Goodness-of-Fit Tests for the Bivariate Skew-Normal Distribution Based on a new Characterization

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Abstract
In this article, we propose test statistics to test the conformity of bivariate data to the bivariate skew-normal distribution (BSN). The tests are based on partial functional mean characterization. We will use this characterization to introduce two tests, defined as the integrated deviation (ID) or integrated squared deviation (ISD) between the sample and the population partial functional means. The performance of the two tests is compared to that of the Meintanis and Hlávka (MH) and Balakrishnan et al. (BCS) tests. Using a bootstrap procedure, the proposed tests, as well as MH and BCS tests, were applied to real data. It turned out that the computational forms of the proposed tests are much simpler than that of the MH test. However, depending on the BSN shape parameters and the alternative model chosen, the proposed tests either outperform MH or the MH test outperforms the proposed tests. Except for a few cases, the proposed tests and the MH test outperform the BCS test.
Keywords: - Partial functional mean, characterization, power.

1. Introduction

The normal distribution, which is a symmetric distribution, is a probabilistic model that can be fitted to data sets from many application domains. However, many other data sets can be skewed to the right or left. Therefore, the normal distribution, which has many interesting properties, may not be a suitable model for such data. It is therefore necessary to define a class of probability distributions that can be fitted to asymmetric data and at the same time has interesting mathematical properties. Azzalini [1, 2] introduced such a class of distributions, called the skew-normal distribution (SND), which includes the normal distribution as a special case. The probability density function for this class is given by

$$f(x;\alpha) = 2\phi(x)\Phi(\alpha x), -\infty < x < \infty,$$

where $\phi(x)$ and $\Phi(x)$ are the normal density and distribution functions, respectively. Due to its wide range of applications, the univariate SND has attracted many researchers; among these are [3, 4, 5, 6], to name a few.

Azzalini and Dalla Valle [7] extended this family to the multivariate skew-normal (MSN) distribution. They proposed the following joint probability density function for the k-dimensional variable $Y = (Y_1, \dots, Y_k)^T$:

$$f_k(y) = 2\phi_k(y;\mathbf{\Omega})\Phi(\alpha^T y), \tag{1}$$

where

$$\alpha^T = \frac{\lambda^T \mathbf{\Psi}^{-1}\mathbf{\Delta}^{-1}}{\sqrt{1+\lambda^T\mathbf{\Psi}^{-1}\lambda}}, \tag{2}$$

$$\mathbf{\Delta} = \operatorname{diag}\left(\sqrt{1-\delta_1^2}, \dots, \sqrt{1-\delta_k^2}\right), \tag{3}$$

$$\mathbf{\Omega} = \mathbf{\Delta}(\mathbf{\Psi} + \lambda\lambda^T)\mathbf{\Delta}, \tag{4}$$

$$\boldsymbol{\lambda} = \left(\lambda(\delta_1), \dots, \lambda(\delta_k)\right)^T, \tag{5}$$

where $\lambda(\delta_i) = \delta_i(1-\delta_i^2)^{-1/2}$and $\phi_k(y;\mathbf{\Omega})$ is k-dimensional multivariate normal distribution (MND) with standardized marginals and correlation matrix $\mathbf{\Omega}$. In particular, the Bivariate Skew Normal (BSN($\omega, \alpha_1, \alpha_2$)) density is given as

$$f(y_1, y_2) = 2\phi_2(y_1, y_2; \omega)\Phi(\alpha_1 y_1 + \alpha_2 y_2), \tag{6}$$

where

$\phi_2(z_1, z_2; \omega)$ is the Bivariate Normal Density (BND) with standardized marginals and correlation ω , that is

$$\phi_2(z_1, z_2; \omega) = \frac{1}{2\pi\sqrt{1-\omega^2}} Exp\left[-\frac{1}{2(1-\omega^2)}(z_1^2 - 2\omega z_1 z_2 + z_2^2)\right],$$

$$\Phi_2(z_1, z_2; \omega) = \int_{-\infty}^{z_2}\int_{-\infty}^{z_1} \phi_2(x_1, x_2; \omega)dx_1 dx_2,$$

Let Z_0, Z_1, and Z_2 be independent standard normal distributions, and let $X_1 = Z_1 and\, X_2 = \omega Z_1 + \sqrt{1-\omega^2}Z_2$, then (X_1, X_2) are jointly bivariate normal with standardized marginals and correlation ψ. Consider the transformations

$$Y_1 = \delta_1|Z_0| + \sqrt{1-\delta_1^2}X_1, Y_2 = \delta_2|Z_0| + \sqrt{1-\delta_2^2}X_2.$$

The covariance between Y_1 and Y_2 is $\omega = \delta_1\delta_2 + \psi\sqrt{(1-\delta_1^2)(1-\delta_2^2)}$ where ψ is the generic element of Ψ. Azzalini and Dalla Valle [7] showed that the joint distribution of Y_1 and Y_2 is the bivariate skew-normal with pdf given in (6). The parameters α_1 and α_2 are related to ω, δ_1 and δ_2 via the respective expressions

$$\alpha_1 = \frac{\delta_1 - \delta_2\omega}{\sqrt{(1-\omega^2)(1-\omega^2-\delta_1^2-\delta_2^2+2\delta_1\delta_2\omega)}}, \tag{7}$$

$$\alpha_2 = \frac{\delta_2 - \delta_1\omega}{\sqrt{(1-\omega^2)(1-\omega^2-\delta_1^2-\delta_2^2+2\delta_1\delta_2\omega)}}. \tag{8}$$

Azzalini and Capitanio [8] have examined some properties of the MSN distribution. Several articles have dealt with the MSN Azzalini-Capitanio distribution or have introduced different versions of it. Reference can be made to [9, 10, 11].

Goodness-of-fit procedures are generally used to examine whether a data set can be modeled using a certain probability distribution. Basically, the procedure is a hypothesis testing procedure that tests whether to accept or reject that a sample can be attributed to some assumed distribution. The first known goodness of fit test procedure was the Pearson chi-square test [12]. The procedure measures the discrepancy between the number of observed and expected observations in a category or interval. Different versions of the test have been proposed, for example [13, 14, 15, 16, 17, 18, 19, 20]. While these types of tests are more appropriate for discrete or categorical data, they can nevertheless be used to test for continuous distributions.

Another important class of goodness of fit tests is the class of tests based on the empirical distribution function, called EDF tests. This class is based on discrepancies between the distribution function assumed under the null hypothesis and its empirical counterpart. Beside the classical Kolmogorov-Smirnov and Cramer-von Mises tests, the class also includes [21, 22, 23] tests.

In addition to the two classes of tests mentioned above, many tests have been introduced based on different properties or characterizations of distributions. For example, we find tests based on the empirical characteristic or moment generating function such as [24, 25, 26]. There are also tests based on regression and correlation such as the normality test [27] and tests based on the integrated distribution function such as tests proposed in [28, 29, 30, 31].

Few papers studied goodness of fit for the univariate or multivariate skew normal distributions; Gupta and Chen [32] and Meintanis [33, 34] proposed empirical distribution function based statistics to test a skew-normal distribution. Meintanis and Hlávka [35] and Balakrishnan et al. [36] introduced tests for the MSN distribution.

