \breve{a}_1}{\breve{a}_2 - \breve{a}_1}\right) \mathcal{T}_{\dot{N}}^{l} & \breve{a}_1 \leq x \leq \breve{a}_2 \\ 1 - \left(\frac{x - \breve{a}_2}{\breve{a}_3 - \breve{a}_2}\right)\left(1 - \mathcal{T}_{\dot{N}}^{l}\right) & \breve{a}_2 \leq x \leq \breve{a}_3 \\ 1 & x = \breve{a}_3 \\ 1 - \left(\frac{\breve{a}_4 - x}{\breve{a}_4 - \breve{a}_3}\right)\left(1 - \mathcal{T}_{\dot{N}}^{l}\right) & \breve{a}_3 \leq x \leq \breve{a}_4 \\ \left(\frac{\breve{a}_5 - x}{\breve{a}_5 - \breve{a}_4}\right) \mathcal{T}_{\dot{N}}^{l} & \breve{a}_4 \leq x \leq \breve{a}_5 \\ 0 & \text{otherwise"} \end{cases}$$

Definition 2.2 [11]"An interval-valued fuzzy neutrosophic set (IVFNS) $\dot{A}$ over X takes the form

$$\dot{A} = \{\langle x, [T^l_{\dot{A}}(x), T^u_{\dot{A}}(x)], [I^l_{\dot{A}}(x), I^u_{\dot{A}}(x)], [F^l_{\dot{A}}(x), F^u_{\dot{A}}(x)]\rangle : x \in X\}$$

$T^l_{\dot{A}}(x), T^u_{\dot{A}}(x): X \to [0,1]$, $I^l_{\dot{A}}(x), I^u_{\dot{A}}(x): X \to [0,1]$and $F^l_{\dot{A}}(x), F^u_{\dot{A}}(x): X \to [0,1]$ with

$0 \le T^u_{\dot{A}}(x) + I^u_{\dot{A}}(x) + F^u_{\dot{A}}(x) \le 3$, for all $x \in X$.

An interval-valued fuzzy neutrosophic number (IVFNN) is defined as

$\dot{A} = \{\langle x, [\inf T_{\dot{A}}(x), \sup T_{\dot{A}}(x)], [\inf I_{\dot{A}}(x), \sup I_{\dot{A}}(x)], [\inf F_{\dot{A}}(x), \sup F_{\dot{A}}(x)]\rangle : x \in X\}$"

3. INTERVAL - VALUED PENTAGONAL FUZZY NEUTROSOPHIC SET (IVPFNS)

An interval-valued pentagonal fuzzy neutrosophic number (IVPFNN) $\dot{N}$ is defined as an (IVPFNS) on X is represented by $\dot{N}(x) = [\dot{N}^l(x), \dot{N}^u(x)]$, where $\dot{N}^l$ and $\dot{N}^u$ are lower and upper pentagonal fuzzy neutrosophic sets on $\dot{N}$ such that $\dot{N}^l \subseteq \dot{N}^u$.

$\dot{N}^l = \{[x, \mathcal{T}^l_{\dot{N}}(x), \mathcal{I}^l_{\dot{N}}(x), \mathcal{F}^l_{\dot{N}}(x): x \in X]\}$ where $\mathcal{T}^l_{\dot{N}}(x) \subset [0,1]$, $\mathcal{I}^l_{\dot{N}}(x) \subset [0,1]$ and $\mathcal{F}^l_{\dot{N}}(x) \subset [0,1]$ are lower pentagonal fuzzy neutrosophic numbers.

$$\mathcal{T}^l_{\dot{N}}(x) = [t^{l1}_{\dot{N}}(x), t^{l2}_{\dot{N}}(x), t^{l3}_{\dot{N}}(x), t^{l4}_{\dot{N}}(x), t^{l5}_{\dot{N}}(x)]: X \to [0,1],$$

$\mathcal{I}^l_{\dot{N}}(x) = [i^{l1}_{\dot{N}}(x), i^{l2}_{\dot{N}}(x), i^{l3}_{\dot{N}}(x), i^{l4}_{\dot{N}}(x), i^{l5}_{\dot{N}}(x)]: X \to [0,1]$, and

$\mathcal{F}^l_{\dot{N}}(x) = [f^{l1}_{\dot{N}}(x), f^{l2}_{\dot{N}}(x), f^{l3}_{\dot{N}}(x), f^{l4}_{\dot{N}}(x), f^{l5}_{\dot{N}}(x)]: X \to [0,1]$, which satisfies the

condition $0 \le t^{l5}_{\dot{N}}(x) + i^{l5}_{\dot{N}}(x) + f^{l5}_{\dot{N}}(x) \le 3$.

For convenience of representation, we consider $\mathcal{T}^l_{\dot{N}}(x) = (\breve{\zeta}, \breve{\eta}, \breve{\theta}, \breve{\lambda}, \breve{\mu}): X \to [0,1]$,

$\mathcal{I}^l_{\dot{N}}(x) = (\breve{ß}, \breve{\eth}, \breve{\Psi}, \breve{\phi}, \breve{\Omega}): X \to [0,1]$ and $\mathcal{F}^l_{\dot{N}}(x) = (\breve{\xi}, \breve{\rho}, \breve{\varsigma}, \breve{\sigma}, \breve{\tau}): X \to [0,1]$.

Therefore $\dot{N}^l = \{[(\breve{\zeta}, \breve{\eta}, \breve{\theta}, \breve{\lambda}, \breve{\mu}), (\breve{ß}, \breve{\eth}, \breve{\Psi}, \breve{\phi}, \breve{\Omega}), (\breve{\xi}, \breve{\rho}, \breve{\varsigma}, \breve{\sigma}, \breve{\tau})]: X \to [0,1]\}$

Definition 3.1 Let $\dot{N}^l$ be the lower pentagonal fuzzy neutrosophic number. Then $\mathcal{T}^l_{\dot{N}}(x), \mathcal{I}^l_{\dot{N}}(x)$ and $\mathcal{F}^l_{\dot{N}}(x)$is defined as follows:

$$\mathcal{T}_{\dot{N}}^{l}(x)=\begin{cases}\left(\frac{x-\breve{\zeta}}{\breve{\eta}-\breve{\zeta}}\right)\mathcal{T}_{\dot{N}}^{l}\breve{\zeta}\le x\le\breve{\eta}\\ 1-\left(\frac{x-\breve{\eta}}{\breve{\theta}-\breve{\eta}}\right)\left(1-\mathcal{T}_{\dot{N}}^{l}\right)\breve{\eta}\le x\le\breve{\theta}\\ 1 & x=\breve{\theta}\\ 1-\left(\frac{\breve{\lambda}-x}{\breve{\lambda}-\breve{\theta}}\right)\left(1-\mathcal{T}_{\dot{N}}^{l}\right)\breve{\theta}\le x\le\breve{\lambda}\\ \left(\frac{\breve{\mu}-x}{\breve{\mu}-\breve{\lambda}}\right)\mathcal{T}_{\dot{N}}^{l}\breve{\lambda}\le x\le\breve{\mu}\\ 0 & \text{otherwise}\end{cases}\quad \mathcal{I}_{\dot{N}}^{l}(x)=\begin{cases}\left(\frac{\breve{\eth}-x}{\breve{\eth}-\breve{ß}}\right)\mathcal{I}_{\dot{N}}^{l}\breve{ß}\le x\le\breve{\eth}\\ 1-\left(\frac{\breve{\Psi}-x}{\breve{\Psi}-\breve{\eth}}\right)\left(1-\mathcal{I}_{\dot{N}}^{l}\right)\breve{\eth}\le x\le\breve{\Psi}\\ 0 & x=\breve{\Psi}\\ 1-\left(\frac{x-\breve{\Psi}}{\breve{\phi}-\breve{\Psi}}\right)\left(1-\mathcal{I}_{\dot{N}}^{l}\right)\breve{\Psi}\le x\le\breve{\phi}\\ \left(\frac{x-\breve{\phi}}{\breve{o}-\breve{\phi}}\right)\mathcal{I}_{\dot{N}}^{l}\breve{\phi}\le x\le\breve{o}\\ 1 & \text{otherwise}\end{cases}$$

$$\mathcal{F}_{\dot{N}}^{l}(x)=\begin{cases}\left(\frac{\breve{\rho}-x}{\breve{\rho}-\breve{\xi}}\right)\mathcal{F}_{\dot{N}}^{l}\breve{\xi}\le x\le\breve{\rho}\\ 1-\left(\frac{\breve{\varsigma}-x}{\breve{\varsigma}-\breve{\rho}}\right)\left(1-\mathcal{F}_{\dot{N}}^{l}\right)\breve{\rho}\le x\le\breve{\varsigma}\\ 0 & x=\breve{\varsigma}\\ 1-\left(\frac{x-\breve{\varsigma}}{\breve{\sigma}-\breve{\varsigma}}\right)\left(1-\mathcal{F}_{\dot{N}}^{l}\right)\breve{\varsigma}\le x\le\breve{\sigma}\\ \left(\frac{x-\breve{\sigma}}{\breve{\tau}-\breve{\sigma}}\right)\mathcal{F}_{\dot{N}}^{l}\breve{\sigma}\le x\le\breve{\tau}\\ 1 & \text{otherwise}\end{cases}$$

$\dot{N}^{u}=\{[x,\mathcal{T}_{\dot{N}}^{u}(x),\mathcal{I}_{\dot{N}}^{u}(x),\mathcal{F}_{\dot{N}}^{u}(x):x\in X]\}$ where $\mathcal{T}_{\dot{N}}^{u}(x)\subset[0,1]$, $\mathcal{I}_{\dot{N}}^{u}(x)\subset[0,1]$ and $\mathcal{F}_{\dot{N}}^{u}(x)\subset[0,1]$ are upper pentagonal fuzzy neutrosphic numbers.

$$\mathcal{T}_{\dot{N}}^{u}(x)=\left[t_{\dot{N}}^{u1}(x),t_{\dot{N}}^{u2}(x),t_{\dot{N}}^{u3}(x),t_{\dot{N}}^{u4}(x),t_{\dot{N}}^{u5}(x)\right]:X\to[0,1],$$

$\mathcal{I}_{\dot{N}}^{u}(x)=\left[i_{\dot{N}}^{u1}(x),i_{\dot{N}}^{u2}(x),i_{\dot{N}}^{u3}(x),i_{\dot{N}}^{u4}(x),i_{\dot{N}}^{u5}(x)\right]:X\to[0,1]$, and

$\mathcal{F}_{\dot{N}}^{u}(x)=\left[f_{\dot{N}}^{u1}(x),f_{\dot{N}}^{u2}(x),f_{\dot{N}}^{u3}(x),f_{\dot{N}}^{u4}(x),f_{\dot{N}}^{u5}(x)\right]:X\to[0,1]$ which satisfies the condition

$0\le t_{\dot{N}}^{u5}(x)+i_{\dot{N}}^{u5}(x)+f_{\dot{N}}^{u5}(x)\le 3$.

For convenience of representation, we consider $\mathcal{T}_{\dot{N}}^{u}(x)=(\hat{\zeta},\hat{\eta},\hat{\theta},\hat{\lambda},\hat{\mu}):X\to[0,1]$,
$\mathcal{I}_{\dot{N}}^{u}(x)=(\hat{ß},\hat{\eth},\hat{\Psi},\hat{\phi},\hat{\Omega}):X\to[0,1]$ and $\mathcal{F}_{\dot{N}}^{u}(x)=(\hat{\xi},\hat{\rho},\hat{\varsigma},\hat{\sigma},\hat{\tau}):X\to[0,1]$.

Therefore $\dot{N}^{u}=\{[(\hat{\zeta},\hat{\eta},\hat{\theta},\hat{\lambda},\hat{\mu}),(\hat{ß},\hat{\eth},\hat{\Psi},\hat{\phi},\hat{\Omega}),(\hat{\xi},\hat{\rho},\hat{\varsigma},\hat{\sigma},\hat{\tau})]:X\to[0,1]\}$

Definition 3.2 Let $\dot{N}^{u}$ be the upper pentagonal fuzzy neutrosophic number. Then $\mathcal{T}_{\dot{N}}^{u}(x),\mathcal{I}_{\dot{N}}^{u}(x)$ and $\mathcal{F}_{\dot{N}}^{u}(x)$can be defined as follows:

$$\mathcal{T}_{\dot{N}}^{u}(x)=\begin{cases}\left(\frac{x-\hat{\zeta}}{\hat{\eta}-\hat{\zeta}}\right)\mathcal{T}_{\dot{N}}^{u} & \zeta\le x\le\hat{\eta}\\ 1-\left(\frac{x-\hat{\eta}}{\hat{\theta}-\hat{\eta}}\right)\left(1-\mathcal{T}_{\dot{N}}^{u}\right) & \hat{\eta}\le x\le\hat{\theta}\\ 1 & x=\hat{\theta}\\ 1-\left(\frac{\hat{\lambda}-x}{\hat{\lambda}-\hat{\theta}}\right)\left(1-\mathcal{T}_{\dot{N}}^{u}\right) & \hat{\theta}\le x\le\hat{\lambda}\\ \left(\frac{\hat{\mu}-x}{\hat{\mu}-\hat{\lambda}}\right)\mathcal{T}_{\dot{N}}^{u} & \hat{\lambda}\le x\le\hat{\mu}\\ 0 & \text{otherwise}\end{cases}\quad \mathcal{I}_{\dot{N}}^{u}(x)=\begin{cases}\left(\frac{\hat{\eth}-x}{\hat{\eth}-\hat{ß}}\right)\mathcal{I}_{\dot{N}}^{u} & \hat{ß}\le x\le\hat{\eth}\\ 1-\left(\frac{\hat{\Psi}-x}{\hat{\Psi}-\hat{\eth}}\right)\left(1-\mathcal{I}_{\dot{N}}^{u}\right) & \hat{\eth}\le x\le\hat{\Psi}\\ 0 & x=\hat{\Psi}\\ 1-\left(\frac{x-\hat{\Psi}}{\hat{\phi}-\hat{\Psi}}\right)\left(1-\mathcal{I}_{\dot{N}}^{u}\right) & \hat{\Psi}\le x\le\hat{\phi}\\ \left(\frac{x-\hat{\phi}}{\hat{\Omega}-\hat{\phi}}\right)\mathcal{I}_{\dot{N}}^{u} & \hat{\phi}\le x\le\hat{\Omega}\\ 1 & \text{otherwise}\end{cases}$$

$$\mathcal{F}_{\dot{N}}^{u}(x)=\begin{cases}\left(\frac{\hat{\rho}-x}{\hat{\rho}-\hat{\xi}}\right)\mathcal{F}_{\dot{N}}^{u} & \hat{\xi}\le x\le\hat{\rho}\\ 1-\left(\frac{\hat{\varsigma}-x}{\hat{\varsigma}-\hat{\rho}}\right)\left(1-\mathcal{F}_{\dot{N}}^{u}\right) & \hat{\rho}\le x\le\hat{\varsigma}\\ 0 & x=\hat{\varsigma}\\ 1-\left(\frac{x-\hat{\varsigma}}{\hat{\sigma}-\hat{\varsigma}}\right)\left(1-\mathcal{F}_{\dot{N}}^{u}\right) & \hat{\varsigma}\le x\le\hat{\sigma}\\ \left(\frac{x-\hat{\sigma}}{\hat{\tau}-\hat{\sigma}}\right)\mathcal{F}_{\dot{N}}^{u} & \hat{\sigma}\le x\le\hat{\tau}\\ 1 & \text{otherwise}\end{cases}$$

