Families of Weakly Compatible Mappings Satisfying *(E.A)* and Common Limit Range Properties

Rajesh Kumar[1,2*], Sanjay Kumar[1]

[1]Department of Mathematics

Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonepat-131039, Haryana (India)

[2]Department of Mathematics, Institute of Higher Learning, BPS Mahila Vishwavidyalaya, Khanpur Kalan-131305, Sonipat, Haryana (India)

rkdubaldhania@gmail.com, sanjaymudgal2004@yahoo.com

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Abstract

The purpose of this paper is to prove common fixed point theorems for families of weakly compatible mappings satisfying $(E.A)$ property, common limit range property and a weak contraction condition involving cubic terms of distance function. Our results generalize and extend the results by Kumar *et al.* [11]. Results are supported with relevant example.

Keywords and Phrases: Common fixed point, weak contraction, weakly compatible mappings, $(E.A)$ property, common limit range property.

1. Introduction and Preliminaries

Banach's fixed point theorem [14] states that if P is a contraction mapping of a complete metric space (M, d) into itself then P has a unique fixed point in M. Several authors explored some new type contraction and proved various fixed point theorems in order to generalize the Banach fixed point theorem (see [2],[6],[9],[12],[13]). In 1976, for generalization of Banach's fixed point theorem, Jungck [3] used the notion of commuting maps to prove a common fixed point theorem.

In 1982, Sessa [15] generalized the notion of commutativity to weak commutativity and proved some common fixed point theorems for weakly commuting mappings. Further, in 1996, Jungck [5] extended the notion of compatible mappings to a larger class of mappings known as weakly compatible. Infact weakly compatible mappings relax the condition of continuity of the mappings.

Definition 1.1. [5] Let P and Q be two mappings from a metric space (M, d) into itself. If P and Q commute at their coincidence point, i.e., if $Pt = Qt$ for some $t \in M$ implies $PQt = QPt$, then P and Q are called weakly compatible.

In the general setting, the notion of property $(E.A)$, which requires the closedness of the subspace, was introduced by Aamri and El- Moutawakil [7].

Definition 1.2. [7] A pair of self-mappings P and Q on a metric space (M, d) is said to satisfy property $(E.A)$ if there exists a sequence $\{u_n\} \in M$ such that $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = t$ for some $t \in M$.

Remark 1.3. [7] One can note that weakly compatibility and property $(E.A)$ are independent concepts.

In 2011, Sintunavarat and Kumam [17] introduced common limit range property (CLR_Q) as follows:

Definition 1.4. [17] A pair of self-mappings (P, Q) on a metric space (M, d) is said to satisfy common limit range property with respect to Q, denoted by (CLR_Q), if there exists a sequence $\{u_n\} \in M$ such that $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = t$ for some $t \in Q(M)$.

Thus, one can note that a pair (P, Q) satisfying the property $(E.A)$ along with the closedness of the subspace $Q(M)$ always enjoys the (CLR_Q) property with respect to the mapping Q (see Examples 2.16-2.17 of [8]).

Imdad *et al.* [8] extend this notion of common limit range property for two pairs of mappings.

Definition 1.5. [8] Two pairs (P, S) and (Q, T) of a metric space (M, d) are said to satisfy the common limit range property with respect to the mappings S and T, denoted by (CLR_{ST}), if there exists sequences $\{u_n\}$ and $\{v_n\}$ in M such that

$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Su_n = \lim_{n\to\infty} Qv_n = \lim_{n\to\infty} Tv_n = t$, for some $t \in S(M) \cap T(M)$.

In 2017, a new type of common limit range property is introduced by Popa [16].

Definition 1.6. [16] Let P, S and T be self mappings of a metric space (M, d). The pair (P, S) is said to satisfy a common limit range property with respect to T, denoted by, $(CLR)_{(P,S)T}$, if there exist a sequence $\{u_n\}$ in M such that

$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Su_n = u$, for some $u \in S(M) \cap T(M)$.

In this paper, we prove common fixed point theorems for families of weakly compatible mappings enjoying $(E.A)$ and $(CLR)_{(P,S)T}$ properties.

2. Main Results

In 2021, Kumar *et al.* [11] introduced a new weak contraction condition that involves cubic terms of distance function and proved common fixed point theorems for compatible mappings and its variants.

