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Properties and Applications of the Kumaraswamy Distribution Based on Upper Record Values

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Abstract

The Kumaraswamy distribution has attracted the attention of researchers from many fields of application because of its mathematical tractability and its flexibility to accommodate data in various shapes. The distribution is a beta-type distribution, however, the closed-form of its distribution function gives it an advantage over the beta distribution. The aim of this article is to study record values from the Kumaraswamy distribution. The distribution, conditional distribution, moments, mixed moments, and conditional moments of the upper records and, moregenerally, the k^{th} upper records are derived. We also prove some characterizations of the Kumaraswamy distribution based on the upper records. The maximum likelihood, the modified maximum likelihood, and the conditional median predictors of future records are given. A bootstrap procedure is applied to fit the Kumaraswamy distribution to a real dataset: rainfall for the month of January from 1982 to 2021, recorded at the Changi climate station in Singapore, and then future rainfall records are predicted.

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1. Introduction

Kumaraswamy (1980) derived a probability distribution function for double-bounded randomprocesses as a model to the storage volume of a reservoir of a given capacity. A random variable
X is said to have a Kumaraswamy distribution function, denoted by $X \sim K(p, q)$, if

$$F(x; p, q) = 1 - (1 - x^p)^q \tag{1.1}$$

and, therefore, the probability density function (pdf) is given by

$$f(x;p,q) = pqx^{p-1}(1-x^p)^{q-1}, p,q > 0 \quad (1.2)$$

In the last few years, many researchers have studied the distribution; Nadarajah (2008) pointed out that the Kumaraswamy distribution is a special case of a more general class of distributions. Jones (2009) studied moments, maximum likelihood, limit distributions, and the relationship of the distribution with other known distributions. Mitnik (2013) examined some properties of the distribution, including setting bounds for the variance and the mean of the absolute deviation from the median. Using the Monte-Carlo simulation, Gholizadeh et al. (2011) studied classical and Bayesian point and interval estimators for the shape parameter of the distribution. Estimation of the parameters under progressive type-II censored data was carried out by Gholizadeh et al. (2013). Further maximum likelihood and Bayesian estimation studies were conducted by Hussian (2014) who used simulation to compare estimators based on simple randomsamples and ranked set samples. El-Morshedy et al. (2020) proposed a new three-parameter generalized model of the inverse Gompertz distribution known as the Kumaraswamy inverse Gompertz distribution. Al-Babtain et al. (2021) considered estimation of different types ofentropies for the Kumaraswamy distribution.

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Records occur naturally in many fields of study, including science, sports, engineering, medicine, economics, and industry. Their applications have attracted scientists from a variety of disciplines. An upper (lower) record value in a data sequence is the value that is greater (less) thanall previous values. That is, if $\{X_n\ n \geq 1\}$ is a sequence of random variables and $Y_n = max\{X_1, X_2, \dots, X_n\}, n \geq 1$, then X_j is an upper record value if $X_j > Y_{j-1}$ and is a lower record value if $X_j < Y_{j-1}, j > 1$. It is sufficient to handle either upper records or lower records, because if X_j is an upper record value, then $-X_j$ is a lower record value. In this article, we will only deal with upper record values and simply refer to them as record values. The times at which records appear are called record times and are defined by $U(n) = \min\{j: j > U(n-1), X_j > X_{U(n-1)}, n > 1\}$, with $U(1) = 1$ with probability 1. By definition X_1 is an upper as well as a lower record value.The sequence $X_{U(n)}, n \geq 1$ then defines an upper record value sequence.

A natural extension of record values is the k^{th} records. Suppose $X_{1:1} \leq \cdots X_{n:n}$ be the order statistic of the sample $X_1, X_2, \dots, X_n$. For fixed $k \geq 1$, the k^{th} upper record times $U_k(n), n \geq 1$, of the sequence $\{X_i, i \geq 1\}$, are defined as $U_k(1) = 1$, and

$$U_k(n+1) = \min\{j > U_k(n): X_{j:j+k-1} > X_{U_k(n):U_k(n)+k-1}\}, n = 1,2,...,$$

and the k^{th} upper record values as $Y_n^{(k)} = X_{U_k(n):U_k(n)+k-1,} n = 1, 2, ...$

In addition to the statistical inference research that was conducted based on random samples from the KD, many papers also examined statistical inference based on record values fromthis distribution. Nadar et al. (2012) obtained MLE and Bayesian estimates of the KD shape parameters based on a sequence of upper record values. They also used Bayesian and non-Bayesianprediction methods to predict future record values. Similar estimation and prediction problems were examined by Kizilaslan and Nadar (2015) based on record values and inter-record times. Abou-Elheggag et al. (2018) studied the estimation of $P(X < Y)$ when X and Y are independent record values of the KD.

This article is organized as follows. In Section 2, we derive the distribution, conditional distribution, moments, and conditional moments of ordinary, and more generally, of k^{th} upper record values. In Section 3, new characterization of the Kumaraswamy distribution based on record values will be proved. Maximum likelihood, modified maximum likelihood, and conditionalmedian point predictors will derived.