The objective of this article is to introduce new goodness-of-fit tests for the skew-normal bivariate distribution (BSN). The tests will be based on what we call the empirical *partial functional mean*. In Section 2, we present the test statistics and its computational forms. In Section 3, we simulate the necessary critical values, carry out power calculations and compare the performance of the proposed tests with other test statistics. In Section 5, we illustrate the new tests to real data sets, and in Section 6, we summarize some concluding remarks.

2. Proposed Tests

In this section, we prove a new characterization of the BSN based on the partial functional mean, then we introduce test statistics based on this characterization and derive their computational forms.

2.1. Characterization of the BSN Distribution and Tests Proposal

Let $Y_1, \dots, Y_n$ be a random sample from an absolutely continuous distribution function F and probability density function f. It is straightforward to show that the *partial (incomplete) mean*

$$\mu(t) = \int_{-\infty}^{t} y f(y)\, dy\,,$$

when exists, characterizes the distribution of Y. Based on this characterization, test statistics can be introduced to test whether the random sample has come from a specified distribution. For example, tests may be based on deviations of $\mu(t)$ from its empirical counterpart

$$\hat{\mu}(t) = \frac{1}{n}\sum_{i=1}^{n} Y_i I_{Y_i \le t}.$$

We further notice that if

$$\mu(t) = \int_{-\infty}^{t} \psi(y) f(y)\, dy, \tag{9}$$

then $\mu'(t) = \psi(t)f(t)$. Thus, for any $\mu(t)$ and $\psi(t)$ satisfying (9), $f(t) = \mu'(t)/\psi(t)$, is the *pdf* of some random variable Y, and therefore, $\mu(t)$ uniquely determines the distribution of Y.

The empirical counterpart of (9) is

$$\hat{\mu}(t) = \frac{1}{n}\sum_{i=1}^{n} \psi(Y_i) I_{Y_i \le t}.$$

Now, we will extend this to the bivariate case. Assume $(Y_{11}, Y_{12}) \dots, (Y_{n1,} Y_{n2})$ is a random sample from the joint density $f_{Y_1,Y_2}(y_1, y_2)$, then for any function $\psi(y_1, y_2)$, we define the *joint partial functional mean* by

$$\mu(t_1, t_2) = \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} \Psi(\mathrm{y}_1,\mathrm{y}_2) f(\mathrm{y}_1,\mathrm{y}_2) d\mathrm{y}_1 d\mathrm{y}_2, \tag{10}$$

provided the integral exists.

For our purpose, it is judicious to choose $\Psi(\mathrm{y}_1,\mathrm{y}_2)$ so that the integral has a closed form.

For example if…

$$\Psi(\mathrm{y}_1,\mathrm{y}_2) = \frac{Exp(-\rho^2 \omega \mathrm{y}_1 \mathrm{y}_2)}{\Phi(\alpha_1 y_1 + \alpha_2 y_2)} \ where\ \boldsymbol{\rho} = \frac{1}{\sqrt{1-\omega^2}}, \tag{11}$$

and $f(\mathrm{y}_1,\mathrm{y}_2)$ is the BSN density given in (6), then

$$\begin{aligned}\mu(t_1, t_2) &= \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} \Psi(\mathrm{y}_1,\mathrm{y}_2) f(\mathrm{y}_1,\mathrm{y}_2) d\mathrm{y}_1 d\mathrm{y}_2 \\ &= 2\int_{-\infty}^{t_1}\int_{-\infty}^{t_2} \frac{Exp(-\rho^2 \omega \mathrm{y}_1 \mathrm{y}_2)}{\Phi(\alpha_1 y_1 + \alpha_2 y_2)} \phi_2(y_1, y_2; \omega) \Phi(\alpha_1 y_1 + \alpha_2 y_2) d\mathrm{y}_1 d\mathrm{y}_2 \\ &= 2\int_{-\infty}^{t_2}\int_{-\infty}^{t_1} \phi_2(y_1, y_2; \omega) Exp(-\frac{\rho^2}{2}\omega \mathrm{y}_1 \mathrm{y}_2) d\mathrm{y}_1 d\mathrm{y}_2 \\ &= \frac{\rho}{\pi}\int_{-\infty}^{t_2}\int_{-\infty}^{t_1} Exp\left[-\frac{\rho^2}{2}(y_1^2 + y_2^2)\right] d\mathrm{y}_1 d\mathrm{y}_2 \\ &= \frac{2}{\rho}\ \Phi(\rho t_1)\ \Phi(\rho t_2).\end{aligned} \tag{12}$$

Conversely, if

$$\mu(t_1, t_2) = \frac{2}{\rho}\ \Phi(\rho t_1)\ \Phi(\rho t_2) \text{ and } \Psi(\mathrm{y}_1,\mathrm{y}_2) = \frac{Exp(-\rho^2 \omega \mathrm{y}_1 \mathrm{y}_2)}{\Phi(\alpha_1 y_1 + \alpha_2 y_2)},$$

then

$$\int_{-\infty}^{t2}\int_{-\infty}^{t_1}\frac{Exp(-\rho^2\omega \mathrm{y}_1\mathrm{y}_2)}{\Phi(\alpha_1 y_1+\alpha_2 y_2)}f(\mathrm{y}_1,\mathrm{y}_2)d\mathrm{y}_1 d\mathrm{y}_2=\frac{2}{\rho}\,\Phi(\rho t_1)\,\Phi(\rho t_2) \tag{13}$$

Differentiation of both sides of (13) with respect to t_1 and then with respect to t_2 yields

$$\frac{Exp(-\rho^2\omega t_1 t_2)}{\Phi(\alpha_1 t_1+\alpha_2 t_2)}f(t_1,t_2)=2\rho\phi(\rho t_1)\,\phi(\rho t_2),$$

or

$$f(t_1,t_2)=2\rho\; exp(\rho^2\omega t_1 t_2)\phi(\rho t_1)\,\phi(\rho t_2)\Phi(\alpha_1 t_1+\alpha_2 t_2),$$

$$=2\phi_2(t_1,t_2,\omega)\Phi(\alpha_1 t_1+\alpha_2 t_2).$$

Thus, we have proved the following theorem.

***Theorem**:* Let $f(y_1,y_2)$ be a differentiable function on $\mathbb{R}^2$, if t_1,t_2 are non-zero real numbers, then for $\Psi(\mathrm{y}_1,\mathrm{y}_2)=\frac{Exp(-\rho^2\omega \mathrm{y}_1\mathrm{y}_2)}{\Phi(\alpha_1 y_1+\alpha_2 y_2)}$,

$$\int_{-\infty}^{t_1}\int_{-\infty}^{t_2}\Psi(\mathrm{y}_1,\mathrm{y}_2)f(\mathrm{y}_1,\mathrm{y}_2)d\mathrm{y}_1 d\mathrm{y}_2=\frac{2}{\rho}\Phi(\rho t_1)\,\Phi(\rho t_2)$$

iff

$$f(y_1,y_2)\;=2\phi_2(y_1,y_2,\omega)\Phi(\alpha_1 y_1+\alpha_2 y_2).$$

Now, we will develop test statistics based on the characterization proved in the above theorem. An immediate choice could be a measure of deviation between $\mu(t_1,t_2)$ and its empirical counterpart

$$\widehat{\mu(t_1,t_2)}=\frac{1}{n}\sum_{j=1}^{n}\Psi(Y_{1\mathrm{j}},\mathrm{Y}_{2\mathrm{j}})I_{Y_{1\mathrm{j}}\le t_1}\,I_{Y_{2\mathrm{j}}\le t_2}. \tag{14}$$

Here, we propose the following two tests

$$T_{1,n}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\widehat{\mu(t_1,t_2)}-\mu(t_1,t_2)\right)\omega(t_1,t_2)\,dt_1dt_2 \tag{15}$$

$$T_{2,n}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\widehat{\mu(t_1,t_2)}-\mu(t_1,t_2)\right)^2\omega(t_1,t_2)\,dt_1dt_2 \tag{16}$$

where $\omega(t_1,t_2)$ is an arbitrary weight function chosen such that each of the integrals (15) and (16) exists and has a closed form. A convenient choice of $\omega(t_1,t_2)$ would be

$$\omega(t_1,t_2)=Exp\left[-\frac{a^2}{2}(t_1^2+t_2^2)\right], \tag{17}$$

for some constant a.