Theorem 2.1. [11] Let f, g, S and T be four mappings of a complete metric space (M,d) into itself satisfying the following conditions:

$(C1)$ $f(M) \subset T(M), g(M) \subset S(M)$,

$$(C2)\ d^3(fx,gy) \le p\,max\{\frac{1}{2}[d^2(Sx,fx)d(Ty,gy) + d(Sx,fx)d^2(Ty,gy)], d(Sx,fx)d(Sx,gy)d(Ty,fx), d(Sx,gy)d(Ty,fx)d(Ty,gy)\} -\phi\big(m(Sx,Ty)\big),$$

for all $x, y \in M$, where

$$m(Sx,Ty) = \max\{d^2(Sx,Ty), d(Sx,fx)d(Ty,gy), d(Sx,gy)d(Ty,fx), \frac{1}{2}[d(Sx,fx)d(Sx,gy) + d(Ty,fx)d(Ty,gy)]\},$$

where p is a real number satisfying $0 < p < 1$ and $\phi: [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ iff $t = 0$ and $\phi(t) > 0\ for\ each\ t > 0$.

$(C3)$One of $f, g, S\ and\ T$ is continuous.

Assume that the pairs (f,S) and (g,T) are compatible or compatible of type (A) or compatible of type (B) or compatible of type (C) or compatible of type (P), then $f, g, S\ and\ T$ have a unique common fixed point in M.

Now we prove our main results for families of weakly compatible mappings enjoying $(E.A)$ property.

Theorem 2.2. Let $Q_1, Q_2, \dots, Q_{2n}$, P_0 and P_1 be self mappings on a metric space (M,d), satisfying the following conditions:

$$(C4) Q_2(Q_4 \dots Q_{2n}) = (Q_4 \dots Q_{2n})Q_2,\ Q_2Q_4(Q_6 \dots Q_{2n}) = (Q_6 \dots Q_{2n})Q_2Q_4, \dots, Q_2 \dots Q_{2n-2}(Q_{2n}) = (Q_{2n})Q_2 \dots Q_{2n-2};$$

$$P_1(Q_4 \dots Q_{2n}) = (Q_4 \dots Q_{2n})P_1,\ P_1(Q_6 \dots Q_{2n}) = (Q_6 \dots Q_{2n})P_1, \dots, P_1Q_{2n} = Q_{2n}P_1,$$

$$Q_1(Q_3 \dots Q_{2n-1}) = (Q_3 \dots Q_{2n-1})Q_1,\ Q_1Q_3(Q_5 \dots Q_{2n-1}) = (Q_5 \dots Q_{2n-1})Q_1Q_3, \dots,\ Q_1 \dots Q_{2n-3}(Q_{2n-1}) = (Q_{2n-1})Q_1 \dots Q_{2n-3};$$

$$P_0(Q_3 \dots Q_{2n-1}) = (Q_3 \dots Q_{2n-1})P_0,\ P_0(Q_5 \dots Q_{2n-1}) = (Q_5 \dots Q_{2n-1})P_0, \dots, P_0Q_{2n-1} = Q_{2n-1}P_0,$$

$(C5) P_0(M) \subset Q_2 \dots Q_{2n}(M)$ and the pair $(P_0, Q_1 \dots Q_{2n-1})$ satisfies $(E.A)$ property and $Q_1 \dots Q_{2n-1}(M)$ is a closed subset of M.

or

$(C6) P_1(M) \subset Q_1 \dots Q_{2n-1}(M)$ and the pair $(P_1, Q_2 \dots Q_{2n})$ satisfies $(E.A)$ property and $Q_2 \dots Q_{2n}(M)$ is a closed subset of M.

$(C7)$ $d^3(P_0x, P_1y)$

$$\begin{aligned} &\le p\, max\{\frac{1}{2}[d^2(Q_1 \dots Q_{2n-1}x, P_0x)d(Q_2 \dots Q_{2n}y, P_1y) \\ &+ d(Q_1 \dots Q_{2n-1}x, P_0x)d^2(Q_2 \dots Q_{2n}y, P_1y)], \\ &d(Q_1 \dots Q_{2n-1}x, P_0x)d(Q_1 \dots Q_{2n-1}x, P_1y)d(Q_2 \dots Q_{2n}y, P_0x), \\ &d(Q_1 \dots Q_{2n-1}x, P_1y)d(Q_2 \dots Q_{2n}y, P_0x)d(Q_2 \dots Q_{2n}y, P_1y)\} \\ &-\phi\big(m(x,y)\big), \end{aligned}$$

for all $x, y \in M$, where

$$m(x,y) = \max\{d^2(Q_1 \dots Q_{2n-1}x, Q_2 \dots Q_{2n}y), d(Q_1 \dots Q_{2n-1}x, P_0x)d(Q_2 \dots Q_{2n}y, P_1y), d(Q_1 \dots Q_{2n-1}x, P_1y)d(Q_2 \dots Q_{2n}y, P_0x),$$

$\frac{1}{2}[d(Q_1 \dots Q_{2n-1}x, P_0x)d(Q_1 \dots Q_{2n-1}x, P_1y)\ +$

$dQ2\dots Q2ny,P0xdQ2\dots Q2ny,P1y\}$,

where p is a real number satisfying $0 < p < 1$ and $\phi: [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(t) = 0$ iff $t = 0$ and $\phi(t) > 0$ $for\ each\ t > 0$.