An IVPFNN $\dot{n}$ is represented by $\dot{n}=\left\{\begin{matrix}\langle(\breve{\zeta},\breve{\eta},\breve{\theta},\breve{\lambda},\breve{\mu}:\mathcal{T}_{\dot{n}}^{l}),(\hat{\zeta},\hat{\eta},\hat{\theta},\hat{\lambda},\hat{\mu}:\mathcal{T}_{\dot{n}}^{u})\rangle,\\ \langle(\breve{ß},\breve{\eth},\breve{\Psi},\breve{\phi},\breve{\Omega}:\mathcal{I}_{\dot{n}}^{l}),(\hat{ß},\hat{\eth},\hat{\Psi},\hat{\phi},\hat{\Omega}:\mathcal{I}_{\dot{n}}^{u})\rangle,\\ \langle(\breve{\xi},\breve{\rho},\breve{\varsigma},\breve{\sigma},\breve{\tau}:\mathcal{F}_{\dot{n}}^{l}),(\hat{\xi},\hat{\rho},\hat{\varsigma},\hat{\sigma},\hat{\tau}:\mathcal{F}_{\dot{n}}^{u})\rangle\end{matrix}\right\}$

For convenience, it can also be written as

$$\dot{n}=\left\{\begin{matrix}\langle[(\breve{\zeta},\breve{\eta},\breve{\theta},\breve{\lambda},\breve{\mu}),(\hat{\zeta},\hat{\eta},\hat{\theta},\hat{\lambda},\hat{\mu})]:\mathcal{T}_{\dot{n}}\rangle\\ ,\langle[(\breve{ß},\breve{\eth},\breve{\Psi},\breve{\phi},\breve{\Omega}),(\hat{ß},\hat{\eth},\hat{\Psi},\hat{\phi},\hat{\Omega}):\mathcal{I}_{\dot{n}}]\rangle,\\ \langle[(\breve{\xi},\breve{\rho},\breve{\varsigma},\breve{\sigma},\breve{\tau}),(\hat{\xi},\hat{\rho},\hat{\varsigma},\hat{\sigma},\hat{\tau})]:\mathcal{F}_{\dot{n}}\rangle\end{matrix}\right\}$$

Definition 3.3 Let $\dot{n}_1$ and $\dot{n}_2$ be two IVPFNNs,

$$\dot{n}_1=\left\{\begin{matrix}\langle[(\breve{\zeta}_1,\breve{\eta}_1,\breve{\theta}_1,\breve{\lambda}_1,\breve{\mu}_1),(\hat{\zeta}_1,\hat{\eta}_1,\hat{\theta}_1,\hat{\lambda}_1,\hat{\mu}_1)]:\mathcal{T}_{\dot{n}_1}\rangle\\ \langle[(\breve{ß}_1,\breve{\eth}_1,\breve{\Psi}_1,\breve{\phi}_1,\breve{\Omega}_1),(\hat{ß}_1,\hat{\eth}_1,\hat{\Psi}_1,\hat{\phi}_1,\hat{\Omega}_1)]:\mathcal{I}_{\dot{n}_1}\rangle\\ \langle[(\breve{\xi}_1,\breve{\rho}_1,\breve{\varsigma}_1,\breve{\sigma}_1,\breve{\tau}_1),(\hat{\xi}_1,\hat{\rho}_1,\hat{\varsigma}_1,\hat{\sigma}_1,\hat{\tau}_1)]:\mathcal{F}_{\dot{n}_1}\rangle\end{matrix}\right\}$$

$$\dot{n}_2=\left\{\begin{array}{c}\langle[(\check{\zeta}_2,\check{\eta}_2,\breve{\theta}_2,\check{\lambda}_2,\check{\mu}_2),(\hat{\zeta}_2,\hat{\eta}_2,\widehat{\theta}_2,\hat{\lambda}_2,\hat{\mu}_2)]:\mathcal{T}_{\dot{n}_2}\rangle\\ \langle[(\breve{\text{ß}}_2,\check{\eth}_2,\breve{\Psi}_2,\breve{\phi}_2,\breve{\Omega}_2),(\widehat{\text{ß}}_2,\hat{\eth}_2,\widehat{\Psi}_2,\widehat{\phi}_2,\widehat{\Omega}_2)]:\mathcal{I}_{\dot{n}_2}\rangle\\ \langle[(\check{\xi}_2,\breve{\rho}_2,\check{\varsigma}_2,\breve{\sigma}_2,\check{\tau}_2),(\hat{\xi}_2,\rho_2,\hat{\varsigma}_2,\widehat{\sigma}_2,\hat{\tau}_2)]:\mathcal{F}_{\dot{n}_2}\rangle\end{array}\right\}$$

Then the following operations on IVPFNNs are proposed as:

$$\dot{n}_1+\dot{n}_2$$
$$=\left\{\begin{array}{c}\left[\begin{array}{l}(\check{\zeta}_1+\check{\zeta}_2-\check{\zeta}_1\check{\zeta}_2,\check{\eta}_1+\check{\eta}_2-\check{\eta}_1\check{\eta}_2,\breve{\theta}_1+\breve{\theta}_2-\breve{\theta}_1\breve{\theta}_2,\check{\lambda}_1+\check{\lambda}_2-\check{\lambda}_1\check{\lambda}_2,\check{\mu}_1+\check{\mu}_2-\check{\mu}_1\check{\mu}_2),\\ (\hat{\zeta}_1+\hat{\zeta}_2-\hat{\zeta}_1\hat{\zeta}_2,\hat{\eta}_1+\hat{\eta}_2-\hat{\eta}_1\hat{\eta}_2,\widehat{\theta}_1+\widehat{\theta}_2-\widehat{\theta}_1\widehat{\theta}_2,\hat{\lambda}_1+\hat{\lambda}_2-\hat{\lambda}_1\hat{\lambda}_2,\hat{\mu}_1+\hat{\mu}_2-\hat{\mu}_1\hat{\mu}_2)\end{array}\right]\\ [(\breve{\text{ß}}_1\breve{\text{ß}}_2,\check{\eth}_1\check{\eth}_2,\breve{\Psi}_1\breve{\Psi}_2,\breve{\phi}_1\breve{\phi}_2,\breve{\Omega}_1\breve{\Omega}_2),(\widehat{\text{ß}}_1\widehat{\text{ß}}_2,\hat{\eth}_1\hat{\eth}_2,\widehat{\Psi}_1\widehat{\Psi}_2,\widehat{\phi}_1\widehat{\phi}_2,\widehat{\Omega}_1\widehat{\Omega}_2)],\\ [(\check{\xi}_1\check{\xi}_2,\breve{\rho}_1\breve{\rho}_2,\check{\varsigma}_1\check{\varsigma}_2,\breve{\sigma}_1\breve{\sigma}_2,\check{\tau}_1\check{\tau}_2),(\hat{\xi}_1\hat{\xi}_2,\hat{\rho}_1\hat{\rho}_2,\hat{\varsigma}_1\hat{\varsigma}_2,\widehat{\sigma}_1\widehat{\sigma}_2,\hat{\tau}_1\hat{\tau}_2)]\end{array}\right\}$$

$$\dot{n}_1\times\dot{n}_2$$
$$=\left\{\begin{array}{c}[(\check{\zeta}_1\check{\zeta}_2,\check{\eta}_1\check{\eta}_2,\breve{\theta}_1\breve{\theta}_2,\check{\lambda}_1\check{\lambda}_2,\check{\mu}_1\check{\mu}_2),(\hat{\zeta}_1\hat{\zeta}_2,\hat{\eta}_1\hat{\eta}_2,\widehat{\theta}_1\widehat{\theta}_2,\hat{\lambda}_1\hat{\lambda}_2,\hat{\mu}_1\hat{\mu}_2)],\\ \left[\begin{array}{c}(\breve{\text{ß}}_1+\breve{\text{ß}}_2-\breve{\text{ß}}_1\breve{\text{ß}}_2,\check{\eth}_1+\check{\eth}_2-\check{\eth}_1\check{\eth}_2,\breve{\Psi}_1+\breve{\Psi}_2-\breve{\Psi}_1\breve{\Psi}_2,\breve{\phi}_1+\breve{\phi}_2-\breve{\phi}_1\breve{\phi}_2,\breve{\Omega}_1+\breve{\Omega}_2-\breve{\Omega}_1\breve{\Omega}_2),\\ (\widehat{\text{ß}}_1+\widehat{\text{ß}}_2-\widehat{\text{ß}}_1\widehat{\text{ß}}_2,\hat{\eth}_1+\hat{\eth}_2-\hat{\eth}_1\hat{\eth}_2\widehat{\Psi}_1+\widehat{\Psi}_2-\widehat{\Psi}_1\widehat{\Psi}_2,\widehat{\phi}_1+\widehat{\phi}_2-\widehat{\phi}_1\widehat{\phi}_2,\widehat{\Omega}_1+\widehat{\Omega}_2-\widehat{\Omega}_1\widehat{\Omega}_2)\end{array}\right],\\ \left[\begin{array}{l}(\check{\xi}_1+\check{\xi}_2-\check{\xi}_1\check{\xi}_2,\breve{\rho}_1+\breve{\rho}_2-\breve{\rho}_1\breve{\rho}_2,\check{\varsigma}_1+\check{\varsigma}_2-\check{\varsigma}_1\check{\varsigma}_2,\breve{\sigma}_1+\breve{\sigma}_2-\breve{\sigma}_1\breve{\sigma}_2,\check{\tau}_1+\check{\tau}_2-\check{\tau}_1\check{\tau}_2),\\ (\hat{\xi}_1+\hat{\xi}_2-\hat{\xi}_1\hat{\xi}_2,\hat{\rho}_1+\hat{\rho}_2-\hat{\rho}_1\hat{\rho}_2,\hat{\varsigma}_1+\hat{\varsigma}_2-\hat{\varsigma}_1\hat{\varsigma}_2,\breve{\sigma}_1+\widehat{\sigma}_2-\widehat{\sigma}_1\widehat{\sigma}_2,\hat{\tau}_1+\hat{\tau}_2-\hat{\tau}_1\hat{\tau}_2)\end{array}\right]\end{array}\right\}$$

$$\text{ɤ}\dot{n}_1=\left\{\begin{array}{c}\left[\begin{array}{l}(1-(1-\check{\zeta}_1)^{\text{ɤ}},1-(1-\check{\eta}_1)^{\text{ɤ}},1-(1-\breve{\theta}_1)^{\text{ɤ}},1-(1-\check{\lambda}_1)^{\text{ɤ}},1-(1-\check{\mu}_1)^{\text{ɤ}}),\\ (1-(1-\hat{\zeta}_1)^{\text{ɤ}},1-(1-\hat{\eta}_1)^{\text{ɤ}},1-(1-\widehat{\theta}_1)^{\text{ɤ}},1-(1-\hat{\lambda}_1)^{\text{ɤ}},1-(1-\hat{\mu}_1)^{\text{ɤ}})\end{array}\right],\\ [(\breve{\text{ß}}_1{}^{\text{ɤ}},\check{\eth}_1{}^{\text{ɤ}},\breve{\Psi}_1{}^{\text{ɤ}},\breve{\phi}_1{}^{\text{ɤ}},\breve{\Omega}_1{}^{\text{ɤ}}),(\widehat{\text{ß}}_1{}^{\text{ɤ}},\hat{\eth}_1{}^{\text{ɤ}},\widehat{\Psi}_1{}^{\text{ɤ}},\widehat{\phi}_1{}^{\text{ɤ}},\widehat{\Omega}_1{}^{\text{ɤ}})],\\ [(\check{\xi}_1{}^{\text{ɤ}},\breve{\rho}_1{}^{\text{ɤ}},\check{\varsigma}_1{}^{\text{ɤ}},\breve{\sigma}_1{}^{\text{ɤ}},\check{\tau}_1{}^{\text{ɤ}}),(\hat{\xi}_1{}^{\text{ɤ}},\hat{\rho}_1{}^{\text{ɤ}},\hat{\varsigma}_1{}^{\text{ɤ}},\widehat{\sigma}_1{}^{\text{ɤ}},\hat{\tau}_1{}^{\text{ɤ}})]\end{array}\right\},\ \text{ɤ}>0$$

$$\dot{n}_1{}^{\text{ɤ}}=\left\{\begin{array}{c}[(\check{\zeta}_1{}^{\text{ɤ}},\check{\eta}_1{}^{\text{ɤ}},\breve{\theta}_1{}^{\text{ɤ}},\check{\lambda}_1{}^{\text{ɤ}},\check{\mu}_1{}^{\text{ɤ}}),(\hat{\zeta}_1{}^{\text{ɤ}},\hat{\eta}_1{}^{\text{ɤ}},\widehat{\theta}_1{}^{\text{ɤ}},\hat{\lambda}_1{}^{\text{ɤ}},\hat{\mu}_1{}^{\text{ɤ}})],\\ \left[\begin{array}{l}(1-(1-\breve{\text{ß}}_1)^{\text{ɤ}},1-(1-\check{\eth}_1)^{\text{ɤ}},1-(1-\breve{\Psi}_1)^{\text{ɤ}},1-(1-\breve{\phi}_1)^{\text{ɤ}},1-(1-\breve{\Omega}_1)^{\text{ɤ}}),\\ (1-(1-\widehat{\text{ß}}_1)^{\text{ɤ}},1-(1-\hat{\eth}_1)^{\text{ɤ}},1-(1-\widehat{\Psi}_1)^{\text{ɤ}},1-(1-\widehat{\phi}_1)^{\text{ɤ}},1-(1-\widehat{\Omega}_1)^{\text{ɤ}})\end{array}\right],\\ \left[\begin{array}{l}(1-(1-\check{\xi}_1)^{\text{ɤ}},1-(1-\breve{\rho}_1)^{\text{ɤ}},1-(1-\check{\varsigma}_1)^{\text{ɤ}},1-(1-\breve{\sigma}_1)^{\text{ɤ}},1-(1-\check{\tau}_1)^{\text{ɤ}}),\\ (1-(1-\hat{\xi}_1)^{\text{ɤ}},1-(1-\hat{\rho}_1)^{\text{ɤ}},1-(1-\hat{\varsigma}_1)^{\text{ɤ}},1-(1-\widehat{\sigma}_1)^{\text{ɤ}},1-(1-\hat{\tau}_1)^{\text{ɤ}})\end{array}\right]\end{array}\right\},$$

$\text{ɤ}>0$

3.4 SCORE AND ACCURACY FUNCTIONS

The score and accuracy function of IVPFNN based on the pentagonal neutrosophic numbers$\dot{n}$ are defined as follows

$$Ş(\dot{n}) = \frac{1}{6}\left[4 + \frac{\check{\zeta}+\breve{\eta}+\breve{\theta}+\check{\lambda}+\breve{\mu}}{5} + \frac{\hat{\zeta}+\widehat{\eta}+\widehat{\theta}+\widehat{\lambda}+\widehat{\mu}}{5} - \frac{\check{ß}+\check{ð}+\breve{\varPsi}+\breve{ɸ}+\breve{\Omega}}{5} - \frac{\hat{ß}+\hat{ð}+\widehat{\varPsi}+\widehat{ɸ}+\widehat{\Omega}}{5} - \frac{\check{\xi}+\breve{\rho}+\check{\varsigma}+\breve{\sigma}+\check{\tau}}{5} - \frac{\hat{\xi}+\widehat{\rho}+\hat{\varsigma}+\widehat{\sigma}+\hat{\tau}}{5}\right]$$
(1)

where $S(\dot{n}) \in [0,1]$.