2.2. Computational Forms for the Proposed Tests

To obtain a computational form for $T_{1,n}$, we replace the values of $\mu(t_1,t_2)$, $\widehat{\mu(t_1,t_2)}$, and $\omega(t_1,t_2)$ given, respectively, in (12), (14) and (17), in the integrand of (15)

$$T_{1,n}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1}{n}\sum_{j=1}^{n}\Psi(Y_{1\mathrm{j}},\mathrm{Y}_{2\mathrm{j}})I_{Y_{1\mathrm{j}}\le t_1}\,I_{Y_{2\mathrm{j}}\le t_2}-\frac{2}{\rho}\Phi(\rho t_1)\,\Phi(\rho t_2)\right)exp\left[-\frac{a^2}{2}(t_1^2+t_2^2)\right]dt_1dt_2\quad= \tag{18}$$

$$\frac{1}{n}\sum_{j=1}^{n}\Psi(Y_{1j},Y_{2j})\int_{Y_{2j}}^{\infty}\int_{Y_{1j}}^{\infty}exp\left[-\frac{a^2}{2}(t_1^2+t_2^2)\right]dt_1dt_2\ -$$
$$\frac{2}{\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ \Phi(\rho t_1)\ \Phi(\rho t_2)exp\left[-\frac{a^2}{2}(t_1^2+t_2^2)\right]dt_1dt_2$$
$$=\frac{2\pi}{a^2n}\sum_{j=1}^{n}\Psi(Y_{1j},Y_{2j})\overline{\Phi}(aY_{1j})\overline{\Phi}(aY_{2j})-\frac{2}{\rho}\left[\frac{\sqrt{2\pi}}{a}\int_{-\infty}^{\infty}\ \Phi\left(\frac{\rho t}{a}\right)\phi(t)dt\right]^2=$$
$$\frac{2\pi}{a^2n}\sum_{j=1}^{n}\Psi(Y_{1j},Y_{2j})\overline{\Phi}(aY_{1j})\overline{\Phi}(aY_{2j})-\frac{4\pi}{a^2\rho}E^2\left[\ \Phi\left(\frac{\rho T}{a}\right)\right].$$

where $\Psi(x,y)$ is as defined in (11), $\overline{\Phi}(a)=1-\Phi(a)$, and T is a standard normal variable. For any constants a and b, we have $E[\Phi(aT+b)]=\Phi(\frac{b}{\sqrt{1+a^2}}$ ([37], p 53-54). This gives $E\left[\ \Phi\left(\frac{\rho T}{a}\right)\right]=\Phi(0)=\frac{1}{2}$, and hence

$$\frac{4\pi}{a^2\rho}E^2\left[\ \Phi\left(\frac{\rho T}{a}\right)\right]=\frac{\pi}{a^2\rho}. \tag{19}$$

Replacing (19) in (18), the following computational form for $T_{1,n}$ is obtained

$$T_{1,n}=\frac{2\pi}{a^2n}\sum_{j=1}^{n}\Psi(Y_{1j},Y_{2j})\overline{\Phi}(aY_{1j})\overline{\Phi}(aY_{2j})-\frac{\pi}{\rho a^2}, \tag{20}$$

To derive a computational form for $T_{n,2}$, we have

$$\begin{aligned}T_{n,2}&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\widehat{\mu(t_1,t_2)}-\mu(t_1,t_2)\right]^2\omega(t_1,t_2)dt_1dt_2\\&=\underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\widehat{\mu^2(t_1,t_2)}\omega(t_1,t_2)dt_1dt_2}_{I_1}-\\&\underbrace{2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mu(t_1,t_2)\widehat{\mu(t_1,t_2)}\omega(t_1,t_2)dt_1dt_2}_{I_2}+\underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mu^2(t_1,t_2)\omega(t_1,t_2)dt_1dt_2}_{I_3}.\end{aligned} \tag{21}$$

We now evaluate each of I_1, I_2, and I_3.

$$\begin{aligned}I_1&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\widehat{\mu^2(t_1,t_2)}\omega(t_1,t_2)dt_1dt_2\\&=\frac{1}{n^2}\sum_{i,j=1}^{n}\Psi(Y_{1i},Y_{2i})\Psi(Y_{1j},Y_{2j})\int_{Y_{1i}\vee Y_{1j}}^{+\infty}\int_{Y_{2i}\vee Y_{2j}}^{+\infty}e^{-\frac{a^2}{2}(t_1{}^2+t_2{}^2)}dt_1dt_2\\&=\frac{2\pi}{n^2a^2}\sum_{i,j=1}^{n}\Psi(Y_{1i},Y_{2i})\Psi(Y_{1j},Y_{2j})\overline{\Phi}(a(Y_{1i}\vee Y_{1j}))\ \overline{\Phi}(a(Y_{2i}\vee Y_{2j})),\end{aligned} \tag{22}$$

where $x\vee y=\max(x,y)$.

$$I_2=\frac{4}{an}\sum_{i=1}^{n}\Psi(Y_{1i},Y_{2i})\int_{Y_{1i}}^{+\infty}\int_{Y_{2i}}^{+\infty}\Phi(\rho t_1)\ \Phi(\rho t_2)e^{-\frac{a^2}{2}(t_1{}^2+t_2{}^2)}dt_1dt_2.$$

Applying the change of variables $u=at_1, v=at_2$, one has

$$\begin{aligned}I_2&=\frac{2\pi}{an}\sum_{i=1}^{n}\Psi(Y_{1i},Y_{2i})\int_{Y_{1i}}^{+\infty}2\ \phi(u)\Phi\left(\frac{\rho}{a}u\right)\left(\int_{Y_{2i}}^{+\infty}2\phi(v)\Phi\left(\frac{\rho}{a}v\right)\ dv\right)du\\&=\frac{2\pi}{\rho na^2}\sum_{i=1}^{n}\Psi(Y_{1i},Y_{2i})\overline{G}\left(aY_{1i},\frac{\rho}{a}\right)\ \overline{G}\left(aY_{2i},\frac{\rho}{a}\right).\end{aligned} \tag{23}$$

Where G is the CDF of a skew-normal distribution with skewness parameter $\frac{\rho}{a}$ and $\overline{\mathrm{G}} = 1 - G$.

