Some Results of Character for Finite Groups

1st Salam Abdulkareem Manthoor

Department of Mathematics, College of

Education for Pure Science

Tikrit University

Tikrit, Iraq

salam.a.manthoor@st.tu.edu.iq

2nd Sinan O. Al-Salihi

Department of Mathematics, College of

Education for Pure Science

Tikrit University

Tikrit, Iraq

somer@tu.edu.iq

3rd Taghreed Hur Majeed

Department of Mathematics

Education College

Mustansiriyah University

Baghdad, Iraq

taghreedmajeed@uomustansiriyah.edu.iq

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Abstract

The aim of our work is to study some of theories about the character.

The character theory is a very important theory within the concept of representation theory, especially in our study of finite groups, like symmetric group.

We touched on some basic definitions that we will need in our work, such as the definitions of homomorphism and isomorphism, the definition of character and the definition of class function, in addition to the definitions of tensor product, direct sum, and others.

We have proven that the character of direct sum is a class function. From this we came to the conclusion that the character of tensor product of the symmetric group is also a class function.

Then we proved that any number of direct sum of representations is an isomorphism if and only if they have the same character.

From the above theories, we were able to prove that the sum of three characters equals to the character of direct sum. And by the same way we also prove that the product of three characters equals to the character of tensor product. Representation theory is concerned with how to write a group as a collection of matrices.

The theory is not only beautiful in and of itself, but it also provides one of the keys to a proper understanding of finite groups. We have reached a critical point in the theory of finite group representation. The characters of a representation that we will study in this work will illustrate a great important about a representation. Also we have seen that a group can be represented by a set of invertible n×n matrices that show the same relationships as the group. Furthermore, we've seen that character and representation theories of finite
groups are extremely useful for studying and proving a variety of group theory
results. Obtaining the character table of a finite group is a difficult operation in general. This is deeply true when the degree of representation or the order for the finite group is very large, and the character only has one number for every matrix.

Character theory is the fundamental finding in group representation theory that establishes that linear representations of a finite group over the complex numbers disintegrate into irreducible parts. For instance, character table application relies on this essential concept. The theorem describes the excellent situations, for which the requisite behaviour will hold, in terms of the characteristics of the underlying field of scalars as an abstract algebraic construct. The real number field, for instance, has a similar pattern of behaviour; however the representations that are irreducible vary.

Keywords: Character theory, representation theory, tensor product, direct sum

INTRODUCTION

A group K is referred to as finite if its underlying set is finite. Unless K is infinite, in which case we say that K is infinite if its order is infinite, the order |K| of a finite group K is the order (number of elements) of the underlying set. The representation of a group K over the field F is characterized by the homomorphism ϕ: K $\rightarrow$GL(n, F) for some n. The degree or dimension of has the numerical value n.

If n = 1, then there is said to be a linear representation of. If ϕ: K $\rightarrow$GL(V) is a representation of a group K in finite dimensions on a vector space V. The character of the representation is the function χ_ϕ: K$\rightarrow$ C defined by $\chi_\phi(k)$ = tr($\phi(k)$). A character is considered to be irreducible if it is provided by an irreducible representation. The character that corresponds to the trivial representation of K is referred to as the trivial character of K. This character has value 1 for every k ϵ K, i.e., $\phi(k) = 1$, over all the parts of K. A linear character is a character of degree 1. Often, the K character table, which gives a thorough description of the structure of the group K, is created by first writing down all the linear characters. It is crucial to recognize derived subgroup of K as a first step. The Trace of A, denoted by the symbol tr, is the sum of the diagonal components of a square matrix A. When k_1 and k_2 are members of the same conjugacy class of K, if $\phi(k_1) = \phi(k_2)$ then the function ϕ is called a class function of

K. Assume that K is a group with the elements k_1 and k_2 as its members. The two variables k_1 and k_2 are said to be conjugate in K if $xk_1x^{-1} = k_2$ for some x in K. (and refer to k_2 as the conjugate of k_1). Finally, as a reminder we shall use the two important operations in Representation theory, the tensor product and the direct sum of linear representations in our proofs as follows.