Then $Q_1, Q_2, \dots, Q_{2n}$, P_0 and P_1 have a unique common fixed point in M given that the pairs $(P_0, Q_1 \dots Q_{2n-1})$ and $(P_1, Q_2 \dots Q_{2n-2})$ are weakly compatible.

Proof. Let $Q_1' = Q_1Q_3 \dots Q_{2n-1}$ and $Q_2' = Q_2Q_4 \dots Q_{2n}$. First we assume that the pair $(P_0, Q_1 \dots Q_{2n-1})$ satisfies $(E.A)$property, then there exists a sequence $\{u_n\}$ in M such that $\lim_{n\to\infty} P_0\, u_n = \lim_{n\to\infty} Q_1 \dots Q_{2n-1}u_n = u$, for some $u \in M$.

Since $P_0(M) \subset Q_2 \dots Q_{2n}(M)$, there exists a sequence $\{v_n\}$ in M such that $Q_2 \dots Q_{2n}(v_n) = P_0u_n$. Hence $\lim_{n\to\infty} Q_2 \dots Q_{2n}v_n = u$.

Step (i): First we show that $\lim_{n\to\infty} P_1\, v_n = u$. On putting $x = u_n$ and $y = v_n$ in (C7), we have

$$d^3(P_0u_n, P_1v_n)$$
$$\leq p\, max\{\frac{1}{2}[d^2(Q_1 \dots Q_{2n-1}u_n, P_0u_n)d(Q_2 \dots Q_{2n}v_n, P_1v_n)$$
$$+ d(Q_1 \dots Q_{2n-1}u_n, P_0u_n)d^2(Q_2 \dots Q_{2n}v_n, P_1v_n)],$$
$$d(Q_1 \dots Q_{2n-1}u_n, P_0u_n)d(Q_1 \dots Q_{2n-1}u_n, P_1v_n)d(Q_2 \dots Q_{2n}v_n, P_0u_n), d(Q_1 \dots Q_{2n-1}u_n,$$
$$-\phi\big(m(u_n, v_n)\big),$$

where

$$m(u_n, v_n) = \max\{d^2(Q_1 \dots Q_{2n-1}u_n, Q_2 \dots Q_{2n}v_n), d(Q_1 \dots Q_{2n-1}u_n, P_0u_n)$$
$$.d(Q_2 \dots Q_{2n}v_n, P_1v_n), d(Q_1 \dots Q_{2n-1}u_n, P_1v_n)d(Q_2 \dots Q_{2n}v_n, P_0u_n),$$
$$\frac{1}{2}[d(Q_1 \dots Q_{2n-1}u_n, P_0u_n)d(Q_1 \dots Q_{2n-1}u_n, P_1v_n) +$$
dQ2…Q2nvn,P0undQ2…Q2nvn,P1vn}.

Taking limits as $n \to \infty$, we have

$$d^3(u, P_1v_n) \leq p \max\{\frac{1}{2}[d^2(u,u)d(u, P_1v_n) + d(u,u)d^2(u, P_1v_n)],$$
$$d(u,u)d(u, P_1v_n)d(u,u), d(u, P_1v_n)d(u,u)d(u, P_1v_n)\}$$
$$-\phi\big(m(u_n, v_n)\big),$$

where

$$m(u_n, v_n) = \{d^2(u,u), d(u,u)d(u, P_1v_n), d(u, P_1v_n)d(u,u),$$
$$\frac{1}{2}[d(u,u)d(u, P_1v_n) + d(u,u)d(u, P_1v_n)]\}.$$

which implies that $\lim_{n\to\infty} d^3(u, P_1v_n) = 0$. Hence

$$\lim_{n\to\infty} P_0\, u_n = \lim_{n\to\infty} Q_1 \dots Q_{2n-1}u_n = \lim_{n\to\infty} P_1\, v_n = \lim_{n\to\infty} Q_2 \dots Q_{2n}v_n = u.$$

Now suppose that $Q_1 \dots Q_{2n-1}(M)$ is closed subset of M, then exists $v \in M$ such that $Q_1 \dots Q_{2n-1}v = u$.