$$\begin{aligned} I_3 &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mu^2(\mathrm{t}_1,\mathrm{t}_2)\omega(\mathrm{t}_1,\mathrm{t}_2)dt_1 dt_2 \\ &= \frac{4}{\rho^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi^2(\rho t_1)\,\Phi^2(\rho t_2) e^{-\frac{a^2}{2}({\mathrm{t}_1}^2+{\mathrm{t}_2}^2)} dt_1 dt_2 \\ &= \frac{8\pi}{(a\rho)^2}\int_{-\infty}^{+\infty} \Phi^2\left(\frac{\rho}{a}u\right)\phi(u)\left(\int_{-\infty}^{+\infty}\Phi^2\left(\frac{\rho}{a}v\right)\phi(v)dv\right)du. \end{aligned}$$

From [38], we have

$$\int_{-\infty}^{+\infty}\Phi^2(bv)\phi(v)dv = \frac{1}{\pi}\arctan\left(\sqrt{1+2b^2}\right)$$

which brings I_3 to the simple form

$$I_3 = \frac{8}{(a\rho)^2\pi}\left(arctan\sqrt{1+2\left(\frac{\rho}{a}\right)^2}\right)^2. \tag{24}$$

Replacing (22), (23), and (24) in (21), we obtain the following form for $T_{2,n}$.

$$\begin{aligned} T_{2,n} = &\frac{2\pi}{n^2a^2}\sum_{i,j=1}^{n}\Psi(Y_{1i},Y_{2i})\Psi(Y_{1j},Y_{2j})\overline{\Phi}\big(a(Y_{1i}\vee Y_{1j})\big)\,\overline{\Phi}\big(a(Y_{2i}\vee Y_{2j})\big) \\ &-\frac{2\pi}{\rho n a^2}\sum_{i=1}^{n}\Psi(Y_{1i},Y_{2i})\bar{G}\left(aY_{1i},\frac{\rho}{a}\right)\,\bar{G}\left(aY_{2i},\frac{\rho}{a}\right) \\ &+\frac{8}{(a\rho)^2\pi}\left(arctan\sqrt{1+2\left(\frac{\rho}{a}\right)^2}\right)^2. \end{aligned} \tag{25}$$

When the parameters α_1, α_2 and ω are unknown, which is the case of our study, they must be replaced by their estimates. Based on a random sample from the BSN density given in (6), the maximum likelihood estimators of the parameters ω, α_1, and α_2 are solutions of the following likelihood equations:

$$\frac{\partial l}{\partial\omega} = \omega^3 - \frac{1}{n}\sum_{i,j=1}^{n}Y_{1j}Y_{2j}\ \omega^2 - \left(1-\frac{1}{n}\sum_{j=1}^{n}(Y_{1j}^2+Y_{2j}^2)\right)\omega - \frac{1}{n}\sum_{j=1}^{n}Y_{1j}Y_{2j} = 0 \tag{26}$$

$$\frac{\partial l}{\partial\alpha_1} = \sum_{j=1}^{n}\frac{Y_{1j}\phi\big(\alpha_1Y_{1j}+\alpha_2Y_{2j}\big)}{\Phi\big(\alpha_1Y_{1j}+\alpha_2Y_{2j}\big)} = 0, \tag{27}$$

$$\frac{\partial l}{\partial\alpha_2} = \sum_{j=1}^{n}\frac{Y_{2j}\phi\big(\alpha_1Y_{1j}+\alpha_2Y_{2j}\big)}{\Phi\big(\alpha_1Y_{1j}+\alpha_2Y_{2j}\big)} = 0. \tag{28}$$

The first likelihood equation is a cubic equation in ω, the other two equations are nonlinear equations in α_1 and α_2. These equations can be solved numerically. Caution must be taken so that the solution of ω must satisfy $|\omega - \delta_1\delta_2| < \sqrt{(1-\delta_1^2)(1-\delta_2^2)}$.

3. Critical Values and Power Simulation

Azzalini and Dalla Valle [7] showed that if Z_0 and Z_1 are independent $N(0,1)$ random variables, and $\delta \in (-1,1)$, then $Y = \delta|Z_0| + \sqrt{1-\delta^2}Z_1$ is distributed $SN\left(\frac{\delta}{\sqrt{1-\delta^2}}\right)$. They further proved that if $\boldsymbol{Z} = (Z_1, \dots, Z_K)^T$ is a k-dimensional MVN with standardized marginals, independent of $Z_0 \sim N(0,1)$, and if $Y_j = \delta_j|Z_0| + \sqrt{1-\delta_j^2}Z_j, j = 1, \dots, k$, then $Y = (Y_1, \dots, Y_K)^T$ is a k-dimensional MSN with density given in (6). Based on this result, we will generate bivariate skew-normal samples.

To generate a random sample of size n from a BSN distribution, we first generate a sample, say $z_{01}, \dots, z_{0n}$, from N(0,1), and then generate another sample, independent of the first, from bivariate normal , say $\binom{z_{11}}{z_{21}}, \dots, \binom{z_{1\mathrm{n}}}{z_{2\mathrm{n}}}$ with correlation coefficient ρ and standardized marginals. We then compute $\mathrm{y}_{1j} = \delta_1|z_{0j}| + \sqrt{1-\delta_1^2}z_{1j}$ and $\mathrm{y}_{2j} = \delta_2|z_{0j}| + \sqrt{1-\delta_2^2}z_{2\mathrm{j}}, j = 1, \dots, n$. Thus $\binom{y_{11}}{y_{21}}, \dots, \binom{y_{1\mathrm{n}}}{y_{2\mathrm{n}}}$ is a random sample from BSN with parameters $\theta = (\alpha_{1,}\alpha_2, \omega)$ where $\alpha_{1,}\alpha_2$ are defined in (7) and (8), and ω is the off diagonal element of Ω in (4). Based on this random sample, we numerically estimate θ by solving the likelihood equations (26) to (28) and then evaluate the proposed test statistics $T_{1,n}(\hat{\theta})$ and $T_{2,n}(\hat{\theta})$. For the sake of comparisons, we will also evaluate Meintanis and Hlávka (MH) test and Balakrishnan et. al. (BCS) canonical form test. The computational form for the MH test is given by

$$\begin{aligned} MH = &\frac{1}{n}\sum_{j,k=1}^{n}\left(Y_{1j}Y_{1k}\delta_2^2 + Y_{2j}Y_{2k}\delta_1^2 - 2Y_{1j}Y_{2k}\delta_1\delta_2\right)I_0(Y_{1jk})I_0(Y_{2jk}) \\ &+ \frac{1}{n}\sum_{j,k=1}^{n}\kappa_2^2 I_2(Y_{1jk})I_0(Y_{2jk}) + \kappa_1^2 I_2(Y_{2jk})I_0(Y_{1jk}) \\ &- 2\kappa_1\kappa_2 I_1(Y_{1jk})I_1(Y_{2jk}) \\ &+ \frac{2}{n}\sum_{j,k=1}^{n}\left((\delta_2\kappa_1 Y_{1j} - \delta_1\kappa_1 Y_{2j})I_1(Y_{2jk})I_0(Y_{1jk})\right) + (\delta_1\kappa_2 Y_{2j} \\ &- \delta_2\kappa_2 Y_{1j})I_1(Y_{1jk})I_0(Y_{2jk}). \end{aligned} \tag{29}$$

Where $Y_{mjk} = Y_{mj} + Y_{mk}, m = 1,2$, $\kappa_1 = \delta_1 - \omega\delta_2, \kappa_2 = \delta_2 - \omega\delta_1$, and

$$I_s(Z) = \int_{-\infty}^{+\infty} t^s e^{tZ} w(t)dt \,, w(t) = e^{at^2}, a > 0, s = 0,1,2.$$

According to Meintanis and Hlávka [35], the best power of the test in (29) was attained at $a = 2$. We notice that $T_{1,n}$ and $T_{2,n}$ have simpler computational forms compared to M-H $T_{n,\omega}$ test. It is worth noting that all test values are calculated for the MLEs of the parameters obtained from likelihood equations (26)-(28).