Larger value of $Ş(\dot{n})$ implies higher IVPFNN $\dot{n}$.

If $Ş(\dot{n}) = 1$, then $\dot{n} = \{[(1,1,1,1,1),(1,1,1,1,1)],[(0,0,0,0,0),(0,0,0,0,0)],[(0,0,0,0,0),(0,0,0,0,0)]\}$, which is the largest IVPFNN.

If $Ş(\dot{n}) = 0$, then $\dot{n} = \{[(0,0,0,0,0),(0,0,0,0,0)],[(1,1,1,1,1),(1,1,1,1,1)],[(1,1,1,1,1),(1,1,1,1,1)]\}$, which is the smallest IVPFNN.

Accuracy function for IVPFNN $\dot{n}$ is given by

$$A(\dot{n}) = \frac{1}{2}\left[\frac{\check{\zeta}+\breve{\eta}+\breve{\theta}+\check{\lambda}+\breve{\mu}}{5} + \frac{\hat{\zeta}+\hat{\eta}+\widehat{\theta}+\hat{\lambda}+\hat{\mu}}{5} - \frac{\check{\xi}+\breve{\rho}+\check{\varsigma}+\breve{\sigma}+\check{\tau}}{5} - \frac{\hat{\xi}+\widehat{\rho}\ +\hat{\varsigma}+\widehat{\sigma}+\hat{\tau}}{5}\right]$$

Where $A(\dot{n}) \in [-1,1]$.

Definition 3.5

Let $\dot{n}_1$ and $\dot{n}_2$ be two IVPFNNs, $Ş(\dot{n}_1), Ş(\dot{n}_2)$ be the score functions and $A(\dot{n}_1), A(\dot{n}_2)$ be the accuracy functions respectively.

- If $Ş(\dot{n}_1) > Ş(\dot{n}_2)$, then $\dot{n}_1 > \dot{n}_2$
- If $Ş(\dot{n}_1) = Ş(\dot{n}_2)$ and

a) If $A(\dot{n}) = A(\dot{n}_2)$, then $\dot{n}_1 = \dot{n}_2$;

b) If $A(\dot{n}_1) > A(\dot{n}_2)$, then $\dot{n}_1 > \dot{n}_2$.

4. INTERVAL-VALUED PENTAGONAL FUZZY NEUTROSOPHIC WEIGHTED ARITHMETIC AVERAGING (IVPFNWAA)OPERATOR.

Let $\dot{n}_j = \left\{ \begin{array}{c} \langle[(\check{\zeta}_j, \check{\eta}_j, \check{\theta}_j, \check{\lambda}_j, \check{\mu}_j), (\hat{\zeta}_j, \hat{\eta}_j, \hat{\theta}_j, \hat{\lambda}_j, \hat{\mu}_j)]: \mathcal{T}_{\dot{n}_j}\rangle, \\ \langle\left[\left(\breve{ß}_j, \breve{ð}_j, \breve{\Psi}_j, \breve{ф}_j, \breve{\Omega}_j\right), \left(\widehat{ß}_j, \hat{ð}_j, \widehat{\Psi}_j, \widehat{ф}_j, \widehat{\Omega}_j\right)\right]: \mathcal{I}_{\dot{n}_j}\rangle, \\ \langle[(\check{\xi}_j, \check{\rho}_j, \check{\varsigma}_j, \check{\sigma}_j, \check{\tau}_j), (\hat{\xi}_j, \hat{\rho}_j, \hat{\varsigma}_j, \hat{\sigma}_j, \hat{\tau}_j)]: \mathcal{F}_{\dot{n}_j}\rangle \end{array} \right\}$ where $j = 1,2,3, \dots, n$ be the collection of IVPFNNs is the set of real numbers given by $IVPFNWAA: (Re)^n \to Re$.

Let $IVPFNWAA$ operator is represented by $IVPFNWAA(\dot{n}_1, \dot{n}_1, \dots, \dot{n}_1)$ is defined as

$IVPFNWAA(\dot{n}_1, \dot{n}_1, \dots, \dot{n}_1) = w_1\dot{n}_1 + w_2\dot{n}_2 + \cdots + w_n\dot{n}_n = \sum_{j=1}^{n} w_j \dot{n}_j$, whereas w_j $(j = 1,2, \dots, n)$ denotes the weightage of IVPFNNs $\dot{n}_j\,(j = 1,2, \dots, n)$and also $\sum_{j=1}^{n} w_j = 1$, whereas $w_j \in [0,1]$.

We now propose the following theorem by making use of the basic operations of IVPFNNs

Theorem 1. Let
$\dot{n}_j = \left\{ \begin{array}{c} \langle[(\check{\zeta}_j, \check{\eta}_j, \check{\theta}_j, \check{\lambda}_j, \check{\mu}_j), (\hat{\zeta}_j, \hat{\eta}_j, \hat{\theta}_j, \hat{\lambda}_j, \hat{\mu}_j)]: \mathcal{T}_{\dot{n}_j}\rangle \\ \langle\left[\left(\breve{ß}_j, \breve{ð}_j, \breve{\Psi}_j, \breve{ф}_j, \breve{\Omega}_j\right), \left(\widehat{ß}_j, \hat{ð}_j, \widehat{\Psi}_j, \widehat{ф}_j, \widehat{\Omega}_j\right)\right]: \mathcal{I}_{\dot{n}_j}\rangle \\ \langle[(\check{\xi}_j, \check{\rho}_j, \check{\varsigma}_j, \check{\sigma}_j, \check{\tau}_j), (\hat{\xi}_j, \hat{\rho}_j, \hat{\varsigma}_j, \hat{\sigma}_j, \hat{\tau}_j)]: \mathcal{F}_{\dot{n}_j}\rangle \end{array} \right\}$ $(j = 1,2, \dots, n))$ be a collection of interval-valued pentagonal fuzzy neutrosophic value (IVPFNV)s in the set of real numbers. The aggregated value of IVPFNWAA is also an IVPFNV, and

$$IVPFNWAA_n(\dot{n}_1, \dot{n}_2, \dots, \dot{n}_n) = w_1\dot{n}_1 + w_2\dot{n}_2 + \cdots + w_n\dot{n}_n = \sum_{j=1}^{n} w_j \dot{n}_j$$

$$= \left\{ \begin{array}{c} \left[\begin{array}{c} \left(1 - \prod_{j=1}^{n}(1-\check{\zeta}_j)^{w_j}, 1 - \prod_{j=1}^{n}(1-\check{\eta}_j)^{w_j}, 1 - \prod_{j=1}^{n}(1-\check{\theta}_j)^{w_j}, 1 - \prod_{j=1}^{n}(1-\check{\lambda}_j)^{w_j}, 1 - \prod_{j=1}^{n}(1-\check{\mu}_j)^{w_j}\right), \\ \left(1 - \prod_{j=1}^{n}(1-\hat{\zeta}_j)^{w_j}, 1 - \prod_{j=1}^{n}(1-\hat{\eta}_j)^{w_j}, 1 - \prod_{j=1}^{n}(1-\hat{\theta}_j)^{w_j}, 1 - \prod_{j=1}^{n}(1-\hat{\lambda}_j)^{w_j}, 1 - \prod_{j=1}^{n}(1-\hat{\mu}_j)^{w_j}\right) \end{array} \right], \\ \left[\left(\prod_{j=1}^{n}\breve{ß}_j^{\,w_j}, \prod_{j=1}^{n}\breve{ð}_j^{\,w_j}, \prod_{j=1}^{n}\breve{\Psi}_j^{\,w_j}, \prod_{j=1}^{n}\breve{ф}_j^{\,w_j}, \prod_{j=1}^{n}\breve{\Omega}_j^{\,w_j}\right), \left(\prod_{j=1}^{n}\widehat{ß}_j^{\,w_j}, \prod_{j=1}^{n}\breve{ð}_j^{\,w_j}, \prod_{j=1}^{n}\widehat{\Psi}_j^{\,w_j}, \prod_{j=1}^{n}\widehat{ф}_j^{\,w_j}, \prod_{j=1}^{n}\widehat{\Omega}_j^{\,w_j}\right)\right] \\ \left[\left(\prod_{j=1}^{n}\check{\xi}_j^{\,w_j}, \prod_{j=1}^{n}\check{\rho}_j^{\,w_j}, \prod_{j=1}^{n}\check{\varsigma}_j^{\,w_j}, \prod_{j=1}^{n}\check{\sigma}_j^{\,w_j}, \prod_{j=1}^{n}\check{\tau}_j^{\,w_j}\right), \left(\prod_{j=1}^{n}\hat{\xi}_j^{\,w_j}, \prod_{j=1}^{n}\hat{\rho}_j^{\,w_j}, \prod_{j=1}^{n}\hat{\varsigma}_j^{\,w_j}, \prod_{j=1}^{n}\hat{\sigma}_j^{\,w_j}, \prod_{j=1}^{n}\hat{\tau}_j^{\,w_j}\right)\right] \end{array} \right. \tag{2}$$

whereas the weight of IVPFNNV $\dot{n}_j\,(j = 1,2,\dots,n)$ is $w_j \in [0,1]$, with the condition $\sum_{j=1}^{n} w_j = 1$.

Proof: By mathematical induction we prove this theorem,

For $n = 1$, it is trivial.

For $n = 2$, $\sum_{j=1}^{2} w_j \dot{n}_j = w_1\dot{n}_1 + w_2\dot{n}_2$.

$$
= \left\{ \begin{matrix} \left[\left(\begin{matrix} \left[\begin{matrix} \left(1-\left(1-\check{\zeta}_1\right)^{w_1}, 1-(1-\check{\eta}_1)^{w_1}, 1-\left(1-\breve{\theta}_1\right)^{w_1}, 1-\left(1-\check{\lambda}_1\right)^{w_1}, 1-(1-\breve{\mu}_1)^{w_1}\right), \\ \left(1-\left(1-\hat{\zeta}_1\right)^{w_1}, 1-(1-\hat{\eta}_1)^{w_1}, 1-\left(1-\hat{\theta}_1\right)^{w_1}, 1-\left(1-\hat{\lambda}_1\right)^{w_1}, 1-(1-\hat{\mu}_1)^{w_1}\right) \end{matrix} \right], \\ \left[\left(\breve{\text{ß}}_1^{\,w_1}, \check{\eth}_1^{\,w_1}, \breve{\Psi}_1^{\,w_1}, \breve{\phi}_1^{\,w_1}, \breve{\Omega}_1^{\,w_1}\right), \left(\hat{\text{ß}}_1^{\,w_1}, \hat{\eth}_1^{\,w_1}, \widehat{\Psi}_1^{\,w_1}, \widehat{\phi}_1^{\,w_1}, \widehat{\Omega}_1^{\,w_1}\right)\right], \\ \left[\left(\check{\xi}_1^{\,w_1}, \breve{\rho}_1^{\,w_1}, \check{\varsigma}_1^{\,w_1}, \breve{\sigma}_1^{\,w_1}, \check{\tau}_1^{\,w_1}\right), \left(\hat{\xi}_1^{\,w_1}, \hat{\rho}_1^{\,w_1}, \hat{\varsigma}_1^{\,w_1}, \hat{\sigma}_1^{\,w_1}, \hat{\tau}_1^{\,w_1}\right)\right] \end{matrix} \right) \right] \\ + \\ \left[\left(\begin{matrix} \left[\begin{matrix} \left(1-\left(1-\check{\zeta}_2\right)^{w_2}, 1-(1-\check{\eta}_2)^{w_2}, 1-\left(1-\breve{\theta}_2\right)^{w_2}, 1-\left(1-\check{\lambda}_2\right)^{w_2}, 1-(1-\breve{\mu}_2)^{w_2}\right), \\ \left(1-\left(1-\hat{\zeta}_2\right)^{w_2}, 1-(1-\hat{\eta}_2)^{w_2}, 1-\left(1-\hat{\theta}_2\right)^{w_2}, 1-\left(1-\hat{\lambda}_2\right)^{w_2}, 1-(1-\hat{\mu}_2)^{w_2}\right) \end{matrix} \right], \\ \left[\left(\breve{\text{ß}}_2^{\,w_2}, \check{\eth}_2^{\,w_2}, \breve{\Psi}_2^{\,w_2}, \breve{\phi}_2^{\,w_2}, \breve{\Omega}_2^{\,w_2}\right), \left(\hat{\text{ß}}_2^{\,w_2}, \hat{\eth}_2^{\,w_2}, \widehat{\Psi}_2^{\,w_2}, \widehat{\phi}_2^{\,w_2}, \widehat{\Omega}_2^{\,w_2}\right)\right], \\ \left[\left(\check{\xi}_2^{\,w_2}, \breve{\rho}_2^{\,w_2}, \check{\varsigma}_2^{\,w_2}, \breve{\sigma}_2^{\,w_2}, \check{\tau}_2^{\,w_2}\right), \left(\hat{\xi}_2^{\,w_2}, \hat{\rho}_2^{\,w_2}, \hat{\varsigma}_2^{\,w_2}, \hat{\sigma}_2^{\,w_2}, \hat{\tau}_2^{\,w_2}\right)\right] \end{matrix} \right) \right] \end{matrix} \right\}
$$