Step (ii): Now we show that $P_0v = u$. Using $(C7)$ with $x = v$ and $y = v_n$, we get

$$d^3(P_0v, P_1v_n) \leq p\, max\{\frac{1}{2}[d^2(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_2 \dots Q_{2n}v_n, P_1v_n)$$
$$+ d(Q_1 \dots Q_{2n-1}v, P_0v)d^2(Q_2 \dots Q_{2n}v_n, P_1v_n)],$$
$$d(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_1 \dots Q_{2n-1}v, P_1v_n)d(Q_2 \dots Q_{2n}v_n, P_0v),$$
$$d(Q_1 \dots Q_{2n-1}v, P_1v_n)d(Q_2 \dots Q_{2n}v_n, P_0v)d(Q_2 \dots Q_{2n}v_n, P_1v_n)\}$$
$$-\phi\big(m(v, v_n)\big),$$

where

$$m(v,v_n) = \max\{d^2(Q_1 \dots Q_{2n-1}v, Q_2 \dots Q_{2n}v_n), d(Q_1 \dots Q_{2n-1}v, P_0v)$$
$$.d(Q_2 \dots Q_{2n}v_n, P_1v_n), d(Q_1 \dots Q_{2n-1}v, P_1v_n)d(Q_2 \dots Q_{2n}v_n, P_0v),$$
$$\frac{1}{2}[d(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_1 \dots Q_{2n-1}v, P_1v_n) +$$
$$dQ2\dots Q2nvn,P0vdQ2\dots Q2nvn,P1vn\}.$$

Approaching limits as $n \to \infty$, we get $d^3(P_0v,u) \leq 0$, i.e., $P_0v = u$. Hence $P_0v = Q_1Q_3 \dots Q_{2n-1}v = u$. Since $P_0(M) \subset Q_2 \dots Q_{2n}(M)$, there exists $w \in M$ such that $P_0v = Q_2 \dots Q_{2n}w = u$.

Step (iii): Next we show that $Q_2Q_4 \dots Q_{2n}w = u$. On using (C7) with $x = v$ and $y = w$, we get

$$d^3(P_0v, P_1w) \leq p\, max\{\frac{1}{2}[d^2(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_2 \dots Q_{2n}w, P_1w)$$
$$+ d(Q_1 \dots Q_{2n-1}v, P_0v)d^2(Q_2 \dots Q_{2n}w, P_1w)],$$
$$d(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_1 \dots Q_{2n-1}v, P_1w)d(Q_2 \dots Q_{2n}w, P_0v),$$
$$d(Q_1 \dots Q_{2n-1}v, P_1w)d(Q_2 \dots Q_{2n}w, P_0v)d(Q_2 \dots Q_{2n}w, P_1w)\}$$
$$-\phi\big(m(v,w)\big),$$

where

$$m(v,w) = \max\{d^2(Q_1 \dots Q_{2n-1}v, Q_2 \dots Q_{2n}w), d(Q_1 \dots Q_{2n-1}v, P_0v)$$
$$.d(Q_2 \dots Q_{2n}w, P_1w), d(Q_1 \dots Q_{2n-1}v, P_1w)d(Q_2 \dots Q_{2n}w, P_0v),$$
$$\frac{1}{2}[d(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_1 \dots Q_{2n-1}v, P_1w) +$$
$$dQ2\dots Q2nw,P0vdQ2\dots Q2nw,P1w\}.$$

which implies that $d^3(u, P_1w) \leq 0$, i.e., $P_1w = u$ and hence $P_0v = Q_1Q_3 \dots Q_{2n-1}v = P_1w = Q_2Q_4 \dots Q_{2n}w = u$. Since the pairs $(P_0, Q_1Q_3 \dots Q_{2n-1})$ and $(P_1, Q_2Q_4 \dots Q_{2n})$ are weakly compatible, we have

$$Q_1Q_3 \dots Q_{2n-1}u = Q_1Q_3 \dots Q_{2n-1}(P_0v) = P_0(Q_1Q_3 \dots Q_{2n-1}v) = P_0u$$

and

$$Q_2Q_4 \dots Q_{2n}u = Q_2Q_4 \dots Q_{2n}(P_1w) = P_1(Q_2Q_4 \dots Q_{2n}w) = P_1u.$$

Step (iv): In this step, we prove that $P_0v = P_1u$. Suppose that $P_0v \neq P_1u$. On putting $x = v$ and $y = u$ in (C7), we obtain

$$d^3(P_0v, P_1u) \leq p\, max\{\frac{1}{2}[d^2(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_2 \dots Q_{2n}u, P_1u)$$
$$+ d(Q_1 \dots Q_{2n-1}v, P_0v)d^2(Q_2 \dots Q_{2n}u, P_1u)],$$

$$d(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_1 \dots Q_{2n-1}v, P_1u)d(Q_2 \dots Q_{2n}u, P_0v),$$
$$d(Q_1 \dots Q_{2n-1}v, P_1u)d(Q_2 \dots Q_{2n}u, P_0v)d(Q_2 \dots Q_{2n}u, P_1u)\}$$
$$-\phi\big(m(v,u)\big),$$

where

$$m(v,u) = \max\{d^2(Q_1 \dots Q_{2n-1}v, Q_2 \dots Q_{2n}u), d(Q_1 \dots Q_{2n-1}v, P_0v)$$
$$.d(Q_2 \dots Q_{2n}u, P_1u), d(Q_1 \dots Q_{2n-1}v, P_1u)d(Q_2 \dots Q_{2n}u, P_0v),$$
$$\frac{1}{2}[d(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_1 \dots Q_{2n-1}v, P_1u) +$$
$$dQ2 \dots Q2nu,P0vdQ2 \dots Q2nu,P1u\}.$$

On simplification, we have $d^3(P_0v, P_1u) \leq -\phi(d^2(P_0v, P_1u))$, a contradiction. Thus, we have $P_0v = P_1u = u$, i.e., $P_1u = Q_2Q_4 \dots Q_{2n}u = u$.