Balakrishnan et al. [36] developed three tests based on the so-called canonical form of the MSN distribution, in which the location is the zero vector and the scale is the identity matrix. For the purpose of comparisons to our proposed tests, we will consider only one of the three tests; the sum test, because it generally performed better than the other two tests. In the bivariate case, the test is the sum of the rations of the second component divided by the absolute value of the first component. For a sample of *n* bivariate observations, the test has the following form:

$$BCS = \sum_{j=1}^{n} \frac{y_{2j}^{*}}{\left|y_{1j}^{*}\right|} \tag{30}$$

Mathematica codes will be developed to compute critical values for $T_{1,n}$, $T_{2,n}$, MH and BCS and to calculate the powers of these tests when testing against some competing bivariate alternatives. Simulated critical values for the proposed tests $T_{1,n}$ and $T_{2,n}$ were evaluated for some fixed parameters but treated as unknowns parameters, the results are displayed in Table 1.

Table 1. Critical values for $T_{1,n}^{a}$ and $T_{2,n}^{a}$ at nominal level $\alpha = 0.05$ based on 1000 samples of size 20 and 50 simulated from BSN distribution for ω =0.3 and different choices of $\delta_1, \delta_2,$ and for $a = 0.1, 0.5,$ and 1.0.

				$T_{1,n}^{a}$				$T_{2,n}^{a}$	
n	a	δ_1	δ_2	**2.5%**	**5%**	**95%**	**97.5%**	**90%**	**95%**
20		0.20	0.30	-51.32	-41.14	21.22	48.30	8.40	13.78
		0.20	0.20	-44.58	-35.82	22.33	34.23	6.91	11.26
		0.70	0.80	-171.65	-169.62	-16.80	3.53	147.27	157.41
		0.80	0.80	-173.47	-172.47	-87.71	-74.58	159.99	163.00
	0.1	0.10	0.90	-166.71	163.59	-14.68	3.34	139.77	151.24
		0.20	0.30	-2.48	-2.01	1.32	1.74	0.61	0.77
		0.20	0.20	-2.20	-1.77	1.19	1.65	0.55	0.73
		0.70	0.80	-9.05	-8.86	-1.22	-0.53	6.35	6.82
		0.80	0.80	-9.18	-9.08	-5.04	-4.54	6.90	7.19
	0.5	0.10	0.90	-8.04	-7.79	-0.63	-0.15	5.10	5.66
		0.20	0.30	-0.81	-0.63	0.45	0.54	0.21	0.28
		0.20	0.20	-0.82	-0.61	0.41	0.55	0.20	0.26
		0.70	0.80	-2.64	-2.58	-0.41	-0.15	1.53	1.70
		0.80	0.80	-2.72	-2.69	-1.86	-1.40	1.78	1.84
	1.0	0.10	0.90	-2.34	-2.24	-1.54	0.04	1.13	1.32
50		0.20	0.30	-21.21	-16.36	13.64	19.67	2.20	3.06
		0.20	0.20	-21.03	-16.12	12.49	17.16	1.99	2.88
		0.70	0.80	-135.81	-121.58	-6.14	11.03	64.27	79.03
		0.80	0.80	-169.99	-168.40	-99.91	-84.41	150.44	155.47
	0.1	0.10	0.90	-126.59	-117.55	0.54	22.26	65.17	81.84
		0.20	0.30	-1.09	-0.96	-0.86	0.74	1.02	0.27
		0.20	0.20	-0.96	-0.75	-0.71	0.91	0.20	0.24

	0.70	0.80	-7.41	-6.95	-0.84	0.32	3.11	4.04
	0.80	0.80	-8.84	-8.71	-5.50	-4.94	6.33	6.57
0.5	0.10	0.90	-6.31	-5.82	-0.08	1.46	2.68	3.47
	0.20	0.30	-0.30	-0.26	0.19	0.31	0.07	0.09
	0.20	0.20	-0.34	-0.26	0.22	0.28	0.07	0.08
	0.70	0.80	-2.26	-2.15	-0.23	0.21	0.94	1.15
	0.80	0.80	-2.62	-2.58	-1.71	-1.59	1.60	1.67
1.0	0.10	0.90	-1.82	-1.73	-0.09	0.18	0.63	0.76

The calculations are based on samples of sizes 20 and 50. Due to the time it took to run the simulation codes, we only simulate 1000 samples. The simulated power value, which is the proportion of rejections, is calculated at the significance level $\alpha = 0.05$. Thus, the rejection region for the two-sided test $T_{1,n}$is the region outside its 2.5$^{\text{th}}$ and 97.5$^{\text{th}}$ percentiles, and for the one-sided test $T_{2,n}$ is the interval to the right of its 95$^{\text{th}}$ percentile.

To verify whether the proposed tests recover the nominal value $\alpha = 0.05$, we calculated the power of testing the BSN against itself and presented the results in Table 2. We can see from this table that the two proposed tests mostly recover the nominal value 0.05, however, in some cases the tests are somewhat liberal or slightly conservative.

Table 2. Calculated power for $T_{1,n}^{a}$ and $T_{2,n}^{a}$ based on samples of sizes 20 and 50 when testing the BSN against itself for various values of the parameters $\boldsymbol{\delta} = (\boldsymbol{\delta_1}, \boldsymbol{\delta_2})$ and $\boldsymbol{\omega}$ values and for $\boldsymbol{a} = \mathbf{0.1}, \mathbf{0.5}$, and $\mathbf{1.0}$.

n		20						50					
(δ_1, δ_2)	ω	$T_{1,n}^{0.1}$	$T_{2,n}^{0.1}$	$T_{1,n}^{0.5}$	$T_{2,n}^{0.5}$	$T_{1,n}^{1}$	$T_{2,n}^{1}$	$T_{1,n}^{0.1}$	$T_{2,n}^{0.1}$	$T_{1,n}^{0.5}$	$T_{2,n}^{0.5}$	$T_{1,n}^{1}$	$T_{2,n}^{1}$
	0.3	0.05	0.05	0.05	0.05	0.03	0.03	0.06	0.06	0.05	0.06	0.07	0.07
(0.2, 0.2)	0.4	0.06	0.06	0.05	0.05	0.06	0.06	0.05	0.05	0.05	0.05	0.05	0.05
	0.5	0.05	0.05	0.05	0.04	0.04	0.05	0.04	0.04	0.05	0.05	0.05	0.05
	0.3	0.06	0.05	0.06	0.06	0.06	0.05	0.05	0.06	0.05	0.04	0.07	0.07
(0.1, 0.9)	0.4	0.04	0.06	0.05	0.04	0.07	0.04	0.05	0.06	0.04	0.05	0.04	0.05
	0.5	0.04	0.04	0.06	0.05	0.05	0.06	0.03	0.04	0.05	0.06	0.06	0.06
	0.3	0.004	0.05	0.04	0.04	0.05	0.05	0.04	0.05	0.05	0.06	0.04	0.06
(0.8, 0.8)	0.4	0.07	0.06	0.05	0.07	0.05	0.04	0.06	0.05	0.06	0.05	0.05	0.04
	0.5	0.06	0.06	0.06	0.05	0.04	0.05	0.05	0.06	0.05	0.05	0.05	0.06