$$
= \left\{ \begin{matrix} \left[\begin{matrix} \left(\begin{matrix} 1-\left(1-\check{\zeta}_1\right)^{w_1} + 1-\left(1-\check{\zeta}_2\right)^{w_2} - \left(1-\left(1-\check{\zeta}_1\right)^{w_1}\right)\left(1-\left(1-\check{\zeta}_2\right)^{w_2}\right), \\ 1-(1-\check{\eta}_1)^{w_1} + 1-(1-\check{\eta}_2)^{w_2} - (1-(1-\check{\eta}_1)^{w_1})(1-(1-\check{\eta}_2)^{w_2}), \\ 1-\left(1-\breve{\theta}_1\right)^{w_1} + 1-\left(1-\breve{\theta}_2\right)^{w_2} - \left(1-\left(1-\breve{\theta}_1\right)^{w_1}\right)\left(1-\left(1-\breve{\theta}_2\right)^{w_2}\right), \\ 1-\left(1-\check{\lambda}_1\right)^{w_1} + 1-\left(1-\check{\lambda}_2\right)^{w_2} - \left(1-\left(1-\check{\lambda}_1\right)^{w_1}\right)\left(1-\left(1-\check{\lambda}_2\right)^{w_2}\right), \\ 1-(1-\breve{\mu}_1)^{w_1} + 1-(1-\breve{\mu}_2)^{w_2} - (1-(1-\breve{\mu}_1)^{w_1})(1-(1-\breve{\mu}_2)^{w_2}) \end{matrix} \right) \\ \left(\begin{matrix} 1-\left(1-\hat{\zeta}_1\right)^{w_1} + 1-\left(1-\hat{\zeta}_2\right)^{w_2} - \left(1-\left(1-\hat{\zeta}_1\right)^{w_1}\right)\left(1-\left(1-\hat{\zeta}_2\right)^{w_2}\right), \\ 1-(1-\hat{\eta}_1)^{w_1} + 1-(1-\hat{\eta}_2)^{w_2} - (1-(1-\hat{\eta}_1)^{w_1})(1-(1-\hat{\eta}_2)^{w_2}), \\ 1-\left(1-\hat{\theta}_1\right)^{w_1} + 1-\left(1-\hat{\theta}_2\right)^{w_2} - \left(1-\left(1-\hat{\theta}_1\right)^{w_1}\right)\left(1-\left(1-\hat{\theta}_2\right)^{w_2}\right), \\ 1-\left(1-\hat{\lambda}_1\right)^{w_1} + 1-\left(1-\hat{\lambda}_2\right)^{w_2} - \left(1-\left(1-\hat{\lambda}_1\right)^{w_1}\right)\left(1-\left(1-\hat{\lambda}_2\right)^{w_2}\right), \\ 1-(1-\hat{\mu}_1)^{w_1} + 1-(1-\hat{\mu}_2)^{w_2} - (1-(1-\hat{\mu}_1)^{w_1})(1-(1-\hat{\mu}_2)^{w_2}) \end{matrix} \right) \end{matrix} \right], \\ \left[\begin{matrix} \left(\breve{\text{ß}}_1^{\,w_1}\breve{\text{ß}}_2^{\,w_2}, \check{\eth}_1^{\,w_1}\check{\eth}_2^{\,w_2}, \breve{\Psi}_1^{\,w_1}\breve{\Psi}_2^{\,w_2}, \breve{\phi}_1^{\,w_1}\breve{\phi}_2^{\,w_2}, \breve{\Omega}_1^{\,w_1}\breve{\Omega}_2^{\,w_2}\right), \\ \left(\hat{\text{ß}}_1^{\,w_1}\hat{\text{ß}}_2^{\,w_2}, \hat{\eth}^{w_1}\hat{\eth}_2^{\,w_2}, \widehat{\Psi}_1^{\,w_1}\widehat{\Psi}_2^{\,w_2}, \widehat{\phi}_1^{\,w_1}\widehat{\phi}_2^{\,w_2}, \widehat{\Omega}_1^{\,w_1}\widehat{\Omega}_2^{\,w_2}\right) \end{matrix} \right], \\ \left[\begin{matrix} \left(\breve{\text{ß}}_1^{\,w_1}\breve{\text{ß}}_2^{\,w_2}, \check{\eth}_1^{\,w_1}\hat{\eth}_2^{\,w_2}, \breve{\Psi}_1^{\,w_1}\breve{\Psi}_2^{\,w_2}, \breve{\phi}_1^{\,w_1}\breve{\phi}_2^{\,w_2}, \breve{\Omega}_1^{\,w_1}\breve{\Omega}_2^{\,w_2}\right), \\ \left(\hat{\text{ß}}_1^{\,w_1}\hat{\text{ß}}_2^{\,w_2}, \hat{\eth}_1^{\,w_1}\hat{\eth}_2^{\,w_2}, \widehat{\Psi}_1^{\,w_1}\widehat{\Psi}_2^{\,w_2}, \widehat{\phi}_1^{\,w_1}\widehat{\phi}_2^{\,w_2}, \widehat{\Omega}_1^{\,w_1}\widehat{\Omega}_2^{\,w_2}\right) \end{matrix} \right] \end{matrix} \right.
$$

$$= \left\{ \begin{array}{l} \left[\begin{array}{l} \left(1-\prod_{j=1}^{2}\left(1-\check{\zeta}_j\right)^{w_j}, 1-\prod_{j=1}^{2}\left(1-\check{\eta}_j\right)^{w_j}, 1-\prod_{j=1}^{2}\left(1-\check{\theta}_j\right)^{w_j}, 1-\prod_{j=1}^{2}\left(1-\check{\lambda}_j\right)^{w_j}, 1-\prod_{j=1}^{2}\left(1-\check{\mu}_j\right)^{w_j}\right), \\ \left(1-\prod_{j=1}^{2}\left(1-\hat{\zeta}_j\right)^{w_j}, 1-\prod_{j=1}^{2}\left(1-\hat{\eta}_j\right)^{w_j}, 1-\prod_{j=1}^{2}\left(1-\hat{\theta}_j\right)^{w_j}, 1-\prod_{j=1}^{2}\left(1-\hat{\lambda}_j\right)^{w_j}, 1-\prod_{j=1}^{2}\left(1-\hat{\mu}_j\right)^{w_j}\right) \end{array} \right], \\ \left[\left(\prod_{j=1}^{2}\check{\text{ß}}_j^{\,w_j}, \prod_{j=1}^{2}\check{\eth}_j^{\,w_j}, \prod_{j=1}^{2}\check{\varPsi}_j^{\,w_j}, \prod_{j=1}^{2}\check{\text{ф}}_j^{\,w_j}, \prod_{j=1}^{2}\check{\Omega}_j^{\,w_j}\right), \left(\prod_{j=1}^{2}\hat{\text{ß}}_j^{\,w_j}, \prod_{j=1}^{2}\hat{\eth}_j^{\,w_j}, \prod_{j=1}^{2}\hat{\varPsi}_j^{\,w_j}, \prod_{j=1}^{2}\hat{\text{ф}}_j^{\,w_j}, \prod_{j=1}^{2}\hat{\Omega}_j^{\,w_j}\right)\right], \\ \left[\left(\prod_{j=1}^{2}\check{\xi}_j^{\,w_j}, \prod_{j=1}^{2}\check{\rho}_j^{\,w_j}, \prod_{j=1}^{2}\check{\varsigma}_j^{\,w_j}, \prod_{j=1}^{2}\check{\sigma}_j^{\,w_j}, \prod_{j=1}^{2}\check{\tau}_j^{\,w_j}\right), \left(\prod_{j=1}^{2}\hat{\xi}_j^{\,w_j}, \prod_{j=1}^{2}\hat{\rho}_j^{\,w_j}, \prod_{j=1}^{2}\hat{\varsigma}_j^{\,w_j}, \prod_{j=1}^{2}\hat{\sigma}_j^{\,w_j}, \prod_{j=1}^{2}\hat{\tau}_j^{\,w_j}\right)\right] \end{array} \right.$$

Hence it satisfies for $n = 2$.

For $n = k$, we assume that the theorem holds good.

Therefore, $IVPFNWAA((\dot{n}_1, \dot{n}_2, \dots, \dot{n}_k) = w_1\dot{n}_1 + w_2\dot{n}_2 + \cdots + w_k\dot{n}_k = \sum_{j=1}^{k} w_j\dot{n}_j$

$$= \left\{ \begin{array}{l} \left[\begin{array}{l} \left(1-\prod_{j=1}^{k}\left(1-\check{\zeta}_j\right)^{w_j}, 1-\prod_{j=1}^{k}\left(1-\check{\eta}_j\right)^{w_j}, 1-\prod_{j=1}^{k}\left(1-\check{\theta}_j\right)^{w_j}, 1-\prod_{j=1}^{k}\left(1-\check{\lambda}_j\right)^{w_j}, 1-\prod_{j=1}^{k}\left(1-\check{\mu}_j\right)^{w_j}\right), \\ \left(1-\prod_{j=1}^{k}\left(1-\hat{\zeta}_j\right)^{w_j}, 1-\prod_{j=1}^{k}\left(1-\hat{\eta}_j\right)^{w_j}, 1-\prod_{j=1}^{k}\left(1-\hat{\theta}_j\right)^{w_j}, 1-\prod_{j=1}^{k}\left(1-\hat{\lambda}_j\right)^{w_j}, 1-\prod_{j=1}^{k}\left(1-\hat{\mu}_j\right)^{w_j}\right) \end{array} \right], \\ \left[\left(\prod_{j=1}^{k}\check{\text{ß}}_j^{\,w_j}, \prod_{j=1}^{k}\check{\eth}_j^{\,w_j}, \prod_{j=1}^{k}\check{\varPsi}_j^{\,w_j}, \prod_{j=1}^{k}\check{\text{ф}}_j^{\,w_j}, \prod_{j=1}^{k}\check{\Omega}_j^{\,w_j}\right), \left(\prod_{j=1}^{k}\hat{\text{ß}}_j^{\,w_j}, \prod_{j=1}^{k}\hat{\eth}_j^{\,w_j}, \prod_{j=1}^{k}\hat{\varPsi}_j^{\,w_j}, \prod_{j=1}^{k}\hat{\text{ф}}_j^{\,w_j}, \prod_{j=1}^{k}\hat{\Omega}_j^{\,w_j}\right)\right] \\ \left[\left(\prod_{j=1}^{k}\check{\xi}_j^{\,w_j}, \prod_{j=1}^{k}\check{\rho}_j^{\,w_j}, \prod_{j=1}^{k}\check{\varsigma}_j^{\,w_j}, \prod_{j=1}^{k}\check{\sigma}_j^{\,w_j}, \prod_{j=1}^{k}\check{\tau}_j^{\,w_j}\right), \left(\prod_{j=1}^{k}\hat{\xi}_j^{\,w_j}, \prod_{j=1}^{k}\hat{\rho}_j^{\,w_j}, \prod_{j=1}^{k}\hat{\varsigma}_j^{\,w_j}, \prod_{j=1}^{k}\hat{\sigma}_j^{\,w_j}, \prod_{j=1}^{k}\hat{\tau}_j^{\,w_j}\right)\right] \end{array} \right.$$

For $n = k + 1$,

$$IVPFNWA((\dot{n}_1, \dot{n}_2, \dots, \dot{n}_{k+1}) = \sum_{j=1}^{k} w_j\dot{n}_j + w_{k+1}\dot{n}_{k+1}$$