Further on putting $x = y = u$ in $(C7)$, we have $P_0u = u$ hence $P_0u = Q_1Q_3 \dots Q_{2n-1}u = u$.

Step (v): On putting $x = u$ and $y = Q_4 \dots Q_{2n}u$ in $(C7)$ and using condition $(C4)$ and $Q_1' = Q_1Q_3 \dots Q_{2n-1}$and $Q_2' = Q_2Q_4 \dots Q_{2n}$, we have

$$d^3(P_0u, P_1Q_4 \dots Q_{2n}u)$$
$$\leq p\, max\{\frac{1}{2}[d^2(Q_1'u, P_0u)d(Q_2'Q_4 \dots Q_{2n}u, P_1Q_4 \dots Q_{2n}u)$$
$$+ d(Q_1'u, P_0u)d^2(Q_2'Q_4 \dots Q_{2n}u, P_1Q_4 \dots Q_{2n}u)],$$
$$d(Q_1'u, P_0u)d(Q_1'u, P_1Q_4 \dots Q_{2n}u)d(Q_2'Q_4 \dots Q_{2n}u, P_0u),$$
$$d(Q_1'u, P_1Q_4 \dots Q_{2n}u)d(Q_2'Q_4 \dots Q_{2n}u, P_0Q_4 \dots Q_{2n}u)$$
$$.\, d(Q_2'Q_4 \dots Q_{2n}u, P_1u)\} - \phi\big(m(u, Q_4 \dots Q_{2n}u)\big),$$

where

$$m(u, Q_4 \dots Q_{2n}u) = \max\{d^2(Q_1'u, Q_2'Q_4 \dots Q_{2n}u), d(Q_1'u, P_0u)$$
$$.d(Q_2'Q_4 \dots Q_{2n}u, P_1u), d(Q_1'u, P_1Q_4 \dots Q_{2n}u)d(Q_2'Q_4 \dots Q_{2n}u, P_0u),$$
$$\frac{1}{2}[d(Q_1'u, P_0u)d(Q_1'u, P_1Q_4 \dots Q_{2n}u) +$$
$$dQ2'Q4 \dots Q2nu,P0udQ2'Q4 \dots Q2nu,P1Q4 \dots Q2nu\}.$$

On simplification, we get $d^3(u, Q_4 \dots Q_{2n}u) \leq -\phi(d^2(u, Q_4 \dots Q_{2n}u))$, i.e., $Q_4 \dots Q_{2n}u = u$. Hence $Q_2Q_4 \dots Q_{2n}u = Q_2u = u$. Continuing like this, we have

$$P_0u = Q_2u = Q_4u = \cdots = Q_{2n}u = u.$$

Therefore,

$$P_0u = P_1u = Q_1u = Q_2u = \cdots = Q_{2n-1}u = Q_{2n}u = u.$$

Hence u is a common fixed point of the given mappings. Uniqueness follows easily from condition $(C7)$. Similarly, the proof holds for the condition $(C6)$. This completes the proof.

Now we prove the following theorem for families of mappings which is a slight generalization of Theorem 2.2.

Theorem 2.3. Let $\{P_\alpha\}_{\alpha\in J}$ and $\{Q_i\}_{i=1}^{2p}$ be two families of self-mappings on a metric space (M,d). Suppose that there exists a fixed $\beta\in J$ such that:

$$(C8) Q_2(Q_4 \dots Q_{2n}) = (Q_4 \dots Q_{2n})Q_2,\ \ Q_2Q_4(Q_6 \dots Q_{2n}) = (Q_6 \dots Q_{2n})Q_2Q_4,\ \dots,\ Q_2 \dots Q_{2n-2}(Q_{2n}) = (Q_{2n})Q_2 \dots Q_{2n-2}\ ;$$

$$P_\beta(Q_4 \dots Q_{2n}) = (Q_4 \dots Q_{2n})P_\beta,\ P_\beta(Q_6 \dots Q_{2n}) = (Q_6 \dots Q_{2n})P_\beta, \dots, P_\beta Q_{2n} = Q_{2n}P_\beta,$$