The power calculations of $T_{1,n}^{a}$and $T_{2,n}^{a}$; $a = 0.1, 0.5,$ and $1.0, n = 20$ and $50,$ when testing a BSN with $\delta_1 = 0.2, \delta_2 = 0.3$ and $\omega = 0.3$, against each of BSL (bivariate skew-Laplace), BST_4 (Bivariate skew-T with 4 degrees of freedom), and BST_9 (Bivariate skew-T with 9 degrees of freedom) for distinct parameters δ_1, δ_2 and ω, are displayed in Tables 3-5, respectively. We note that for all values of ω and a considered, the power increases considerably when at least one of the two components of δ increases. We also see that the power generally tends to decrease with ω. Although there is no obvious pattern for the power of the two tests when a

varies, nevertheless, the average power of each of $T^a_{1,n}$and $T^a_{2,n}$ is higher when $a = 0.5$ than when $a = 0.1$ or $a = 1.0$. Unsurprisingly, Tables 3-5 show that the power increases with n which reveals the consistency of the tests. The performance of $T^a_{1,n}$ and $T^a_{2,n}$ is roughly the same when $= 20$, but, $T^a_{2,n}$ performs significantly better than $T^a_{1,n}$for most cases when $n = 50$.

Table 3. Calculated power for $T^a_{1,n}$ and $T^a_{2,n}$, $\mathbf{a = 0.1, 0.5, and\ 1.0}$, based on samples of sizes 20 and 50 when testing the BSN, with different values of $\boldsymbol{\delta} = (\boldsymbol{\delta_1}, \boldsymbol{\delta_2})$ and $\boldsymbol{\omega}$, against BSL.

n		20						50					
(δ_1, δ_2)	ω	$T^{0.1}_{1,n}$	$T^{0.1}_{2,n}$	$T^{0.5}_{1,n}$	$T^{0.5}_{2,n}$	$T^{1}_{1,n}$	$T^{1}_{2,n}$	$T^{0.1}_{1,n}$	$T^{0.1}_{2,n}$	$T^{0.5}_{1,n}$	$T^{0.5}_{2,n}$	$T^{1}_{1,n}$	$T^{1}_{2,n}$
	0.3	0.16	0.14	0.13	0.17	0.10	0.13	0.23	0.27	0.18	0.39	0.13	0.44
(0.2, 0.2)	0.4	0.18	0.14	0.15	0.18	0.11	0.13	0.31	0.36	0.25	0.43	0.18	0.44
	0.5	0.19	0.16	0.16	0.18	0.12	0.14	0.31	0.36	0.25	0.43	0.18	0.44
	0.3	0.54	0.59	0.56	0.61	0.47	0.40	0.81	0.92	0.84	0.91	0.79	0.83
(0.1, 0.9)	0.4	0.62	0.66	0.62	0.64	0.51	0.44	0.85	0.94	0.87	0.94	0.83	0.77
	0.5	0.73	0.76	0.76	0.74	0.64	0.58	0.81	0.92	0.91	0.96	0.88	0.92
	0.3	0.84	0.85	1.00	1.00	1.00	1.00	1.00	1.00	0.96	1.00	0.96	0.98
(0.8, 0.8)	0.4	0.86	0.87	1.00	0.99	0.99	0.99	1.00	1.00	0.96	1.00	0.97	0.99
	0.5	0.86	0.87	0.96	0.96	0.96	0.92	0.97	0.99	0.96	1.00	0.97	0.99

Table 4. Calculated power for $T^a_{1,n}$ and $T^a_{2,n}$, $\mathbf{a = 0.1, 0.5, and\ 1.0}$, based on samples of sizes 20 and 50 when testing the BSN, with different values of $\boldsymbol{\delta} = (\boldsymbol{\delta_1}, \boldsymbol{\delta_2})$ and $\boldsymbol{\omega}$, against BST_{υ}, for $\boldsymbol{\upsilon = 4}$.

n		20						50					
(δ_1, δ_2)	ω	$T^{0.1}_{1,n}$	$T^{0.1}_{2,n}$	$T^{0.5}_{1,n}$	$T^{0.5}_{2,n}$	$T^{1}_{1,n}$	$T^{1}_{2,n}$	$T^{0.1}_{1,n}$	$T^{0.1}_{2,n}$	$T^{0.5}_{1,n}$	$T^{0.5}_{2,n}$	$T^{1}_{1,n}$	$T^{1}_{2,n}$
	0.3	0.10	0.12	0.11	0.16	0.13	0.13	0.19	0.32	0.22	0.36	0.33	0.27
(0.2, 0.2)	0.4	0.11	0.14	0.09	0.15	0.13	0.12	0.18	0.28	0.18	0.38	0.32	0.26
	0.5	0.13	0.16	0.10	0.16	0.11	0.11	0.38	0.51	0.21	0.40	0.34	0.28
	0.3	0.78	0.83	0.81	0.86	0.75	0.76	0.92	0.98	0.92	0.99	0.93	0.99
(0.1, 0.9)	0.4	0.83	0.87	0.83	0.87	0.77	0.77	0.93	0.99	0.94	0.99	0.93	0.99
	0.5	0.88	0.91	0.87	0.90	0.82	0.91	0.95	0.99	0.95	1.00	0.95	0.99
	0.3	0.88	0.89	1.00	1.00	0.77	0.78	1.00	1.00	0.87	0.93	0.89	0.93
(0.8, 0.8)	0.4	0.89	0.90	0.96	0.96	0.78	0.79	0.96	0.99	0.89	0.94	0.90	0.94
	0.5	0.82	0.85	0.91	0.94	0.91	0.91	0.92	0.97	0.90	0.94	0.90	0.94

Table 5. Calculated power for $T_{1,n}^{a}$ and $T_{2,n}^{a}$, $\mathbf{a = 0.1, 0.5}$, and $\mathbf{1.0}$, based on samples of sizes 20 and 50 when testing the BSN, with different values of $\boldsymbol{\delta = (\delta_1, \delta_2)}$ and $\boldsymbol{\omega}$, against BST_{υ}, for $\boldsymbol{\upsilon = 9}$.

n		20						50					
(δ_1, δ_2)	ω	$T_{1,n}^{0.1}$	$T_{2,n}^{0.1}$	$T_{1,n}^{0.5}$	$T_{2,n}^{0.5}$	$T_{1,n}^{1}$	$T_{2,n}^{1}$	$T_{1,n}^{0.1}$	$T_{2,n}^{0.1}$	$T_{1,n}^{0.5}$	$T_{2,n}^{0.5}$	$T_{1,n}^{1}$	$T_{2,n}^{1}$
	0.3	0.11	0.13	0.08	0.13	0.05	0.05	0.13	0.21	0.15	0.33	0.29	0.23
(0.2, 0.2)	0.4	0.10	0.12	0.09	0.14	0.06	0.05	0.17	0.25	0.17	0.29	0.26	0.20
	0.5	0.10	0.12	0.09	0.15	0.06	0.05	0.31	0.39	0.21	0.27	0.25	0.19
	0.3	0.86	0.89	0.81	0.85	0.75	0.74	0.94	0.98	0.92	0.98	0.95	0.99
(0.1, 0.9)	0.4	0.86	0.89	0.84	0.87	0.78	0.77	0.95	0.98	0.94	0.99	0.95	0.99
	0.5	0.88	0.91	0.89	0.91	0.84	0.83	0.96	0.99	0.96	0.99	0.95	0.99
	0.3	0.91	0.92	1.00	1.00	1.00	1.00	0.94	0.98	0.89	0.95	0.91	0.95
(0.8, 0.8)	0.4	0.84	0.85	0.96	0.96	0.96	0.95	0.94	0.98	0.90	0.95	0.92	0.95
	0.5	0.80	0.82	0.89	0.89	0.89	0.87	0.92	0.96	0.90	0.95	0.92	0.96