2. Distribution and Moments of Record Values from $K(p, q)$

In subsection 2.1, we derive the pdf of record values and the k^{th}record values from the $K(p, q)$ distribution. In addition, joint and conditional pdfs of record values and k^{th} records will be presented. The moments and conditional moments of records will be derived in subsection 2.2.

2.1 Distribution of Record Values from $K(p, q)$

If $F(x)$ is an absolutely continuous function of a random variable X, then the distribution of the record value $Y_n, n = 1,2, ...$ is given by (Arnold et al. 1998, p.10)

$$F_{Y_n}(y) = \Gamma(n, H(y)), \tag{2.1}$$

where $H(y) = -\log[1 - F(y)]$ is the cumulative hazard rate function, and $\Gamma(a, y)$ is the incomplete gamma function given by

$$\Gamma(a, x) = \int_0^y t^{a-1} e^{-at} dt, \tag{2.2}$$

Therefore, the pdf of $Y_n,\ n = 1,2,\ldots,$ is

$$f_{Y_n}(y) = \frac{h(y)}{\Gamma(n)}[H(y)]^{n-1}e^{-H(y)}, y > 0, \tag{2.3}$$

where $h(y)$ is the hazard rate function

$$h(y) = \frac{f(y)}{1 - F(y)} \tag{2.4}$$

Since $e^{-H(y)} = 1 - F(y)$, (2.3) is equivalent to

$$H(y) = -q\log(1 - y^p), \tag{2.5}$$

$$f_{Y_n}(y) = \frac{[H(y)]^{\,n-1}}{\Gamma(n)}f(y). \tag{2.6}$$

Thus,

$$f_{Y_n}(y) = \frac{q^n p\,[-\log(1 - y^p)]^{n-1}(1 - y^p)^{q-1}y^{p-1}}{\Gamma(n)}, 0 < y < 1. \tag{2.7}$$

Obviously, for $n = 1$, we obtain the PDF of the KD distribution.
The PDF of the k^{th} record value $Y_n^{(k)}, n \geq 1$, is given by (Ahsanullah, page 75)

$$\begin{aligned} f_{Y_n^{(k)}}(y) &= \frac{k^n}{\Gamma(n)}[H(y)]^{n-1}(1 - F(y))^{k-1}f(y) \\ &= \frac{k^n}{\Gamma(n)}[H(y)]^{n-1}[1 - F(y)]^k h(y), \end{aligned} \tag{2.8}$$

and the joint PDF of $Y_m^{(k)}$ and $Y_n^{(k)}, 1 \leq m < n$, is

$$\begin{aligned} f_{Y_m^{(k)},Y_n^{(k)}}(y_m, y_n) = \frac{k^n}{\Gamma(m)\Gamma(n-m)}[H(y_m)]^{m-1}[H(y_n) - H(y_m)]^{n-m-1} \times \\ [1 - F(y_n)]^{k-1}h(y_m)f(y_n), -\infty < y_m < y_n. \end{aligned} \tag{2.9}$$

The conditional PDF of $Y_n^{(k)}$ given $Y_m^{(k)}, n > m \geq k,$ computed directly from Error! Reference source not found. and **Error! Reference source not found.**, is

$$f_{Y_n^{(k)}|Y_m^{(k)}}(y_n|y_m) = \frac{k^{n-m}}{\Gamma(n-m)}\left[\frac{1 - F(y_n)}{1 - F(y_m)}\right]^k [H(y_n) - H(y_m)]^{n-m-1}h(y_n), n > m \geq 1. \tag{2.10}$$

Applying $(2.8)-(2.10)$ to sequence of independent random variables $\{X_i, i \geq 1\}$ from $K(p,q)$, the pdf of $Y_n^{(k)}, n \geq k \geq 1$, the joint PDF of $Y_m^{(k)}$and $Y_n^{(k)}, n > m$, and the conditional pdf of $Y_n^{(k)}$given $Y_m^{(k)}, n > m \geq k$, respectively, are

$$f_{Y_n^{(k)}}(y) = \frac{k^n q^n p}{\Gamma(n)}[-\log(1-y^p)]^{n-1}(1-y^p)^{qk-1}y^{p-1}, \tag{2.11}$$

$$f_{Y_m^{(k)},Y_n^{(k)}}(y_m, y_n) = \frac{(kq)^n p^2}{\Gamma(m)\Gamma(n-m)}\left[-\log\left(1-y_m^p\right)\right]^{m-1} \times \left[-\log\frac{\left(1-y_n^p\right)}{\left(1-y_m^p\right)}\right]^{n-m-1} \frac{{y_m}^{p-1}}{\left(1-y_m^p\right)}\left(1-y_n^p\right)^{kq-1} y_n^{p-1}, \tag{2.12}$$