$$Q_1(Q_3 \dots Q_{2n-1}) = (Q_3 \dots Q_{2n-1})Q_1,\ \ Q_1Q_3(Q_5 \dots Q_{2n-1}) = (Q_5 \dots Q_{2n-1})Q_1Q_3,\ \dots,\ Q_1 \dots Q_{2n-3}(Q_{2n-1}) = (Q_{2n-1})Q_1 \dots Q_{2n-3};$$

$$P_\alpha(Q_3 \dots Q_{2n-1}) = (Q_3 \dots Q_{2n-1})P_\alpha,\ P_\alpha(Q_5 \dots Q_{2n-1}) = (Q_5 \dots Q_{2n-1})P_\alpha, \dots, P_\alpha Q_{2n-1} = Q_{2n-1}P_\alpha,$$

$(C9) P_\alpha(M)\subset Q_2 \dots Q_{2n}(M)$ and the pair $(P_\alpha, Q_1 \dots Q_{2n-1})$ satisfies $(E.A)$ property and $Q_1 \dots Q_{2n-1}(M)$ is a closed subset of M.

or

$(C10) P_\beta(M)\subset Q_1 \dots Q_{2n-1}(M)$ and the pair $(P_\beta, Q_2 \dots Q_{2n})$ satisfies $(E.A)$ property and $Q_2 \dots Q_{2n}(M)$ is a closed subset of M.

$$(C11)\ d^3(P_\alpha x, P_\beta y) \le p\, max\{\frac{1}{2}[d^2(Q_1 \dots Q_{2n-1}x, P_\alpha x)d(Q_2 \dots Q_{2n}y, P_\beta y) + d(Q_1 \dots Q_{2n-1}x, P_\alpha x)d^2(Q_2 \dots Q_{2n}y, P_\beta y)],$$
$$d(Q_1 \dots Q_{2n-1}x, P_\alpha x)d(Q_1 \dots Q_{2n-1}x, P_\beta y)d(Q_2 \dots Q_{2n}y, P_\alpha x),$$
$$d(Q_1 \dots Q_{2n-1}x, P_\beta y)d(Q_2 \dots Q_{2n}y, P_\alpha x)d(Q_2 \dots Q_{2n}y, P_\beta y)\} - \phi(m(x,y)),$$

for all $x, y\in M$, where

$$m(x,y) = \max\{d^2(Q_1 \dots Q_{2n-1}x, Q_2 \dots Q_{2n}y), d(Q_1 \dots Q_{2n-1}x, P_\alpha x)d(Q_2 \dots Q_{2n}y, P_\beta y),$$
$$d(Q_1 \dots Q_{2n-1}x, P_\beta y)d(Q_2 \dots Q_{2n}y, P_\alpha x),$$

$\frac{1}{2}\left[d(Q_1 \dots Q_{2n-1}x, P_\alpha x)d\big(Q_1 \dots Q_{2n-1}x, P_\beta y\big) + \right.$

dQ2…Q2ny,PαxdQ2…Q2ny,Pβy},

where p is a real number satisfying $0 < p < 1$ and $\phi: [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ iff $t = 0$ and $\phi(t) > 0$ $for\ each\ t > 0$.

Then all Q_i and P_α have a unique common fixed point in M given that the pairs $(P_\alpha, Q_1 \dots Q_{2n-1})$ and $(P_\beta, Q_2 \dots Q_{2n-2})$ are weakly compatible.

Proof. Let $P_{\alpha 0}$be a fixed element in $\{P_\alpha\}_{\alpha\in J}$. By Theorem 2.2 with $P_\alpha = P_{\alpha 0}$ and $P_1 = P_\beta$ it follows that there exists some $u \in M$ such that

$$P_\beta u = P_{\alpha 0} u = Q_2 Q_4 \dots Q_{2n} u = Q_1 Q_3 \dots Q_{2n-1} u = u.$$

Let$\alpha \in J$ be arbitrary. Then from condition (C11), we have

$$\begin{aligned} d^3\big(P_\alpha u, P_\beta u\big) \leq p\, max\{&\frac{1}{2}[d^2(Q_1 \dots Q_{2n-1}u, P_\alpha u)d\big(Q_2 \dots Q_{2n}u, P_\beta u\big) \\ &+ d(Q_1 \dots Q_{2n-1}u, P_\alpha u)d^2\big(Q_2 \dots Q_{2n}u, P_\beta u\big)], \\ &d(Q_1 \dots Q_{2n-1}u, P_\alpha u)d\big(Q_1 \dots Q_{2n-1}u, P_\beta u\big)d(Q_2 \dots Q_{2n}u, P_\alpha u), \\ &d\big(Q_1 \dots Q_{2n-1}u, P_\beta u\big)d(Q_2 \dots Q_{2n}u, P_\alpha u)d(Q_2 \dots Q_{2n}u, P_\beta u)\} \\ &-\phi\big(m(u,u)\big), \end{aligned}$$