Now that we have compared $T_{1,n}^{a}$ and $T_{2,n}^{a}$ tests at different parameter's values and found that the second test is generally better than the first and noted that both tests perform best when $a = 0.5$ compared to $a = 0.1$ and $a = 1.0$, we will now compare the performance of $T_{1,n}^{0.5}$ and $T_{2,n}^{0.5}$ with that of MH and BCS tests in (29) and (30). In addition to BSL, BST_4, and BST_9, which were defined earlier on page 10, we consider the following alternatives :

1) BN1: Bivariate normal distribution with marginal means 0, marginal variances 1, and correlation $\rho = -0.7$.
   BN2: Bivariate normal distribution with marginal means 0, marginal variances 1, and correlation $\rho = 0.8$
2) BL1: Bivariate Laplace distribution with marginal mean 0**,** marginal variances 1, and correlation $\rho = -0.7$.
   BL2: Bivariate Laplace distribution with marginal mean 0**,** marginal variances 1, and correlation $\rho = 0.8$.
3) The bivariate Feller- Pareto distribution for the following cases:
   a) BF1: $X_i = \left(\frac{W_i}{U}\right)^{c_i}, i = 1, 2$, where U, W_1 , and W_2 are independent random variables with $U \sim gamma(3,1), W_i \sim gamma(1,1)$, and $c_i = 1; i = 1,2$.
   b) BF2: $X_i = \left(\frac{W_i}{U}\right)^{c_i}, i = 1, 2$, where U, W_1 , and W_2 are independent random variables with $U \sim gamma(2,1), W_i \sim gamma(1,1)$, and $c_i = 0.5; i = 1,2$.
   c) BF3: $X_i = \left(\frac{W_i}{U}\right)^{c_i}, i = 1, 2$, where U, W_1 , and W_2 are independent random variables with $U \sim gamma(3,1), W_1 \sim gamma(0.5,1)$, $W_2 \sim gamma(1.5,1)$ and $c_i = 0.5; i = 1,2$.

Table 6 displays simulated powers when testing the BSN against the BSL. Tables 7 and 8 display the powers when testing the BSN against BST_{υ} for $\upsilon = 4$ and $\upsilon = 9$, respectively. The powers of $T_{1,n}^{0.5}$, $T_{2,n}^{0.5}$, and MH when testing the BSN with $\delta_1 = 0.2, \delta_2 = 0.3$ and $\omega = 0.3$ against each of BN1, BN2, BL1, BL2, BF1, BF2, and BF3 are shown in Table 9.

Table 6. The calculated power for $T_{1,n}^{0.5}$, $T_{2,n}^{0.5}$, MH and BCS, based on samples of sizes 20 and 50 when testing the BSN against BSL.

(δ_1, δ_2)	ω	$T_{1,n}^{0.5}$	$T_{1,n}^{0.5}$	MH	BCT	$T_{1,n}^{0.5}$	$T_{1,n}^{0.5}$	MH	BCT
	0.3	0.13	0.17	0.38	0.056	0.18	0.39	0.61	0.058
(0.2, 0.2)	0.4	0.15	0.18	0.38	0.041	0.25	0.43	0.61	0.045
	0.5	0.16	0.18	0.39	0.038	0.25	0.43	0.62	0.052
	0.3	0.56	0.61	0.81	0.038	0.84	0.91	0.98	0.050
(0.1, 0.9)	0.4	0.62	0.64	0.83	0.042	0.87	0.94	0.99	0.063
	0.5	0.76	0.74	0.83	0.042	0.91	0.96	0.99	0.057
	0.3	1.00	1.00	0.95	0.056	0.96	1.00	1.00	0.080
(0.8, 0.8)	0.4	1.00	0.99	0.95	0.045	0.96	1.00	1.00	0.085
	0.5	0.96	0.96	0.94	0.066	0.96	1.00	1.00	0.070

Table 7. The calculated power for $T_{1,n}^{0.5}$, $T_{2,n}^{0.5}$, and MH, based on samples of sizes 20 and 50 when testing the BSN with different choices of δ_1, δ_2 , and ω against BST_υ, for $\upsilon = 4$.

(δ_1, δ_2)	ω	$T_{1,n}^{0.5}$	$T_{1,n}^{0.5}$	MH	BCT	$T_{1,n}^{0.5}$	$T_{1,n}^{0.5}$	MH	BCT
	0.3	0.11	0.16	0.66	0.052	0.22	0.36	0.88	0.057
(0.2, 0.2)	0.4	0.09	0.15	0.53	0.034	0.18	0.38	0.88	0.058
	0.5	0.10	0.16	0.50	0.040	0.21	0.40	0.87	0.049
	0.3	0.81	0.86	0.81	0.058	0.92	0.99	0.98	0.062
(0.1, 0.9)	0.4	0.83	0.87	0.71	0.056	0.94	0.99	0.98	0.070
	0.5	0.87	0.90	0.69	0.08	0.95	1.00	0.97	0.092
	0.3	1.00	1.00	0.82	0.056	0.87	0.93	0.90	0.102
(0.8, 0.8)	0.4	0.96	0.96	0.84	0.057	0.89	0.94	0.91	0.111
	0.5	0.91	0.94	0.84	0.065	0.90	0.94	0.92	0.093

Table 8. The calculated power for $T_{1,n}^{0.5}$, $T_{2,n}^{0.5}$, and MH, based on samples of sizes 20 and 50 when testing the BSN with different choices of δ_1, δ_2 , and ω against BST_{υ}, for $\upsilon = 9$.

n		20				50			
(δ_1, δ_2)	ω	$T_{1,n}^{0.5}$	$T_{2,n}^{0.5}$	MH	BCT	$T_{1,n}^{0.5}$	$T_{2,n}^{0.5}$	MH	BCT
	0.3	0.08	0.13	0.47	0.039	0.15	0.33	0.77	0.05
(0.2, 0.2)	0.4	0.09	0.14	0.48	0.032	0.17	0.29	0.71	0.056
	0.5	0.09	0.15	0.47	0.038	0.21	0.27	0.66	0.053
	0.3	0.81	0.85	0.64	0.058	0.92	0.98	0.80	0.069
(0.1, 0.9)	0.4	0.84	0.87	0.66	0.072	0.94	0.99	0.81	0.081
	0.5	0.89	0.91	0.66	0.068	0.96	0.99	0.82	0.092
	0.3	1.00	1.00	0.64	0.057	0.89	0.95	0.92	0.10
(0.8, 0.8)	0.4	0.96	0.96	0.60	0.064	0.90	0.95	0.91	0.09
	0.5	0.89	0.89	0.59	0.072	0.90	0.95	0.90	0.092