$$
= \left\{ \begin{array}{c}
\left\langle \begin{array}{c}
\left[\begin{array}{l}
1-\prod_{j=1}^{k}\left(1-\check{\zeta}_j\right)^{w_j}+1-\left(1-\check{\zeta}_{k+1}\right)^{w_{k+1}}-(1-\prod_{j=1}^{k}\left(1-\check{\zeta}_j\right)^{w_j})(1-\left(1-\check{\zeta}_{k+1}\right)^{w_{k+1}}),\\
1-\prod_{j=1}^{k}\left(1-\check{\eta}_j\right)^{w_j}+1-(1-\check{\eta}_{k+1})^{w_{k+1}}-(1-\prod_{j=1}^{k}\left(1-\check{\eta}_j\right)^{w_j})(1-(1-\check{\eta}_{k+1})^{w_{k+1}}),\\
1-\prod_{j=1}^{k}\left(1-\check{\theta}_j\right)^{w_j}+1-\left(1-\check{\theta}_{k+1}\right)^{w_{k+1}}-(1-\prod_{j=1}^{k}\left(1-\check{\theta}_j\right)^{w_j})(1-\left(1-\check{\theta}_{k+1}\right)^{w_{k+1}}),\\
1-\prod_{j=1}^{k}\left(1-\check{\lambda}_j\right)^{w_j}+1-(1-\lambda_{k+1})^{w_{k+1}}-(1-\prod_{j=1}^{k}\left(1-\lambda_j\right)^{w_j})(1-\left(1-\check{\lambda}_{k+1}\right)^{w_{k+1}}),\\
1-\prod_{j=1}^{k}\left(1-\check{\mu}_j\right)^{w_j}+1-(1-\check{\mu}_{k+1})^{w_{k+1}}-(1-\prod_{j=1}^{k}\left(1-\check{\mu}_j\right)^{w_j})(1-(1-\check{\mu}_{k+1})^{w_{k+1}})
\end{array} \right] \\
\left[\begin{array}{l}
1-\prod_{j=1}^{k}\left(1-\hat{\zeta}_j\right)^{w_j}+1-\left(1-\hat{\zeta}_{k+1}\right)^{w_{k+1}}-(1-\prod_{j=1}^{k}\left(1-\hat{\zeta}_j\right)^{w_j})(1-\left(1-\hat{\zeta}_{k+1}\right)^{w_{k+1}}),\\
1-\prod_{j=1}^{k}\left(1-\hat{\eta}_j\right)^{w_j}+1-(1-\hat{\eta}_{k+1})^{w_{k+1}}-(1-\prod_{j=1}^{k}\left(1-\hat{\eta}_j\right)^{w_j})(1-(1-\hat{\eta}_{k+1})^{w_{k+1}}),\\
1-\prod_{j=1}^{k}\left(1-\hat{\theta}_j\right)^{w_j}+1-\left(1-\hat{\theta}_{k+1}\right)^{w_{k+1}}-(1-\prod_{j=1}^{k}\left(1-\hat{\theta}_j\right)^{w_j})(1-\left(1-\hat{\theta}_{k+1}\right)^{w_{k+1}}),\\
1-\prod_{j=1}^{k}\left(1-\hat{\lambda}_j\right)^{w_j}+1-\left(1-\hat{\lambda}_{k+1}\right)^{w_{k+1}}-(1-\prod_{j=1}^{k}\left(1-\hat{\lambda}_j\right)^{w_j})(1-\left(1-\hat{\lambda}_{k+1}\right)^{w_{k+1}}),\\
1-\prod_{j=1}^{k}\left(1-\hat{\mu}_j\right)^{w_j}+1-(1-\hat{\mu}_{k+1})^{w_{k+1}}-(1-\prod_{j=1}^{k}\left(1-\hat{\mu}_j\right)^{w_j})(1-(1-\hat{\mu}_{k+1})^{w_{k+1}})
\end{array} \right]
\end{array} \right\rangle \\
\left[\begin{array}{l}
\left(\prod_{j=1}^{k}\check{k}_j^{\,w_j}.\check{k}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\check{l}_j^{\,w_j}.\check{l}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\check{m}_j^{\,w_j}.\check{m}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\check{n}_j^{\,w_j}.\check{n}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\check{o}_j^{\,w_j}.\check{o}_{k+1}^{\,w_{k+1}}\right),\\
\left(\prod_{j=1}^{k}\hat{k}_j^{\,w_j}.\hat{k}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\hat{l}_j^{\,w_j}.\hat{l}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\widehat{m}_j^{\,w_j}.\widehat{m}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\hat{n}_j^{\,w_j}.\hat{n}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\hat{o}_j^{\,w_j}.\hat{o}_{k+1}^{\,w_{k+1}}\right)
\end{array} \right] \\
\left[\begin{array}{l}
\left(\prod_{j=1}^{k}\check{p}_j^{\,w_j}.\check{p}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\check{q}_j^{\,w_j}.\check{q}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\check{r}_j^{\,w_j}.\check{r}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\check{s}_j^{\,w_j}.\check{s}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\check{t}_j^{\,w_j}.\check{t}_{k+1}^{\,w_{k+1}}\right),\\
\left(\prod_{j=1}^{k}\hat{p}_j^{\,w_j}.\hat{p}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\hat{q}_j^{\,w_j}.\hat{q}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\hat{r}_j^{\,w_j}.\hat{r}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\hat{s}_j^{\,w_j}.\hat{s}_{k+1}^{\,w_{k+1}},\prod_{j=1}^{k}\hat{t}_j^{\,w_j}.\hat{t}_{k+1}^{\,w_{k+1}}\right)
\end{array} \right]
\end{array} \right\}
$$

$$= \left\{ \begin{array}{l} \left[\begin{array}{l} \left(1-\prod_{j=1}^{k+1}\left(1-\check{\zeta}_j\right)^{w_j}, 1-\prod_{j=1}^{k+1}\left(1-\check{\eta}_j\right)^{w_j}, 1-\prod_{j=1}^{k+1}\left(1-\check{\theta}_j\right)^{w_j}, 1-\prod_{j=1}^{k+1}\left(1-\check{\lambda}_j\right)^{w_j}, 1-\prod_{j=1}^{k+1}\left(1-\check{\mu}_j\right)^{w_j} \right), \\ \left(1-\prod_{j=1}^{k+1}\left(1-\hat{\zeta}_j\right)^{w_j}, 1-\prod_{j=1}^{k+1}\left(1-\hat{\eta}_j\right)^{w_j}, 1-\prod_{j=1}^{k+1}\left(1-\hat{\theta}_j\right)^{w_j}, 1-\prod_{j=1}^{k+1}\left(1-\hat{\lambda}_j\right)^{w_j}, 1-\prod_{j=1}^{k+1}\left(1-\hat{\mu}_j\right)^{w_j} \right) \end{array} \right], \\ \left[\left(\prod_{j=1}^{k+1}\breve{ß}_j^{\,w_j}, \prod_{j=1}^{k+1}\check{ð}_j^{\,w_j}, \prod_{j=1}^{k+1}\breve{\Psi}_j^{\,w_j}, \prod_{j=1}^{k+1}\check{ɸ}_j^{\,w_j}, \prod_{j=1}^{k+1}\breve{\Omega}_j^{\,w_j} \right), \left(\prod_{j=1}^{k+1}\widehat{ß}_j^{\,w_j}, \prod_{j=1}^{k+1}\hat{ð}_j^{\,w_j}, \prod_{j=1}^{k+1}\widehat{\Psi}_j^{\,w_j}, \prod_{j=1}^{k+1}\widehat{ɸ}_j^{\,w_j}, \prod_{j=1}^{k+1}\widehat{\Omega}_j^{\,w_j} \right) \right] \\ \left[\left(\prod_{j=1}^{k+1}\check{\xi}_j^{\,w_j}, \prod_{j=1}^{k+1}\check{\rho}_j^{\,w_j}, \prod_{j=1}^{k+1}\check{\varsigma}_j^{\,w_j}, \prod_{j=1}^{k+1}\check{\sigma}_j^{\,w_j}, \prod_{j=1}^{k+1}\check{\tau}_j^{\,w_j} \right), \left(\prod_{j=1}^{k+1}\hat{\xi}_j^{\,w_j}, \prod_{j=1}^{k+1}\hat{\rho}_j^{\,w_j}, \prod_{j=1}^{k+1}\hat{\varsigma}_j^{\,w_j}, \prod_{j=1}^{k+1}\hat{\sigma}_j^{\,w_j}, \prod_{j=1}^{k+1}\hat{\tau}_j^{\,w_j} \right) \right] \end{array} \right.$$

Hence, we observe that the theorem holds good for $n = k + 1$.

By mathematical induction, the theorem is verified for all values of n.

The three membership functions of $\dot{n}_j$ lies between $[0,1]$ which satisfies the condition

$0 \leq \left(1-\prod_{j=1}^{n}\left(1-\check{\mu}_j\right)^{w_j}\right) \leq 1,\ 0 \leq \left(\prod_{j=1}^{n}\breve{\Omega}_j^{\,w_j}\right) \leq 1,\ 0 \leq \left(\prod_{j=1}^{n}\check{\tau}_j^{\,w_j}\right) \leq 1,$

$0 \leq \left(1-\prod_{j=1}^{n}\left(1-\hat{\mu}_j\right)^{w_j}\right) \leq 1,\ 0 \leq \left(\prod_{j=1}^{n}\widehat{\Omega}_j^{\,w_j}\right) \leq 1,\ 0 \leq \left(\prod_{j=1}^{n}\hat{\tau}_j^{\,w_j}\right) \leq 1$, further it holds the relation

$0 \leq \left(1-\prod_{j=1}^{n}\left(1-\check{\mu}_j\right)^{w_j} + \prod_{j=1}^{n}\breve{\Omega}_j^{\,w_j} + \prod_{j=1}^{n}\check{\tau}_j^{\,w_j}\right) \leq 3$ and

$0 \leq \left(1-\prod_{j=1}^{n}\left(1-\hat{\mu}_j\right)^{w_j} + \prod_{j=1}^{n}\widehat{\Omega}_j^{\,w_j} + \prod_{j=1}^{n}\hat{\tau}_j^{\,w_j}\right) \leq 3.$

Hence the theorem is proved.

We now discuss some of the necessary properties of IVPFNWAA operator.

Property 1.[Idempotency]

If all $\dot{n}_j\,(j = 1,2,\dots,n)$ are equal, then $\dot{n}_j = \dot{n} = \left\{ \begin{array}{c} \langle\left[\left(\check{\zeta},\check{\eta},\check{\theta},\check{\lambda},\check{\mu}\right),\left(\hat{\zeta},\hat{\eta},\hat{\theta},\hat{\lambda},\hat{\mu}\right)\right]:\mathcal{T}_{\dot{n}}\rangle, \\ \langle\left[\left(\breve{ß},\check{ð},\breve{\Psi},\check{ɸ},\breve{\Omega}\right),\left(\widehat{ß},\hat{ð},\widehat{\Psi},\widehat{ɸ},\widehat{\Omega}\right):\mathcal{I}_{\dot{n}}\right]\rangle, \\ \langle\left[\left(\check{\xi},\check{\rho},\check{\varsigma},\check{\sigma},\check{\tau}\right),\left(\hat{\xi},\hat{\rho},\hat{\varsigma},\hat{\sigma},\hat{\tau}\right)\right]:\mathcal{F}_{\dot{n}}\rangle \end{array} \right\}$, then

$IVPFNWAA(\dot{n}_1, \dot{n}_2, \dots \dot{n}_k) = \dot{n}$

Proof: For proving the idempotency we can use the eqn 2 of previous theorem

$$IVPFNWAA(\dot{n}_1, \dot{n}_2, \dots, \dot{n}_n) = IVPFNWAA(\dot{n}, \dot{n}, \dots, \dot{n}) = \sum_{j=1}^{n} w_j \dot{n}$$

$$= \left\{ \begin{array}{c} \left[\begin{array}{c} \left(1-\prod_{j=1}^{n}(1-\check{\zeta})^{w_j}, 1-\prod_{j=1}^{n}(1-\check{\eta})^{w_j}, 1-\prod_{j=1}^{n}(1-\check{\theta})^{w_j}, 1-\prod_{j=1}^{n}(1-\check{\lambda})^{w_j}, 1-\prod_{j=1}^{n}(1-\check{\mu})^{w_j}\right), \\ \left(1-\prod_{j=1}^{n}(1-\hat{\zeta})^{w_j}, 1-\prod_{j=1}^{n}(1-\hat{\eta})^{w_j}, 1-\prod_{j=1}^{n}(1-\hat{\theta})^{w_j}, 1-\prod_{j=1}^{n}(1-\hat{\lambda})^{w_j}, 1-\prod_{j=1}^{n}(1-\hat{\mu})^{w_j}\right) \end{array} \right], \\ \left[\left(\prod_{j=1}^{n}\breve{ß}^{w_j}, \prod_{j=1}^{n}\check{ð}^{w_j}, \prod_{j=1}^{n}\breve{\Psi}^{w_j}, \prod_{j=1}^{n}\breve{ɸ}^{w_j}, \prod_{j=1}^{n}\breve{\Omega}^{w_j}\right), \left(\prod_{j=1}^{n}\widehat{ß}^{w_j}, \prod_{j=1}^{n}\hat{ð}^{w_j}, \prod_{j=1}^{n}\widehat{\Psi}^{w_j}, \prod_{j=1}^{n}\widehat{ɸ}^{w_j}, \prod_{j=1}^{n}\widehat{\Omega}^{w_j}\right)\right], \\ \left[\left(\prod_{j=1}^{n}\check{\xi}^{w_j}, \prod_{j=1}^{n}\breve{\rho}^{w_j}, \prod_{j=1}^{n}\check{\varsigma}^{w_j}, \prod_{j=1}^{n}\breve{\sigma}^{w_j}, \prod_{j=1}^{n}\check{\tau}^{w_j}\right), \left(\prod_{j=1}^{n}\hat{\xi}^{w_j}, \prod_{j=1}^{n}\hat{\rho}^{w_j}, \prod_{j=1}^{n}\hat{\varsigma}^{w_j}, \prod_{j=1}^{n}\hat{\sigma}^{w_j}, \prod_{j=1}^{n}\hat{\tau}^{w_j}\right)\right] \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left[\begin{array}{c} \left(1-(1-\check{\zeta})^{\sum_{j=1}^{n} w_j}, 1-(1-\check{\eta})^{\sum_{j=1}^{n} w_j}, 1-(1-\check{\theta})^{\sum_{j=1}^{n} w_j}, 1-(1-\check{\lambda})^{\sum_{j=1}^{n} w_j}, 1-(1-\check{\mu})^{\sum_{j=1}^{n} w_j}\right), \\ \left(1-(1-\hat{\zeta})^{\sum_{j=1}^{n} w_j}, 1-(1-\hat{\eta})^{\sum_{j=1}^{n} w_j}, 1-(1-\hat{\theta})^{\sum_{j=1}^{n} w_j}, 1-(1-\hat{\lambda})^{\sum_{j=1}^{n} w_j}, 1-(1-\hat{\mu})^{\sum_{j=1}^{n} w_j}\right) \end{array} \right] \\ \left[\left(\breve{ß}^{\sum_{j=1}^{n} w_j}, \check{ð}^{\sum_{j=1}^{n} w_j}, \breve{\Psi}^{\sum_{j=1}^{n} w_j}, \breve{ɸ}^{\sum_{j=1}^{n} w_j}, \breve{\Omega}^{\sum_{j=1}^{n} w_j}\right), \left(\widehat{ß}^{\sum_{j=1}^{n} w_j}, \hat{ð}^{\sum_{j=1}^{n} w_j}, \widehat{\Psi}^{\sum_{j=1}^{n} w_j}, \widehat{ɸ}^{\sum_{j=1}^{n} w_j}, \widehat{\Omega}^{\sum_{j=1}^{n} w_j}\right)\right], \\ \left[\left(\check{\xi}^{\sum_{j=1}^{n} w_j}, \breve{\rho}^{\sum_{j=1}^{n} w_j}, \check{\varsigma}^{\sum_{j=1}^{n} w_j}, \breve{\sigma}^{\sum_{j=1}^{n} w_j}, \check{\tau}^{\sum_{j=1}^{n} w_j}\right), \left(\hat{\xi}^{\sum_{j=1}^{n} w_j}, \hat{\rho}^{\sum_{j=1}^{n} w_j}, \hat{\varsigma}^{\sum_{j=1}^{n} w_j}, \hat{\sigma}^{\sum_{j=1}^{n} w_j}, \hat{\tau}^{\sum_{j=1}^{n} w_j}\right)\right] \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \langle [(\check{\zeta}, \check{\eta}, \check{\theta}, \check{\lambda}, \check{\mu}), (\hat{\zeta}, \hat{\eta}, \hat{\theta}, \hat{\lambda}, \hat{\mu})] : \mathcal{T}_{\dot{n}} \rangle \\ \langle [(\breve{ß}, \check{ð}, \breve{\Psi}, \breve{ɸ}, \breve{\Omega}), (\widehat{ß}, \hat{ð}, \widehat{\Psi}, \widehat{ɸ}, \widehat{\Omega}) : \mathcal{I}_{\dot{n}}] \rangle, \\ \langle [(\check{\xi}, \breve{\rho}, \check{\varsigma}, \breve{\sigma}, \check{\tau}), (\hat{\xi}, \hat{\rho}, \hat{\varsigma}, \hat{\sigma}, \hat{\tau})] : \mathcal{F}_{\dot{n}} \rangle \end{array} \right\} = \dot{n}$$

Thus, the theorem is proved.