where

$$\begin{aligned} m(u,u) = \max\{&d^2(Q_1 \dots Q_{2n-1}u, Q_2 \dots Q_{2n}u), d(Q_1 \dots Q_{2n-1}u, P_\alpha u)d\big(Q_2 \dots Q_{2n}u, P_\beta u\big), \\ &d\big(Q_1 \dots Q_{2n-1}u, P_\beta u\big)d(Q_2 \dots Q_{2n}u, P_\alpha u), \\ &\frac{1}{2}\left[d(Q_1 \dots Q_{2n-1}u, P_\alpha u)d\big(Q_1 \dots Q_{2n-1}u, P_\beta u\big)\right. \\ &\left. + d(Q_2 \dots Q_{2n}u, P_\alpha u)d\big(Q_2 \dots Q_{2n}u, P_\beta u\big)\right]\} \end{aligned}$$

and hence

$$\begin{aligned} d^3(P_\alpha u, u) \leq p\, max\{&\frac{1}{2}[d^2(u, P_\alpha u)d(u,u) + d(u, P_\alpha u)d^2(u,u)], \\ &d(u, P_\alpha u)d(u,u)d(u, P_\alpha u), \\ &d(u,u)d(u, P_\alpha u)d(u,u)\} - \phi\big(d^2(u,u)\big), \end{aligned}$$

which implies that $d^3(P_\alpha u, u) \leq 0$, i.e., $P_\alpha u = u$ for each $\alpha \in J$. Uniqueness follows easily.

Next, we prove a theorem for any even number of weakly compatible mappings satisfying common limit range property.

Theorem 2.4. Let $Q_1, Q_2, \ldots, Q_{2n}$, P_0 and P_1 be self mappings on a metric space (M, d), satisfying the conditions $(C4)$, $(C7)$ and

$(C12)$ the pair $(P_0, Q_1^{'})$ and $Q_2^{'}$ satisfy $(CLR)_{(P_0\,,\,Q_1^{'})Q_2^{'}}$ property.

Then $Q_1, Q_2, \ldots, Q_{2n}$, P_0 and P_1 have a unique common fixed point in M given that the pairs $(P_0, Q_1^{'})$ and $(P_1, Q_2^{'})$ are weakly compatible, where $Q_1^{'} = Q_1 Q_3 \ldots Q_{2n-1}$ and $Q_2^{'} = Q_2 Q_4 \ldots Q_{2n}$.

Proof. Since the pair $(P_0, Q_1^{'})$ and $Q_2^{'}$satisfy $(CLR)_{(P_0\,,\,Q_1^{'})Q_2^{'}}$ property, there exists a sequence $\{u_n\}$ in M such that

$$\lim_{n\to\infty} P_0\, u_n = \lim_{n\to\infty} Q_1^{'}\, u_n = \lim_{n\to\infty} Q_1 Q_3 \ldots Q_{2n-1}\, u_n = u,$$

for some $u \in Q_1^{'} \cap Q_2^{'} = Q_2 Q_4 \ldots Q_{2n}(M) \cap Q_1 Q_3 \ldots Q_{2n-1}(M)$.

Since $u \in Q_2 Q_4 \ldots Q_{2n}(M)$, this gives $u = Q_2 Q_4 \ldots Q_{2n} v$, for some v in M.

We show that $u = P_1 v$. On putting $x = u_n$ and $y = v$ in (C7), we have

$$\begin{aligned} d^3(P_0 u_n, P_1 v) \le p\, max\{&\frac{1}{2}[d^2(Q_1 \ldots Q_{2n-1} u_n, P_0 u_n) d(Q_2 \ldots Q_{2n} v, P_1 v) \\ &+ d(Q_1 \ldots Q_{2n-1} u_n, P_0 u_n) d^2(Q_2 \ldots Q_{2n} v, P_1 v)], \\ &d(Q_1 \ldots Q_{2n-1} u_n, P_0 u_n) d(Q_1 \ldots Q_{2n-1} u_n, P_1 v) d(Q_2 \ldots Q_{2n} v, P_0 u_n), \\ &d(Q_1 \ldots Q_{2n-1} u_n, P_1 v) d(Q_2 \ldots Q_{2n} v, P_0 u_n) d(Q_2 \ldots Q_{2n} v, P_1 v)\} \\ &-\phi\big(m(u_n, v)\big), \end{aligned}$$

where

$$\begin{aligned} m(u_n, v) = \max\{&d^2(Q_1 \ldots Q_{2n-1} u_n, Q_2 \ldots Q_{2n} v), d(Q_1 \ldots Q_{2n-1} u_n, P_0 u_n) \\ &.d(Q_2 \ldots Q_{2n} v, P_1 v), d(Q_1 \ldots Q_{2n-1} u_n, P_1 v) d(Q_2 \ldots Q_{2n} v, P_0 u_n), \\ &\frac{1}{2}[d(Q_1 \ldots Q_{2n-1} u_n, P_0 u_n) d(Q_1 \ldots Q_{2n-1} u_n, P_1 v) + \\ &dQ2\ldots Q2nvn,P0undQ2\ldots Q2nv,P1v\}. \end{aligned}$$