Table 9. The calculated power for $T_{1,n}^{0.5}$, $T_{2,n}^{0.5}$, and MH, based on samples of sizes 20 and 50 when testing the BSN with $\delta_1 = \delta_2 = 0.2$, and $\omega = 0.3$ against the alternatives BN, BL, and BP distributions with sample sizes $n = 20$ and 50.

n	20				50			
$Alt.$	$T_{1,n}^{0.5}$	$T_{2,n}^{0.5}$	MH	BCT	$T_{1,n}^{0.5}$	$T_{2,n}^{0.5}$	MH	BCT
BN1	0.13	0.11	0.07	0.05	0.20	0.13	0.03	0.07
BN2	0.13	0.11	0.05	0.04	0.18	0.14	0.04	0.052
BL1	0.27	0.25	0.31	0.04	0.40	0.49	0.39	0.05
BL2	0.26	0.25	0.30	0.05	0.42	0.50	0.39	0.06
BF1	0.90	0.85	0.60	0.12	0.96	0.93	0.83	0.16
BF2	0.99	0.99	0.43	0.09	1.00	1.00	0.66	0.12
BF3	0.96	0.95	0.83	0.08	1.00	0.993	0.99	0.11

4. Real data applications

In this section, we apply the suggested procedures, along with the MH and BCS procedures, to test certain real-world data for compliance with the BSN distribution. Two sets of data will be processed; Australian Institute of Sport (AIS) data [39] and Old Faithful Geyser (OFG) data [40]. AIS data consists of 13 biomedical variables measured on 100 female athletes and 102 male athletes. The OFG data consists of 272 observations that measure the duration of eruption and the inter-eruption times for the Old Faithful geyser in Yellowstone National Park, USA [8, 36, 39, 40]. We will apply the T_1, T_2, MH, and BCS tests to eight pairs of AIS variables (male and female analyzed separately), namely, body fat percentage and lean body mass (Bfat, LBM), body mass index and sum of skin folds (BMI, SSF), red blood cell count and white blood cell count (RCC, WCC), and plasma ferritin concentration and Hematocrit (Fe, Hc). We will also apply the tests to the eruption times and the inter-eruption times (ET, IET) of the OFG data.

The bootstrap procedure is used to test the null hypothesis that a given set of data fits the BSN distribution. To compute the p-value for a right-tail test, say T, we take the following steps:

i) For a given sample, estimate the BSN parameters including the location and scale parameters, standardize the data, and then evaluate the statistic for the estimated parameter $\widehat{\boldsymbol{\theta}} = \widehat{(\omega}, \widehat{\alpha_1}, \widehat{\alpha_2})$. Denote this by $T(\widehat{\boldsymbol{\theta}})$.

ii) Simulate B samples from BSN with the parameter $\widehat{\boldsymbol{\theta}} \equiv (\widehat{\omega}, \widehat{\alpha_1}, \widehat{\alpha_2})$ obtained in (i), then for each sample calculate the bootstrap MLE of $\boldsymbol{\theta}$, say $\hat{\boldsymbol{\theta}}^*$, and then calculate $T(\hat{\boldsymbol{\theta}}^*)$.

iii) The approximate p-value is calculated as the proportion of times the value of $T(\hat{\boldsymbol{\theta}}^*)$ exceeds $T(\widehat{\boldsymbol{\theta}})$.

The calculated p-values when applying T_1, T_2, *MH*, and *BCS* test to FOG data and to each of the pairs (Bfat,LBM), (BMI,SSF), (RCC,WCC), (Fe,Hc), and (Ht,Wt) are displayed in Table 10.

For the male athletes data, T_1 and *BCS* lead to the same conclusion in favor of the BSN distribution while T_2 and *MH* lead to the same conclusion but against the BSN. For the female athletes data, T_1 and *BCS* lead to the same conclusion in favor of BSN for (Bfat, LBM), (BMI, SSF) and to the same conclusion against BSN for (Fe, Hc), but have different conclusions for (RCC, WCC). On the other hand, T_2 and *MH* agree against BSN for both male and female (Bfat, LBM) and (BMI, SSF) data, disagree for (RCC, WCC) and (Fe, Hc). The four tests, T_1, T_2, *MH*, and *BCS* lead to the same conclusion in favor of the BSN for the (ET,IET) data.

Table 10. P-values of tests of the BSN for eight pairs of variables from the AIS data set and for FOG data.

Data	(Bfat, LBM)		(BMI, SSF)		(RCC, WCC)		(Fe, Hc)		(ET,IET)
	Male	Female	Male	Female	Male	Female	Male	Female	
$T_{1,n}^{0.5}$	0.99	1.00	0.99	1.00	0.99	0.001	1.00	0.00	0.80
$T_{2,n}^{0.5}$	0.001	0.001	0.003	0.001	0.00	0.001	0.000	0.001	0.25
MH	0.002	0.020	0.002	0.002	0.001	0.42	0.002	0.67	0.49
BCT	0.73	0.48	0.22	0.48	0.28	0.75	0.64	0.04	0.84

5. Conclusions

We have proposed two tests based on a partial functional mean characterization. Simulation were used to calculate critical values for the proposed tests as well as for the *MH* and the *BCS* test. Meintanis and Hlávka [26] calculated critical values and powers of their *MH* test using bootstrap. Balakrishnan et al [8] used bootstrap to calculate p-values of their proposed tests to compare with the *MH* test performance. From the power calculations displayed in Tables 2-8 and in Table 10, we have the following conclusions:

1. The proposed tests could recover the nominal value $\alpha = 0.05$ for all but few cases as can be seen in Table 2.
2. Tables 3-5 tell that there is a slight variation in the powers of $T_{1,n}^{a}$ and $T_{2,n}^{a}$ for the three choices of $a = 0.1, a = 0.5,$ and $a = 1.0$. However, in most cases, $T_{1,n}^{a}$ and $T_{2,n}^{a}$ have better performance when $a = 0.5$.
3. It can be concluded from Table 6 that MH performs better than $T_{1,n}^{0.5}$ and $T_{2,n}^{0.5}$ when testing the BSN with $\delta_1 = \delta_2 = 0.2$ against the BSL distribution. The three tests perform as well when $\delta_1 = \delta_2 = 0.8$.
4. From Tables 7-8, we note that MH outperforms $T_{1,n}^{0.5}$ and $T_{2,n}^{0.5}$ when testing the BSN with $\delta_1 = \delta_2 = 0.2$ against the BST_{υ}, for $\upsilon = 4$ and 9, while $T_{1,n}^{0.5}$ and $T_{2,n}^{0.5}$ outperform MH when $\delta_1 = 0.1$ and $\delta_2 = 0.9$ and when $\delta_1 = \delta_2 = 0.8$.The tests $T_{1,n}^{0.5}$ and $T_{2,n}^{0.5}$ outperform the MH test when testing the BSN with $\delta_1 = \delta_2 = 0.2$ and $\omega = 0.5$ against each of BN1, BN2, BL1, BL2, BF1, and BF2, while MH performs better than $T_{1,n}^{0.5}$ and $T_{2,n}^{0.5}$ when testing against BF3 for $n = 50$ but not for $n = 20$.
5. We note that generally, MH performs better when $\delta_1$and δ_2 are both small, while $T_{1,n}^{0.5}$ and $T_{2,n}^{0.5}$ work better, when either of the two parameters is small and the other is large.
6. Except for few cases, the BCT is outperformed by the other three tests.

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