SOME RESULTS OF CHARACTER FOR FINITE GROUPS

Definition2.1: If the underlying set of a group K is finite, then the group is said to be finite. The order |K| of a finite group K is the order (number of elements) of the underlying set unless K is infinite, in which case we say that K is infinite if its order is infinite. [4]

Definition2.2: The homomorphism K→ GL(n, F) for some n characterizes the representation of a group K over the field F. The degree or dimension of is an integer with the symbol n. A linear representation of is said to exist if n = 1. [7]

Definition 2.3: Let ϕ: K → GL(V) be a finite dimensional representation of the group K on a vector space V. The function χ_ϕ: K→ C defined by

$\chi_\phi(k) = tr(\phi(k))$ is called the character of the representation. [1]

Definition 2.4: If a character χ_ϕ is provided by an irreducible representation ϕ, that character is said to be irreducible. [2]

Definition 2.5: It is referred to as the trivial character of K to identify the character that corresponds to the trivial representation of K. Over all the components of K, this character has value 1, for every k ϵK, i. e. $\chi(k) = 1$. [2]

Definition 2.6: character of degree 1 is called a linear character. Writing down all the linear characters is frequently the first step in creating the K character table, which contains a detailed description of the group K's structure. It is important to identify the derived subgroup of K as a first step. [2]

Definition 2.7: The Trace of A is the total of a square matrix A's diagonal elements, and it is represented by the symbol tr(A). [4]

Definition 2.8: Let K be a group and let k_1, k_2 ϵ K. The function ϕ: K → C is called a class function of K if $\phi(k_1) = \phi(k_2)$ whenever k_1 and k_2 are in the same conjugacy class of K. [5]

Definition 2.9: A one-to-one mapping (or function) from K onto K' that maintains the group operation is known as an isomorphism ϕ from a group K to a group K'. In other words, for any k_2 and k_2 in K, $\phi(k_1 k_2) = \phi(k_1)\phi(k_2)$. We assert that K and K' are isomorphic and write K ≈ K' if there is an isomorphism from K onto K'. [4]

Definition 2.10: Suppose that K is a group and k_1 and k_2 be its elements. If $xk_1x^{-1} = k_2$ for some x in K, we say that the two variables k_1 and k_2 are conjugate in K (and refer to k_2 as the conjugate of k_1). The conjugacy class for k_1 is represented by the set $cl(k_1) = \{xk_1x^{-1} \mid x \,\epsilon K\}$. [4]

Definition 2.11: Let ϕ_1: $K_1 \to GL(V_1)$, ϕ_2: $K_2 \to GL(V_2)$ are linear representations. We are defined a linear representation $\phi_1 \otimes \phi_2$: $K_1 \times K_2 \to GL(V_1 \otimes V_2)$ into the tensor product of V_1 and V_2 by $\phi_1 \otimes \phi_2$ $(k_1, k_2) = \phi_1(k_1) \otimes \phi_2(k_2)$, in which $k_1 \epsilon K_2$, $k_2 \epsilon K_2$ This is called the tensor product of ϕ_1 and ϕ_2. [6], [7]

Definition 2.12: If (ϕ_1, V_1) and (ϕ_2, V_2) be a representation of K_1 and K_2, respectively then the direct sum of the representations is a linear representation and define as. [1]

$$\forall k_1 \in K_1, k_2 \in K_2, v_1 \in V_1, v_2 \in V_2 : \begin{Bmatrix} \phi_1 \oplus \phi_2 : K_1 \times K_2 \to GL(V_1 \oplus V_2) \\ (\phi_1 \oplus \phi_2)(k_1, k_2)(v_1, v_2) = \phi_1(k_1) v_1 \oplus \phi_2(k_2) v_2 \end{Bmatrix}$$

Proposition 2.13: A Character is a class function for a group. [2]

Proposition 2.14: A character of a direct sum of groups is class function

Proof: suppose that χ_ϕ is a character accorded by a representation ϕ, then for all k_1, k_2, g ϵ K, we have:

$$\phi(g\, (k_1 \oplus k_2)\, g^{-1}) = \phi(g)\, \phi(k_1 \oplus k_2)\, \phi(g^{-1})$$

$$= \phi(g)\, \phi(k_1 \oplus k_2)\, (\, \phi(g)\,)^{-1}$$

Since $\chi_\phi(g\, (k_1 \oplus k_2)\, g^{-1})$

$= tr(\phi(g)\, \phi(k_1 \oplus k_2)\, (\phi(g)\,)^{-1}$

$= tr(\phi(k_1 \oplus k_2)\,)$

$= \chi_\phi(k_1 \oplus k_2)$

Proposition 2.15: A character of tensor product of symmetric groups is a class function.

Proof: suppose that χ_ϕ is character accorded by the representation of ϕ then, for all k_1, k_2, g ϵ K, we have

$$\phi(g\, (k_1 \otimes k_2)\, g^{-1}) = \phi(g)\, \phi(k_1 \otimes k_2)\, \phi(g)^{-1}$$

$$= \phi(g)\, \phi(k_1 \otimes k_2)\, (\phi(g)\,)^{-1}$$

Since $\chi_\phi\, (g\, (k_1 \otimes k_2)\, g^{-1})$

$= tr\, (\phi(g)\, \phi(k_1 \otimes k_2)\, (\phi(g)\,)^{-1})$

$= tr\, (\phi(k_1 \otimes k_2)\,)$

$= \chi_\phi\, (k_1 \otimes k_2)$

Proposition 2.16: any numbers of direct sum of representations are isomorphic if and only if they have the same characters.