Taking limits as $n \to \infty$ and on simplification, we have $d^3(u, P_1 v) \le 0$, i.e., $P_1 v = u$ and hence $u = P_1 v = Q_2 Q_4 \ldots Q_{2n} v$.

Also $u \in Q_1 Q_3 \ldots Q_{2n-1}(M)$ which gives $u = Q_1 Q_3 \ldots Q_{2n-1} w$, for some $w \in M$.

If we put $x = w$ and $y = v$ in condition (C7), on simplification we get, $P_0 w = u$.

Hence $P_0 w = Q_1 Q_3 \ldots Q_{2n-1} w = P_1 v = Q_2 Q_4 \ldots Q_{2n} v = u$.

Due to weakly compatibility of the given mappings, we obtain

$$P_0 u = P_0 Q_1 Q_3 \ldots Q_{2n-1} w = Q_1 Q_3 \ldots Q_{2n-1} P_0 w = Q_1 Q_3 \ldots Q_{2n-1} u$$

and

$$P_1 u = P_1 Q_2 Q_4 \dots Q_{2n} v = Q_2 Q_4 \dots Q_{2n} P_1 v = Q_2 Q_4 \dots Q_{2n} u.$$

Further using steps (iv) and (v) of Theorem 2.2, we get u is a unique common fixed of the given mappings.

The following theorem is a slight generalization of Theorem 2.4.

Theorem 2.5. Let $\{P_\alpha\}_{\alpha \in J}$ and $\{Q_i\}_{i=1}^{2p}$ be two families of self-mappings on a metric space (M, d). Suppose that there exists a fixed $\beta \in J$ such that conditions $(C8)$ and $(C11)$ are satisfied. Further,

(C13) the pair $(P_\alpha, Q_1 Q_3 \dots Q_{2n-1})$ and $Q_2 Q_4 \dots Q_{2n}$ satisfy $(CLR)_{(P_\alpha, Q_1 Q_3 \dots Q_{2n-1}) Q_2 Q_4 \dots Q_{2n}}$ property.

Then all Q_i and P_α have a unique common fixed point in M given that the pairs $(P_\alpha, Q_1 \dots Q_{2n-1})$ and $(P_\beta, Q_2 \dots Q_{2n-2})$ are weakly compatible.

Proof. Proof of the Theorem 2.5 follows from the proof of Theorem 2.3.

Corollary 2.6. Let f, g, S and T are self-mappings on a metric space (M, d) satisfying $(C2)$ and the following conditions

$(C14) f(M) \subset T(M)$ and the pair (f, S) satisfies $(E.A)$ property and $S(M)$ is a closed subset of M.

or

$(C15) g(M) \subset S(M)$ and the pair (g, T) satisfies $(E.A)$ property and $T(M)$ is a closed subset of M.

Then f, g, S and T have a unique common fixed point in M provided that the pairs (f, S) and (g, T) are weakly compatible.

Proof. If we take $P_0 = f$, $P_1 = g$, $Q_1 = S$, $Q_2 = T$ and $Q_3 = Q_4 = \cdots = Q_{2n} = I$ (Identity Map) in Theorem 2.2, we get required result.

Remark 2.7. Corollary 2.6 is a generalization of Theorem 2.1 in the sense that conditions of completeness of space and continuity of the mappings are relaxed. Theorems 2.2-2.5 generalize and extend Theorem 2.1 for families of weakly compatible mappings.

Now we give an example in support of our theorems.

Example 2.8. Let $M = [0, 1]$ and d be usual metric on M. Define

$P_\alpha(x) = \frac{x^3}{1+x^3}$ for each $\alpha \in J$ and all $x \in M$,

$Q_i(x) = x^{\sqrt[n]{3}}$ for each $i \in \{1,2, \dots ,2n\}$ and all $x \in M$.
Then $Q_2Q_4 \dots Q_{2n}x = x^3, Q_1Q_3 \dots Q_{2n-1}x = x^3$.
Let $\phi: [0, \infty) \to [0, \infty)$ be a function defined by $\phi(t) = \frac{t}{30}$, for $t \geq 0$. Then all the conditions of Theorems 2.2-2.5 are satisfied for $p = \frac{9}{10}$ and 0 is the unique common fixed point of the mappings.

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