Property 2. (Boundedness)

Let $\dot{n}_j = \left\{ \begin{array}{c} \langle [(\check{\zeta}_j, \check{\eta}_j, \check{\theta}_j, \check{\lambda}_j, \check{\mu}_j), (\hat{\zeta}_j, \hat{\eta}_j, \hat{\theta}_j, \hat{\lambda}_j, \hat{\mu}_j)] : \mathcal{T}_{\dot{n}_j} \rangle \\ \langle [(\breve{ß}_j, \check{ð}_j, \breve{\Psi}_j, \breve{ɸ}_j, \breve{\Omega}_j), (\widehat{ß}_j, \hat{ð}_j, \widehat{\Psi}_j, \widehat{ɸ}_j, \widehat{\Omega}_j)] : \mathcal{I}_{\dot{n}_j} \rangle \\ \langle [(\check{\xi}_j, \breve{\rho}_j, \check{\varsigma}_j, \breve{\sigma}_j, \check{\tau}_j), (\hat{\xi}_j, \hat{\rho}_j, \hat{\varsigma}_j, \hat{\sigma}_j, \hat{\tau}_j)] : \mathcal{F}_{\dot{n}_j} \rangle \end{array} \right\}$ $(j = 1,2, \dots, n)$ be a collection of

IVPFNVs in the set of real numbers.

Consider

$\dot{n}^{+} =$

$$\begin{Bmatrix} \left[\left(max_j(\check{\zeta}_j), max_j(\check{\eta}_j), max_j(\check{\theta}_j), max_j(\check{\lambda}_j), max_j(\check{\mu}_j)\right), \left(max_j(\hat{\zeta}_j), max_j(\hat{\eta}_j), max_j(\hat{\theta}_j), max_j(\hat{\lambda}_j), max_j(\hat{\mu}_j)\right)\right] \\ \left[\left(min_j(\breve{ß}_j), min_j(\check{ð}_j), min_j(\breve{\Psi}_j), min_j(\check{ɸ}_j), min_j(\breve{\Omega}_j)\right), \left(min_j(\widehat{ß}_j), min_j(\hat{ð}_j), min_j(\widehat{\Psi}_j), min_j(\widehat{ɸ}_j), min_j(\widehat{\Omega}_j)\right)\right] \\ \left[\left(min_j(\check{\xi}_j), min_j(\breve{\rho}_j), min_j(\check{\varsigma}_j), min_j(\breve{\sigma}_j), min_j(\check{\tau}_j)\right), \left(min_j(\hat{\xi}_j), min_j(\hat{\rho}_j), min_j(\hat{\varsigma}_j), min_j(\hat{\sigma}_j), min_j(\hat{\tau}_j)\right)\right] \end{Bmatrix}$$

$\dot{n}^{-}$

$$= \begin{Bmatrix} \left[\left(\min_j(\check{\zeta}_j), \min_j(\check{\eta}_j), \min_j(\check{\theta}_j), \min_j(\check{\lambda}_j), \min_j(\check{\mu}_j)\right), \left(\min_j(\hat{\zeta}_j), \min_j(\hat{\eta}_j), \min_j(\hat{\theta}_j), \min_j(\hat{\lambda}_j), \min_j(\hat{\mu}_j)\right)\right] \\ \left[\left(\max_j(\breve{ß}_j), \max_j(\check{ð}_j), \max_j(\breve{\Psi}_j), \max_j(\check{ɸ}_j), \max_j(\breve{\Omega}_j)\right), \left(\max_j(\widehat{ß}_j), \max_j(\hat{ð}_j), \max_j(\widehat{\Psi}_j), \max_j(ɸ), \max_j(\widehat{\Omega}_j)\right)\right] \\ \left[\left(\max_j(\check{\xi}_j), \max_j(\breve{\rho}_j), \max_j(\check{\varsigma}_j), \max_j(\breve{\sigma}_j), \max_j(\check{\tau}_j)\right), \left(\max_j(\hat{\xi}_j), \max_j(\hat{\rho}_j), \max_j(\hat{\varsigma}_j), \max_j(\hat{\sigma}_j), \max_j(\hat{\tau}_j)\right)\right] \end{Bmatrix}$$

Where $j = 1,2, \dots, n$. Then $\dot{n}^{-} \leq IVPFNWAA(\dot{n}_1, \dot{n}_2, \dots, \dot{n}_n) \leq \dot{n}^{+}$.

Proof :

We infer that $min_j(\check{\mu}_j) \leq \check{\mu}_j \leq max_j(\check{\mu}_j)$, $min_j(\breve{\Omega}_j) \leq \breve{\Omega}_j \leq max_j(\breve{\Omega}_j)$, $min_j(\check{\tau}_j) \leq \check{\tau}_j \leq max_j(\check{\tau}_j)$,$min_j(\hat{\mu}_j) \leq \hat{\mu}_j \leq max_j(\hat{\mu}_j)$, $min_j(\widehat{\Omega}_j) \leq \widehat{\Omega}_j \leq max_j(\widehat{\Omega}_j)$, $min_j(\hat{\tau}_j) \leq \hat{\tau}_j \leq max_j(\hat{\tau}_j)$, for $j = 1,2, \dots, n$.

(3)

Then $1 - \prod_{j=1}^{n}\left(1 - min_j(\check{\mu}_j)\right)^{w_j} \leq 1 - \prod_{j=1}^{n}\left(1 - \check{\mu}_j\right)^{w_j} \leq 1 - \prod_{j=1}^{n}\left(1 - max_j(\check{\mu}_j)\right)^{w_j}$

$$1 - \left(1 - \min_j(\check{\mu}_j)\right)^{\sum_{j=1}^{n} w_j} \leq 1 - \prod_{j=1}^{n}\left(1 - \check{\mu}_j\right)^{w_j} \leq 1 - \left(1 - \max_j(\check{\mu}_j)\right)^{\sum_{j=1}^{n} w_j}$$

$$\min_j(\check{\mu}_j) \leq 1 - \prod_{j=1}^{n}\left(1 - \check{\mu}_j\right)^{w_j} \leq \max_j(\check{\mu}_j)$$

By eq.(3), for $j = 1,2, \dots, n$.

$$\prod_{j=1}^{n}\left(\min_j(\breve{\Omega}_j)\right)^{w_j} \leq \prod_{j=1}^{n}(\breve{\Omega}_j)^{w_j} \leq \prod_{j=1}^{n}\left(\max_j(\breve{\Omega}_j)\right)^{w_j}$$

$$\left(\min_j(\breve{\Omega}_j)\right)^{\sum_{j=1}^{n} w_j} \leq \prod_{j=1}^{n}(\breve{\Omega}_j)^{w_j} \leq \left(\max_j(\breve{\Omega}_j)\right)^{\sum_{j=1}^{n} w_j}$$

$min_j\left(\breve{\Omega}_j\right) \leq \prod_{j=1}^{n}\left(\breve{\Omega}_j\right)^{w_j} \leq max_j\left(\breve{\Omega}_j\right)$ and

$$\prod_{j=1}^{n}\left(\min_j\left(\check{\tau}_j\right)\right)^{w_j} \leq \prod_{j=1}^{n}\left(\check{\tau}_j\right)^{w_j} \leq \prod_{j=1}^{n}\left(\max_j\left(\check{\tau}_j\right)\right)^{w_j}$$

$$\left(\min_j\left(\check{\tau}_j\right)\right)^{\sum_{j=1}^{n} w_j} \leq \prod_{j=1}^{n}\left(\check{\tau}_j\right)^{w_j} \leq \left(\max_j\left(\check{\tau}_j\right)\right)^{\sum_{j=1}^{n} w_j}$$

$min_j\left(\check{\tau}_j\right) \leq \prod_{j=1}^{n}\left(\check{\tau}_j\right)^{w_j} \leq max_j\left(\check{\tau}_j\right)$.

In the same way,

$$1-\prod_{j=1}^{n}\left(1-\min_j\left(\hat{\mu}_j\right)\right)^{w_j} \leq 1-\prod_{j=1}^{n}\left(1-\hat{\mu}_j\right)^{w_j} \leq 1-\prod_{j=1}^{n}\left(1-\max_j\left(\hat{\mu}_j\right)\right)^{w_j}$$

$$1-\left(1-\min_j\left(\hat{\mu}_j\right)\right)^{\sum_{j=1}^{n} w_j} \leq 1-\prod_{j=1}^{n}\left(1-\hat{\mu}_j\right)^{w_j} \leq 1-\left(1-\max_j\left(\hat{\mu}_j\right)\right)^{\sum_{j=1}^{n} w_j}$$

$$\min_j\left(\hat{\mu}_j\right) \leq 1-\prod_{j=1}^{n}\left(1-\hat{\mu}_j\right)^{w_j} \leq \max_j\left(\hat{\mu}_j\right)$$

By eq.(3), for $j = 1,2, \dots, n$.

$$\prod_{j=1}^{n}\left(\min_j\left(\widehat{\Omega}_j\right)\right)^{w_j} \leq \prod_{j=1}^{n}\left(\widehat{\Omega}_j\right)^{w_j} \leq \prod_{j=1}^{n}\left(\max_j\left(\widehat{\Omega}_j\right)\right)^{w_j}$$

$$\left(\min_j\left(\widehat{\Omega}_j\right)\right)^{\sum_{j=1}^{n} w_j} \leq \prod_{j=1}^{n}\left(\widehat{\Omega}_j\right)^{w_j} \leq \left(\max_j\left(\widehat{\Omega}_j\right)\right)^{\sum_{j=1}^{n} w_j}$$

$min_j\left(\widehat{\Omega}_j\right) \leq \prod_{j=1}^{n}\left(\widehat{\Omega}_j\right)^{w_j} \leq max_j\left(\widehat{\Omega}_j\right)$ and

$$\prod_{j=1}^{n}\left(\min_j\left(\hat{\tau}_j\right)\right)^{w_j} \leq \prod_{j=1}^{n}\left(\hat{\tau}_j\right)^{w_j} \leq \prod_{j=1}^{n}\left(\max_j\left(\hat{\tau}_j\right)\right)^{w_j}$$

$$\left(\min_j\left(\hat{\tau}_j\right)\right)^{\sum_{j=1}^{n} w_j} \leq \prod_{j=1}^{n}\left(\hat{\tau}_j\right)^{w_j} \leq \left(\max_j\left(\hat{\tau}_j\right)\right)^{\sum_{j=1}^{n} w_j}$$

$min_j(\hat{\tau}_j) \leq \prod_{j=1}^{n}(\hat{\tau}_j)^{w_j} \leq max_j(\hat{\tau}_j)$.

Similarly,

$min_j(\check{\zeta}_j) \leq 1-\prod_{j=1}^{n}(1-\check{\zeta}_j)^{w_j} \leq max_j(\check{\zeta}_j)$, $min_j(\check{\eta}_j) \leq 1-\prod_{j=1}^{n}(1-\check{\eta}_j)^{w_j} \leq max_j(\check{\eta}_j)$,

$min_j(\check{\theta}_j) \leq 1-\prod_{j=1}^{n}(1-\check{\theta}_j)^{w_j} \leq max_j(\check{\theta}_j)$,$min_j(\check{\lambda}_j) \leq 1-\prod_{j=1}^{n}(1-\check{\lambda}_j)^{w_j} \leq max_j(\check{\lambda}_j)$;

$min_j(\check{ß}_j) \leq 1-\prod_{j=1}^{n}(1-\check{ß}_j)^{w_j} \leq max_j(\check{ß}_j)$, $min_j(\check{ð}_j) \leq 1-\prod_{j=1}^{n}(1-\check{ð}_j)^{w_j} \leq max_j(\check{ð}_j)$,

$min_j(\check{\Psi}_j) \leq 1-\prod_{j=1}^{n}(1-\check{\Psi}_j)^{w_j} \leq max_j(\check{\Psi}_j)$,$min_j(\check{\phi}_j) \leq 1-\prod_{j=1}^{n}(1-\check{\phi}_j)^{w_j} \leq max_j(\check{\phi}_j)$;

$min_j(\check{\xi}_j) \leq 1-\prod_{j=1}^{n}(1-\check{\xi}_j)^{w_j} \leq max_j(\check{\xi}_j)$, $min_j(\check{\rho}_j) \leq 1-\prod_{j=1}^{n}(1-\check{\rho}_j)^{w_j} \leq max_j(\check{\rho}_j)$,

$min_j(\check{\varsigma}_j) \leq 1-\prod_{j=1}^{n}(1-\check{\varsigma}_j)^{w_j} \leq max_j(\check{\varsigma}_j)$,$min_j(\check{\sigma}_j) \leq 1-\prod_{j=1}^{n}(1-\check{\sigma}_j)^{w_j} \leq max_j(\check{\sigma}_j)$.