Proof: Let ϕ_1: $K \to GL(\oplus_{i=1}^{n} V_i)$,

ϕ_2: K→GL($\oplus_{i=1}^{n}$ W_i),

and let $\phi_1 = \oplus_{i=1}^{n} A_i$, $\phi_2 = \oplus_{i=1}^{n} B_i$.

Since ϕ_1 and ϕ_2 have same characters, i. e. $\chi_{\phi_1} = \chi_{\phi_2}$

then tr($P^{-1} \oplus_{i=1}^{n} B_i$ P) = tr($P^{-1} \oplus_{i=1}^{n} A_i$ P)

tr($\oplus_{i=1}^{n} B_i$) = tr($\oplus_{i=1}^{n} A_i$)

χ($\oplus_{i=1}^{n} B_i$) = χ($\oplus_{i=1}^{n} A_i$)

that means $A_i = B_i$, for all i.

Thus ϕ_1 is isomorphic to ϕ_2.

Conversely, let ϕ_1 isomorphic ϕ_2. That means $A_i = B_i$ for all i.

Hence χ($\oplus_{i=1}^{n} A_i$) = χ($\oplus_{i=1}^{n} B_i$), therefore tr($\oplus_{i=1}^{n} A_i$) = tr($\oplus_{i=1}^{n} B_i$).

Suppose there is P a vector space isomorphism such that

tr($P^{-1} \oplus_{i=1}^{n} A_i$ P) = tr($P^{-1} \oplus_{i=1}^{n} B_i$ P)

this implies that $\chi_{\phi_1} = \chi_{\phi_2}$.

Proposition 2.17: Let ϕ_1: K→GL(V_1),

ϕ_2: K→GL(V_2),

ϕ_3: K→GL(V_3),

are three representations of K and let χ_1, χ_2, χ_3 are their characters respectively, then we have

i. $\chi_{(V_1)}\chi_{(V_2)}\chi_{(V_3)} = \chi_{(V_1 \oplus V_2 \oplus V_3)}$

ii. $\chi_{(V_1)} \cdot \chi_{(V_2)} \cdot \chi_{(V_3)} = \chi_{(V_1 \otimes V_2 \otimes V_3)}$

Proof:

i. By proposition 2.14

Let $\chi_{(V_1)} + \chi_{(V_2)} + \chi_{(V_3)} = tr\left(\phi_1(V_1)\right) + tr\left(\phi_2(V_2)\right) + tr\left(\phi_3(V_3)\right)$

$$= tr\left(\phi_1(V_1) + \phi_2(V_2) + \phi_3(V_3)\right)$$

$$= \chi_{(V_1 \oplus V_2 \oplus V_3)}$$

ii. By proposition 2.15

Let $\chi_{(V_1)} \cdot \chi_{(V_2)} \cdot \chi_{(V_3)} = tr\left(\phi_1(V_1)\right) \cdot tr\left(\phi_2(V_2)\right) \cdot tr\left(\phi_3(V_3)\right)$

$$= tr\left(\phi_1(V_1) \cdot \phi_2(V_2) \cdot \phi_3(V_3)\right)$$

$$= \chi_{(V_1 \otimes V_2 \otimes V_3)}$$

Definition 2.18: Assume that ϕ_1 and ϕ_2 are functions from K → C. Then we define the inner product of ϕ_1 and ϕ_2, which denoted by ‹ , ›, to be

$$\left\langle \phi_1, \phi_2 \right\rangle = \frac{1}{|K|} \sum_{k \in K} \phi_1(k) \overline{\phi_2(k)}$$

Proposition 2.19: Let K has precisely n conjugacy classes. If ϕ_1, ϕ_2 and ϕ_3 are three functions from K into C, then $\left\langle \phi_1 \oplus \phi_2, \phi_3 \right\rangle = \left\langle \phi_3, \phi_1 \oplus \phi_2 \right\rangle$