And also,

$min_j(\hat{\zeta}_j) \leq 1-\prod_{j=1}^{n}(1-\hat{\zeta}_j)^{w_j} \leq max_j(\hat{\zeta}_j)$, $min_j(\hat{\eta}_j) \leq 1-\prod_{j=1}^{n}(1-\hat{\eta}_j)^{w_j} \leq max_j(\hat{\eta}_j)$,

$min_j(\hat{\theta}_j) \leq 1-\prod_{j=1}^{n}(1-\hat{\theta}_j)^{w_j} \leq max_j(\hat{\theta}_j)$,$min_j(\hat{\lambda}_j) \leq 1-\prod_{j=1}^{n}(1-\hat{\lambda}_j)^{w_j} \leq max_j(\hat{\lambda}_j)$;

$min_j(\widehat{ß}_j) \leq 1-\prod_{j=1}^{n}(1-\widehat{ß}_j)^{w_j} \leq max_j(\widehat{ß}_j)$, $min_j(\hat{ð}_j) \leq 1-\prod_{j=1}^{n}(1-\hat{ð}_j)^{w_j} \leq max_j(\hat{ð}_j)$,

$min_j(\widehat{\Psi}_j) \leq 1-\prod_{j=1}^{n}(1-\widehat{\Psi}_j)^{w_j} \leq max_j(\widehat{\Psi}_j)$,$min_j(\widehat{\phi}_j) \leq 1-\prod_{j=1}^{n}(1-\widehat{\phi}_j)^{w_j} \leq max_j(\widehat{\phi}_j)$;

$min_j(\hat{\xi}_j) \leq 1-\prod_{j=1}^{n}(1-\hat{\xi}_j)^{w_j} \leq max_j(\hat{\xi}_j)$, $min_j(\hat{\rho}_j) \leq 1-\prod_{j=1}^{n}(1-\hat{\rho}_j)^{w_j} \leq max_j(\hat{\rho}_j)$,

$min_j(\hat{\varsigma}_j) \leq 1-\prod_{j=1}^{n}(1-\hat{\varsigma}_j)^{w_j} \leq max_j(\hat{\varsigma}_j)$,$min_j(\hat{\sigma}_j) \leq 1-\prod_{j=1}^{n}(1-\hat{\sigma}_j)^{w_j} \leq max_j(\hat{\sigma}_j)$.

for $j = 1,2,\ldots,n$.

Let $IVPFNWAA_w(\dot{n}_1, \dot{n}_2, \ldots, \dot{n}_n) \leq \dot{n} = \left\{ \begin{array}{c} \langle [(\check{\zeta}, \check{\eta}, \check{\theta}, \check{\lambda}, \check{\mu}), (\hat{\zeta}, \hat{\eta}, \hat{\theta}, \hat{\lambda}, \hat{\mu})] : \mathcal{T}_{\dot{n}} \rangle, \\ \langle [(\check{ß}, \check{ð}, \check{\Psi}, \check{\phi}, \check{\Omega}), (\widehat{ß}, \hat{ð}, \widehat{\Psi}, \widehat{\phi}, \widehat{\Omega}) : \mathcal{I}_{\dot{n}}] \rangle, \\ \langle [(\check{\xi}, \check{\rho}, \check{\varsigma}, \check{\sigma}, \check{\tau}), (\hat{\xi}, \hat{\rho}, \hat{\varsigma}, \hat{\sigma}, \hat{\tau})] : \mathcal{F}_{\dot{n}} \rangle \end{array} \right\}$

The score function of $\dot{n}$ is

$$\text{Ş}(\dot{n})=\frac{1}{6}\Bigg[4+\frac{\check{\zeta}+\check{\eta}+\breve{\theta}+\check{\lambda}+\check{\mu}}{5}+\frac{\hat{\zeta}+\hat{\eta}\ +\hat{\theta}+\hat{\lambda}+\hat{\mu}}{5}-\frac{\breve{\text{ß}}+\breve{l}+\breve{\Psi}+\breve{\phi}+\breve{\Omega}}{5}$$
$$-\frac{\widehat{\text{ß}}+\hat{l}\ +\widehat{\Psi}+\widehat{\phi}+\widehat{\Omega}}{5}-\frac{\check{\xi}+\breve{\rho}+\check{\varsigma}+\breve{\sigma}+\check{\tau}}{5}-\frac{\hat{\xi}+\hat{\rho}\ +\hat{\varsigma}+\hat{\sigma}+\hat{\tau}}{5}\Bigg]$$

$$\leq\frac{1}{6}\left[\begin{array}{l}4+\dfrac{\max\limits_j(\check{\zeta}_j)+\max\limits_j(\check{\eta}_j)+\max\limits_j(\breve{\theta}_j)+\max\limits_j(\check{\lambda}_j)+\max\limits_j(\check{\mu}_j)}{5}\\+\dfrac{\max\limits_j(\hat{\zeta}_j)+\max\limits_j(\hat{\eta}_j)\ +\max\limits_j(\hat{\theta}_j)+\max\limits_j(\hat{\lambda}_j)+\max\limits_j(\hat{\mu}_j)}{5}\\-\dfrac{\min\limits_j(\breve{\text{ß}}_j)+\min\limits_j(\breve{\eth}_j)+\min\limits_j(\breve{\Psi}_j)+\min\limits_j(\breve{\phi}_j)+\min\limits_j(\breve{\Omega}_j)}{5}\\-\dfrac{\min\limits_j(\widehat{\text{ß}}_j)+\min\limits_j(\hat{\eth}_j)\ +\min\limits_j(\widehat{\Psi}_j)+\min\limits_j(\widehat{\phi}_j)+\min\limits_j(\widehat{\Omega}_j)}{5}\\-\dfrac{\min\limits_j(\check{\xi}_j)+\min\limits_j(\breve{\rho}_j)+\min\limits_j(\check{\varsigma}_j)+\min\limits_j(\breve{\sigma}_j)+\min\limits_j(\check{\tau}_j)}{5}\\-\dfrac{\min\limits_j(\hat{\xi}_j)+\min\limits_j(\hat{\rho}_j)\ +\min\limits_j(\hat{\varsigma}_j)+\min\limits_j(\hat{\sigma}_j)+\min\limits_j(\hat{\tau}_j)}{5}\end{array}\right]=\text{Ş}(\dot{n}^{+})$$

In the same way,

$$\text{Ş}(\dot{n})=\frac{1}{6}\Bigg[4+\frac{\check{\zeta}+\check{\eta}+\breve{\theta}+\check{\lambda}+\check{\mu}}{5}+\frac{\hat{\zeta}+\hat{\eta}\ +\hat{\theta}+\hat{\lambda}+\hat{\mu}}{5}-\frac{\breve{\text{ß}}+\breve{\eth}+\breve{\Psi}+\breve{\phi}+\breve{\Omega}}{5}$$
$$-\frac{\widehat{\text{ß}}+\hat{\eth}\ +\widehat{\Psi}+\widehat{\phi}+\widehat{\Omega}}{5}-\frac{\check{\xi}+\breve{\rho}+\check{\varsigma}+\breve{\sigma}+\check{\tau}}{5}-\frac{\hat{\xi}+\hat{\rho}\ +\hat{\varsigma}+\hat{\sigma}+\hat{\tau}}{5}\Bigg]$$

$$\geq\frac{1}{6}\left[\begin{array}{l}4+\dfrac{\min\limits_j(\check{\zeta}_j)+\min\limits_j(\check{\eta}_j)+\min\limits_j(\breve{\theta}_j)+\min\limits_j(\check{\lambda}_j)+\min\limits_j(\check{\mu}_j)}{5}\\+\dfrac{\min\limits_j(\hat{\zeta}_j)+\min\limits_j(\hat{\eta}_j)\ +\min\limits_j(\hat{\theta}_j)+\min\limits_j(\hat{\lambda}_j)+\min\limits_j(\hat{\mu}_j)}{5}\\-\dfrac{\max\limits_j(\breve{\text{ß}}_j)+\max\limits_j(\breve{\eth}_j)+\max\limits_j(\breve{\Psi}_j)+\max\limits_j(\breve{\phi}_j)+\max\limits_j(\breve{\Omega}_j)}{5}\\-\dfrac{\max\limits_j(\widehat{\text{ß}}_j)+\max\limits_j(\hat{\eth}_j)\ +\max\limits_j(\widehat{\Psi}_j)+\max\limits_j(\widehat{\phi}_j)+\max\limits_j(\widehat{\Omega}_j)}{5}\\-\dfrac{\max\limits_j(\check{\xi}_j)+\max\limits_j(\breve{\rho}_j)+\max\limits_j(\check{\varsigma}_j)+\max\limits_j(\breve{\sigma}_j)+\max\limits_j(\check{\tau}_j)}{5}\\-\dfrac{\max\limits_j(\hat{\xi}_j)+\max\limits_j(\hat{\rho}_j)\ +\max\limits_j(\hat{\varsigma}_j)+\max\limits_j(\hat{\sigma}_j)+\max\limits_j(\hat{\tau}_j)}{5}\end{array}\right]=\text{Ş}(\dot{n}^{-})$$

Here we discuss the different cases:

Case (i) If $Ş(\dot{n}) < Ş(\dot{n}^{+})$ and $Ş(\dot{n}) > Ş(\dot{n}^{-})$ then, $\dot{n}^{-} < IVPFNWAA(\dot{n}_1, \dot{n}_2, \ldots, \dot{n}_n) < \dot{n}^{+}$.

Case (ii) If $Ş(\dot{n}) = Ş(\dot{n}^{+})$, we consider

$$Ş(\dot{n}) = \frac{1}{6}\left[4 + \frac{\check{\zeta} + \check{\eta} + \check{\theta} + \check{\lambda} + \check{\mu}}{5} + \frac{\hat{\zeta} + \hat{\eta} + \hat{\theta} + \hat{\lambda} + \hat{\mu}}{5} - \frac{\check{ß} + \check{ð} + \check{\Psi} + \check{ф} + \check{\Omega}}{5} - \frac{\widehat{ß} + \hat{ð} + \widehat{\Psi} + \widehat{ф} + \widehat{\Omega}}{5} - \frac{\check{\xi} + \check{\rho} + \check{\varsigma} + \check{\sigma} + \check{\tau}}{5} - \frac{\hat{\xi} + \hat{\rho} + \hat{\varsigma} + \hat{\sigma} + \hat{\tau}}{5}\right]$$

$$= \frac{1}{6}\left[\begin{array}{l} 4 + \frac{\max_j(\check{\zeta}_j) + \max_j(\check{\eta}_j) + \max_j(\check{\theta}_j) + \max_j(\check{\lambda}_j) + \max_j(\check{\mu}_j)}{5} \\ + \frac{\max_j(\hat{\zeta}_j) + \max_j(\hat{\eta}_j) + \max_j(\hat{\theta}_j) + \max_j(\hat{\lambda}_j) + \max_j(\hat{\mu}_j)}{5} \\ - \frac{\min_j(\check{ß}_j) + \min_j(\check{ð}_j) + \min_j(\check{\Psi}_j) + \min_j(\check{ф}_j) + \min_j(\check{\Omega}_j)}{5} \\ - \frac{\min_j(\widehat{ß}_j) + \min_j(\hat{ð}_j) + \min_j(\widehat{\Psi}_j) + \min_j(\widehat{ф}_j) + \min_j(\widehat{\Omega}_j)}{5} \\ - \frac{\min_j(\check{\xi}_j) + \min_j(\check{\rho}_j) + \min_j(\check{\varsigma}_j) + \min_j(\check{\sigma}_j) + \min_j(\check{\tau}_j)}{5} \\ - \frac{\min_j(\hat{\xi}_j) + \min_j(\hat{\rho}_j) + \min_j(\hat{\varsigma}_j) + \min_j(\hat{\sigma}_j) + \min_j(\hat{\tau}_j)}{5} \end{array}\right]$$

Then,it also follows

$\frac{\check{\zeta}+\check{\eta}+\check{\theta}+\check{\lambda}+\check{\mu}}{5} = \frac{\max_j(\check{\zeta}_j)+\max_j(\check{\eta}_j)+\max_j(\check{\theta}_j)+\max_j(\check{\lambda}_j)+\max_j(\check{\mu}_j)}{5}$,

$\frac{\check{ß}+\check{ð}+\check{\Psi}+\check{ф}+\check{\Omega}}{5} = \frac{\min_j(\check{ß}_j)+\min_j(\check{ð}_j)+\min_j(\check{\Psi}_j)+\min_j(\check{ф}_j)+\min_j(\check{\Omega}_j)}{5}$,

$\frac{\widehat{ß}+\hat{ð}+\widehat{\Psi}+\widehat{ф}+\widehat{\Omega}}{5} = \frac{\min_j(\widehat{ß}_j)+\min_j(\hat{ð}_j)+\min_j(\widehat{\Psi}_j)+\min_j(\widehat{ф}_j)+\min_j(\widehat{\Omega}_j)}{5}$,

$\frac{\check{\xi}+\check{\rho}+\check{\varsigma}+\check{\sigma}+\check{\tau}}{5} = \frac{\min_j(\check{\xi}_j)+\min_j(\check{\rho}_j)+\min_j(\check{\varsigma}_j)+\min_j(\check{\sigma}_j)+\min_j(\check{\tau}_j)}{5}$, and

$\frac{\hat{\xi}+\hat{\rho}+\hat{\varsigma}+\hat{\sigma}+\hat{\tau}}{5} = \frac{\min_j(\hat{\xi}_j)+\min_j(\hat{\rho}_j)+\min_j(\hat{\varsigma}_j)+\min_j(\hat{\sigma}_j)+\min_j(\hat{\tau}_j)}{5}$.

The accuracy function

$$A(\dot{n}) = \frac{1}{2}\left[\frac{\check{\zeta} + \check{\eta} + \check{\theta} + \check{\lambda} + \check{\mu}}{5} + \frac{\hat{\zeta} + \hat{\eta} + \hat{\theta} + \hat{\lambda} + \hat{\mu}}{5} - \frac{\check{\xi} + \check{\rho} + \check{\varsigma} + \check{\sigma} + \check{\tau}}{5} - \frac{\hat{\xi} + \hat{\rho} + \hat{\varsigma} + \hat{\sigma} + \hat{\tau}}{5}\right]$$

$$= \frac{1}{2}\begin{bmatrix} \frac{max_j(\check{\zeta}_j)+max_j(\check{\eta}_j)+max_j(\check{\theta}_j)+max_j(\check{\lambda}_j)+max_j(\check{\mu}_j)}{5} \\ +\frac{max_j(\hat{\zeta}_j)+max_j(\hat{\eta}_j)+max_j(\hat{\theta}_j)+max_j(\hat{\lambda}_j)+max_j(\hat{\mu}_j)}{5} \\ -\frac{min_j(\check{\xi}_j)+min_j(\check{\rho}_j)+min_j(\check{\varsigma}_j)+min_j(\check{\sigma}_j)+min_j(\check{\tau}_j)}{5} \\ -\frac{min_j(\hat{\xi}_j)+min_j(\hat{\rho}_j)+min_j(\hat{\varsigma}_j)+min_j(\hat{\sigma}_j)+min_j(\hat{\tau}_j)}{5} \end{bmatrix} = A(\dot{n}^{+}) \quad (4)$$

which implies $IVPFNWAA(\dot{n}_1, \dot{n}_2, \ldots, \dot{n}_n) \le \dot{n}^{+}$.