Proof: By the property of inner product $\left\langle \phi_1 \oplus \phi_2, \phi_3 \right\rangle = \left\langle \phi_1, \phi_3 \right\rangle \oplus \left\langle \phi_2, \phi_3 \right\rangle$

$$\frac{1}{|K|} \sum_{k \in K} \phi_1(k) \overline{\phi_3(K)} \oplus \frac{1}{|K|} \sum_{k \in K} \phi_2(k) \overline{\phi_3(k)}$$

since $\overline{\phi(k)} = \phi(k^{-1}), \forall k \in K$ then,

$$\frac{1}{|K|} \sum_{k \in K} \phi_1(k) \phi_3(k^{-1}) \oplus \frac{1}{|K|} \sum_{k \in K} \phi_2(k) \phi_3(k^{-1})$$

$$= \frac{1}{|K|} \sum_{k \in K} \phi_1(k^{-1}) \phi_3(k) \oplus \frac{1}{|K|} \sum_{k \in K} \phi_2(k^{-1}) \phi_3(k)$$

$$= \frac{1}{|K|} \sum_{k \in K} \phi_3(k) \phi_1(k^{-1}) \oplus \frac{1}{|K|} \sum_{k \in K} \phi_3(k) \phi_2(k^{-1})$$

$$= \frac{1}{|K|} \sum_{k \in K} \phi_3(k) \overline{\phi_1(K)} \oplus \frac{1}{|K|} \sum_{k \in K} \phi_3(k) \overline{\phi_2(k)}$$

$$= \left\langle \phi_3, \phi_1 \right\rangle \oplus \left\langle \phi_3, \phi_2 \right\rangle$$

$$= \left\langle \phi_3, \phi_1 \oplus \phi_2 \right\rangle$$

Proposition 2.20: Let K has precisely n conjugacy classes. If $\chi_{\phi_1}, \chi_{\phi_2}$, and χ_{ϕ_3} are characters of K, then $\left\langle \chi_{\phi_1} + \chi_{\phi_2}, \chi_{\phi_3} \right\rangle = \left\langle \chi_{\phi_3}, \chi_{\phi_1} + \chi_{\phi_2} \right\rangle$

Proof: By the properties of inner product $\left\langle \chi_{\phi_1} + \chi_{\phi_2}, \chi_{\phi_3} \right\rangle = \left\langle \chi_{\phi_1}, \chi_{\phi_3} \right\rangle + \left\langle \chi_{\phi_2}, \chi_{\phi_3} \right\rangle$

and by the fact that character is class function, we get

$$\frac{1}{|K|}\sum_{k\in K}\chi_{\phi_1}(k)\overline{\chi_{\phi_3}(k)}+\frac{1}{|K|}\sum_{k\in K}\chi_{\phi_2}(k)\overline{\chi_{\phi_3}(k)}$$

since $\overline{\chi(k)} = \chi(k^{-1}), \forall k \in K$ then,

$$\frac{1}{|K|}\sum_{k\in K}\chi_{\phi_1}(k)\chi_{\phi_3}(k^{-1})+\frac{1}{|K|}\sum_{k\in K}\chi_{\phi_2}(k)\chi_{\phi_3}(k^{-1})$$

since this sum is unaltered, we replacing k instead of k^{-1}, so

$$\frac{1}{|K|}\sum_{k\in K}\chi_{\phi_1}(k^{-1})\chi_{\phi_3}(k)+\frac{1}{|K|}\sum_{k\in K}\chi_{\phi_2}(k^{-1})\chi_{\phi_3}(k)$$

$$=\frac{1}{|K|}\sum_{k\in K}\chi_{\phi_3}(k)\chi_{\phi_1}(k^{-1})+\frac{1}{|K|}\sum_{k\in K}\chi_{\phi_3}(k)\chi_{\phi_2}(k^{-1})$$

$$= \left\langle \chi_{\phi_3}, \chi_{\phi_1} \right\rangle + \left\langle \chi_{\phi_3}, \chi_{\phi_2} \right\rangle$$

$$= \left\langle \chi_{\phi_3}, \chi_{\phi_1} + \chi_{\phi_2} \right\rangle$$

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CONCLUSION

The character theories that we were examining and proving in this work will serve as excellent examples of something very significant. We've also seen that a group can be represented by a collection of n-by-n invertible matrices that exhibit the same relationships as the group as a whole. Additionally, it has been demonstrated that character and representation theories of finite groups are very helpful for understanding and demonstrating a number of group theory conclusions.

In general, it is challenging to obtain a finite group's character table.

This is especially true if the character only has one number in each matrix and there is a very high degree of representation or order for the finite group. Basic proof that linear representations of a finite group over complex numbers disintegrate into irreducible parts is provided by character theory. This is essential, for instance, to the use of character tables. The theorem, which is a piece of abstract algebra, describes the good cases those in which the requisite behaviour will hold in terms of the characteristics of the underlying field of scalars. The real number field, for instance, yields the same kind of conclusion; however the representations that are irreducible change.

FUTURE WORKS

- Studying the representation theory of topological group.
- Studying the representation theory of topological groupoid.
- Studying the representation theory of Lie group.

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