In the same way,

$$A(\dot{n}) = \frac{1}{2}\left[\frac{\check{\zeta}+\check{\eta}+\check{\theta}+\check{\lambda}+\check{\mu}}{5} + \frac{\hat{\zeta}+\hat{\eta}+\hat{\theta}+\hat{\lambda}+\hat{\mu}}{5} - \frac{\check{\xi}+\check{\rho}+\check{\varsigma}+\check{\sigma}+\check{\tau}}{5} - \frac{\hat{\xi}+\hat{\rho}+\hat{\varsigma}+\hat{\sigma}+\hat{\tau}}{5}\right]$$

$$= \frac{1}{2}\begin{bmatrix} \frac{min_j(\check{\zeta}_j)+min_j(\check{\eta}_j)+min_j(\check{\theta}_j)+min_j(\check{\lambda}_j)+min_j(\check{\mu}_j)}{5} \\ +\frac{min_j(\hat{\zeta}_j)+min_j(\hat{\eta}_j)+min_j(\hat{\theta}_j)+min_j(\hat{\lambda}_j)+min_j(\hat{\mu}_j)}{5} \\ -\frac{max_j(\check{\xi}_j)+max_j(\check{\rho}_j)+max_j(\check{\varsigma}_j)+max_j(\check{\sigma}_j)+max_j(\check{\tau}_j)}{5} \\ -\frac{max_j(\hat{\xi}_j)+max_j(\hat{\rho}_j)+max_j(\hat{\varsigma}_j)+max_j(\hat{\sigma}_j)+max_j(\hat{\tau}_j)}{5} \end{bmatrix} = A(\dot{n}^{-}) \quad (5)$$

which implies $IVPFNNWAA(\dot{n}_1, \dot{n}_2, \ldots, \dot{n}_n) \le \dot{n}^{-}$.From eq. (4) and (5),
we infer that $\dot{n}^{-} \le IVPFNWAA(\dot{n}_1, \dot{n}_2, \ldots, \dot{n}_n) \le \dot{n}^{+}$.Hence the proof is verified.

5. Multi Attribute decision making using IVPFNWAA Operator.

To resolve MADM technique with pentagonal numbers under interval -valued neutrosophic environment which is represented in the form of IVPFNNs with m alternatives $L = \{L_1, L_2, \ldots, L_m\}$ and attributes are given by $C = \{C_1, C_2, \ldots, C_n\}$ and their weights be $\mathcal{W} = \{w_1, w_2, \ldots w_n\}^T$ with $w_j \ge 0$ and $\sum_{j=1}^{n} w_j = 1$for $j = 1,2, \ldots, n$. The decision matrix is given by

$$\mathfrak{D} = (\dot{d}_{ij})_{m\times n} = \begin{bmatrix} [(\check{\zeta}_{ij}, \check{\eta}_{ij}, \check{\theta}_{ij}, \check{\lambda}_{ij}, \check{\mu}_{ij}), (\hat{\zeta}_{ij}, \hat{\eta}_{ij}, \hat{\theta}_{ij}, \hat{\lambda}_{ij}, \hat{\mu}_{ij})] \\ [(\breve{ß}_{ij}, \check{ð}_{ij}, \breve{\Psi}_{ij}, \check{ɸ}_{ij}, \breve{Ω}_{ij}), (\hat{ß}_{ij}, \hat{ð}_{ij}, \hat{\Psi}_{ij}, \hat{ɸ}_{ij}, \hat{Ω}_{ij})] \\ [(\check{\xi}_{ij}, \check{\rho}_{ij}, \check{\varsigma}_{ij}, \check{\sigma}_{ij}, \check{\tau}_{ij}), (\hat{\xi}_{ij}, \hat{\rho}_{ij}, \hat{\varsigma}_{ij}, \hat{\sigma}_{ij}, \hat{\tau}_{ij})] \end{bmatrix}_{m\times n}$$

where $(\check{\zeta}_{ij}, \check{\eta}_{ij}, \check{\theta}_{ij}, \check{\lambda}_{ij}, \check{\mu}_{ij}), (\hat{\zeta}_{ij}, \hat{\eta}_{ij}, \hat{\theta}_{ij}, \hat{\lambda}_{ij}, \hat{\mu}_{ij}) \subset [0,1]$ represents lower and upper level of truthness.$(\check{ß}_{ij}, \check{ð}_{ij}, \check{\Psi}_{ij}, \check{Ф}_{ij}, \check{\Omega}_{ij}), (\hat{ß}_{ij}, \hat{ð}_{ij}, \hat{\Psi}_{ij}, \hat{Ф}_{ij}, \hat{\Omega}_{ij}) \subset [0,1]$ represents lower and upper level of uncertainty.$(\check{\xi}_{ij}, \check{\rho}_{ij}, \check{\varsigma}_{ij}, \check{\sigma}_{ij}, \check{\tau}_{ij}), (\hat{\xi}_{ij}, \hat{\rho}_{ij}, \hat{\varsigma}_{ij}, \hat{\sigma}_{ij}, \hat{\tau}_{ij}) \subset [0,1]$ represents lower and upper level of falseness. With the conditions $0 \leq \check{\mu}_{ij} + \check{\Omega}_{ij} + \check{\tau}_{ij} \leq 3$ and $0 \leq \hat{\mu}_{ij} + \hat{\Omega}_{ij} + \hat{\tau}_{ij} \leq 3$ for $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$.

Step 1: To find aggregate of the attributes
Step 2: By using INPFNWAA operator, find the aggregate value corresponding to each alternative.
Step 3: For the aggregated values, obtain the score value for each alternative.
Step 4: Valuate the ranking order of each alternate
Step 5: Pertain the best choice in accordance to the ranking order.

5.1 Illustrative example

We consider four women $L_i (i = 1,2,3,4)$ of age group 25-30 years goes for a shopping to select a saree. Their choice of selecting a saree includes colour (S_1), fabric (S_2), cost (S_3), design (S_4), texture (S_5). This has been represented by the decision matrix in terms of IVPFNNs with the weight of the attributes $\mathcal{W} = (0.11,0.26,0.26,0.14,0.23)^T$.

Table 1: Rating values in terms of IVPFNNs

	C1
	0,0.45,0.5,0.55,0.6), (0.65,0.7,0.75,0.8,0.85)], 3,0.3,0.34,0.37,0.38),(0.43,0.5,0.55,0.59,0.6)], 1,0.23,0.32,0.43,0.56), (0.6,0.63,0.7,0.72,0.25)]
	0,0.22,0.24,0.26,0.28), (0.42,0.44,0.46,0.48,0.50)], 5,0.49,0.51,0.53,0.56),(0.72,0.74,0.76,0.78,0.80)], 6,0.39,0.42,0.45,0.48), (0.51,0.53,0.55,0.59,0.62)]
	7,0.29,0.32,0.37,0.41), (0.61,0.64,0.67,0.70,0.72)], [(0.5,0.53,0.55,0.57,0.59), ,0.74,0.77,0.79,0.81)], 1,0.33,0.35,0.37,0.39), (0.52,0.55,0.57,0.62,0.64)]
	1,0.63,0.65,0.67,0.69), (0.82,0.84,0.86,0.88,0.9)], [(0.41,0.43,0.45,0.47,0.49), ,0.64,0.66,0.68,0.7)], 6,0.08,0.11,0.15,0.17), (0.30,0.32,0.34,0.36,0.38)]

C2	
	8,0.10,0.12,0.14,0.16), (0.42,0.44,0.46,0.48,0.50)], [(0.20,0.22,0.24,0.26,0.28), 2,0.64,0.66,0.68,0.70)], 2,0.44,0.46,0.48,0.50), (0.72,0.74,0.76,0.78,0.80)]
	0,0.72,0.74,0.76,0.78), (0.80,0.82,0.84,0.86,0.88)], 1,0.34,0.40,0.42,0.46),(0.62,0.64,0.69,0.71,0.74)], 2,0.55,0.57,0.59,0.61), (0.72,0.74,0.76,0.78,0.80)]
	2,0.54,0.56,0.58,0.60), (0.71,0.74,0.77,0.79,0.81)], 4,0.17,0.20,0.23,0.25),(0.42,0.46,0.48,0.50,0.52)], 1,0.63,0.65,0.67,0.69), (0.74,0.76,0.78,0.80,0.82)]
	2,0.44,0.49,0.52,0.55), (0.61,0.63,0.65,0.67,0.69)], [(0.71,0.73,0.75,0.77,0.79), ,0.83,0.85,0.87,0.89)], 3,0.25,0.27,0.29,0.31), (0.40,0.42,0.44,0.46,0.48)]
C3	
	3,0.65,0.68,0.72,0.75), (0.80,0.83,0.86,0.89,0.92)], [(0.31,0.34,0.37,0.39,0.42), 5,0.49,0.51,0.53,0.56)], 2,0.44,0.46,0.48,0.50), (0.55,0.57,0.59,0.61,0.63)]
	5,0.49,0.51,0.53,0.56), (0.6,0.63,0.7,0.72,0.75)], [(0.30,0.35,0.40,0.45,0.5), 2,0.64,0.66,0.65,0.70)], 1,0.54,0.56,0.58,0.6), (0.71,0.73,0.75,0.77,0.79)]
	1,0.24,0.26,0.29,0.31), (0.52,0.54,0.56,0.58,0.60)], [(0.72,0.74,0.76,0.78,0.80), 4,0.86,0.88,0.90,0.92)], 1,0.54,0.57,0.59,0.61), (0.81,0.83,0.85,0.87,0.91)]
	2,0.74,0.76,0.78,0.8), (0.85,0.87,0.89,0.91,0.93)], [(0.11,0.15,0.20,0.25,0.30), ,0.45,0.5,0.55,0.6)], 1,0.53,0.55,0.57,0.59), (0.65,0.67,0.69,0.71,0.73)]
C4	
	2,0.15,0.18,0.21,0.24), (0.36,0.39,0.42,0.45,0.48)], [(0.31,0.35,0.39,0.43,0.47), ,0.66,0.72,0.75,0.78)], ,0.53,0.55,0.57,0.59), (0.72,0.74,0.76,0.78,0.80)]
	,0.53,0.55,0.57,0.59), (0.72,0.74,0.76,0.78,0.8)], [(0.11,0.13,0.15,0.17,0.19),

	,0.39,0.41,0.43,0.45)], 1,0.73,0.75,0.77,0.79), (0.82,0.84,0.86,0.88,0.9)]
	7,0.49,0.51,0.53,0.55), (0.71,0.73,0.75,0.77,0.79)], [(0.19,0.21,0.23,0.25,0.27), ,0.45,0.45,0.47,0.49)], 2,0.74,0.76,0.78,0.80), (0.91,0.93,0.95,0.97,0.99)]
	3,0.07,0.09,0.14,0.17), (0.23,0.25,0.27,0.29,0.31)], [(0.5,0.53,0.56,0.59,0.62), ,0.73,0.75,0.77,0.79)], 1,0.83,0.85,0.87,0.89), (0.91,0.93,0.95,0.97,0.99)]
C5	
	3,0.25,0.27,0.29,0.31), (0.43,0.45,0.52,0.56,0.6)], [(0.77,0.79,0.82,0.84,0.86), ,0.92,0.94,0.96,0.98)], 2,0.54,0.56,0.58,0.6), (0.72,0.74,0.76,0.78,0.8)]
	3,0.05,0.07,0.09,0.11), (0.31,0.33,0.35,0.37,0.39)], [(0.50,0.53,0.55,0.57,0.6), ,0.67,0.69,0.7,0.72)], 1,0.83,0.85,0.87,0.89), (0.91,0.93,0.95,0.97,0.99)]
	1,0.83,0.85,0.87,0.89), (0.92,0.93,0.94,0.95,0.96)], [(0.03,0.07,0.11,0.15,0.19), ,0.25,0.31,0.33,0.35)], 2,0.54,0.56,0.58,0.6), (0.73,0.75,0.77,0.79,0.81)]
	4,0.56,0.58,0.6,0.62), (0.71,0.73,0.75,0.79,0.81)], [(0.8,0.82,0.84,0.86,0.88), 0.92,0.94,0.96,0.98)], 4,0.16,0.18,0.20,0.22), (0.33,0.35,0.37,0.39,0.41)]

Table 2: Aggregated values of IVPFNNs by using IVPFNWAA operator.

	Aggregated values of IVPFNNs
	4,0.37,0.40,0.43,0.47), (0.60,0.61,0.66,0.70,0.74)], [(0.32,0.37,0.40,0.42,0.45), ,0.64,0.66,0.69,0.71)], 0,0.44,0.47,0.51,0.54), (0.66,0.68,0.71,0.73,0.75)]
	5,0.48,0.50,0.52,0.55), (0.62,0.65,0.69,0.71,0.73)], [(0.31,0.35,0.39,0.41,0.45), ,0.61,0.64,0.66,0.68)], 7,0.60,0.63,0.65,0.67), (0.74,0.76,0.78,0.81,0.83)]
	3,0.56,0.58,0.61,0.64), (0.75,0.77,0.79,0.81,0.83)], [(0.18,0.24,0.28,0.32,0.35),

	,0.49,0.53,0.55,0.57)], 3,0.56,0.58,0.60,0.62), (0.75,0.77,0.79,0.81,0.83)]
	3,0.55,0.58,0.61,0.63), (0.71,0.74,0.76,0.79,0.81)], [(0.40,0.45,0.50,0.54,0.58), ,0.69,0.72,0.76,0.79)], 6,0.29,0.31,0.34,0.37), (0.47,0.49,0.51,0.54,0.56)]

Table 3: Rating values of score and accuracy function.

	ore Value	ccuracy Value
A	0.4723	-0.057
B	0.459	-0.114
C	0.5353	0.003
D	0.5493	0.257

Based upon the score and accuracy functions, it has been inferred that D> C $> A > B$, i.e the woman D chooses the best among the four.

6. Conclusion

In order to deal with imprecise and uncertain data, fuzzy sets have been introduced. As an extension to this, Type 2 fuzzy sets: IVPFNS its operational laws, score and accuracy functions were proposed and also IVPFNWAA operator have been introduced and proved and also some of its properties were proved. MCDM technique has been proved by IVPFNWAA by giving a suitable example for ranking order. Further study will be to use IVPFNWAA in other problem-solving methods, which includes pattern recognition, probabilistic approach, similarity measures etc in order to get more efficient results.

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