and

$$f_{Y_n^{(k)}|Y_m^{(k)}}(y_n|y_m) = \frac{(kq)^{n-m}\,p}{\Gamma(n-m)}\left[-\log\frac{1-y_n^p}{1-y_m^p}\right]^{n-m-1}\left[\frac{1-y_n^p}{1-y_m^p}\right]^{kq-1}\frac{y_n^p}{1-y_m^p}, 0 < y_m < y_n < 1.$$
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$If\ n = m+1,$ $then$ Error! Reference source not found. and Error! Reference source not found., respectively, are reduced to

$$f_{Y_m^{(k)},Y_{m+1}^{(k)}}(y_m, y_{m+1}) = \frac{(kq)^{m+1}p^2}{\Gamma(m)}\left[-\log\left(1-y_m^p\right)\right]^{m-1} \times \frac{{y_m}^{p-1}}{\left(1-y_m^p\right)}\left(1-y_{m+1}^p\right)^{kq-1} y_{m+1}^{p-1}, \tag{2.14}$$

and

$$f_{Y_{m+1}^{(k)}|Y_m^{(k)}}(y_{m+1}|y_m) = kpq\left(\frac{1-y_{m+1}^p}{1-y_m^p}\right)^{kq-1}\frac{y_{m+1}^{p-1}}{\left(1-y_m^p\right)}\ 0 < y_m < y_{m+1} < 1. \tag{2.15}$$

2.2 Moments of k^th record records from KD

The r^th moment, $r = 1,2,\ldots$ of records values $Y_n, n = 1,2,\ldots$ with density given in (2.7) is

$$\mu_r^{(k,n)} \equiv E(Y_n^{(k)})^r = \int_0^1 y^r\, f_{Y_n^{(k)}}(y)dy$$
$$= \frac{k^n q^n p}{\Gamma(n)}\int_0^1 [-\log(1-y^p)]^{n-1}(1-y^p)^{qk-1}y^{r+p-1}dy$$

Let $u = (1-y^p)$, then the above integral reduces to

$$\mu_r^{(k,n)} = \frac{(-1)^{n-1}k^n q^n}{\Gamma(n)}\int_0^1 (\log u)^{n-1}(1-u)^{\frac{r}{p}}u^{kq-1}du$$
$$= \frac{(-1)^{n-1}\,(kq)^n}{\Gamma(n)}\int_0^1 \frac{(1-u)^{r/p}}{k^{n-1}}\frac{\partial^{n-1}}{\partial q^{n-1}}u^{kq-1}du \quad \square. \tag{2.16}$$

For $k \geq 1$ and $n = 1$, 2, and 3, we have, respectively, the following r^{th} moments

$$\mu_r^{(k,1)} = kq\; Be\left(\frac{r}{p}+1, kq\right), \tag{2.17}$$

$$\mu_r^{(k,2)} = -kq^2\;\frac{\partial}{\partial q}Be\left(\frac{r}{p}+1, kq\right)$$
$$= k^2q^2\;Be\left(\frac{r}{p}+1, q\right)\left[\psi\left(\frac{r}{p}+kq+1\right)-\psi(kq)\right]. \tag{2.18}$$

and

$$\mu_r^{(k,3)} = \frac{kq^3}{\Gamma(3)}\frac{\partial^2}{\partial q^2}Be\left(\frac{r}{p}+1, kq\right) \tag{2.19}$$
$$= \frac{k^2q^3}{2}Be\left(1+\frac{r}{p}, kq\right)$$
$$\times\left\{k\left[\psi(kq)-\psi\left(\frac{r}{p}+kq+1\right)\right]^2+\left[\psi'(kq)-k\psi'(\frac{r}{p}+kq+1)\right]\right\},$$

Where $Be(a,b)$ is the usual beta function and $\psi(x) = \frac{\Gamma\prime(x)}{\Gamma(x)}$ is the digamma function.

When $r = 1$, Error! Reference source not found. gives

$$\mu_1^{(k,n)} = \frac{(-1)^{n-1}kq^n}{\Gamma(n)}\left[\frac{\partial^{n-1}}{\partial q^{n-1}}B\left(kq, \frac{1}{p}+1\right)\right]. \tag{2.20}$$

For $k = 1, n = 1$, (2.20) reduces to

$$\mu_1^{(1,1)} = q\;B\left(q, \frac{1}{p}+1\right), \tag{2.21}$$

which is the mean of $K(p,q)$ variable.

The variance of $Y_n^{(k)}$ has the representation

$$V\left(Y_n^{(k)}\right) = \mu_2^{(k,n)} - \left(\mu_1^{(k,n)}\right)^2 \tag{2.22}$$

Where $\mu_r^{(k,n)}$ is as given in (2.16).

The $(r,s)^{\text{th}}$, $r,s = 1,2,\ldots$, mixed moments are given by

$$\mu_{r,s}^{(k,(m,,n))} \equiv E\left[\left(Y_m^{(k)}\right)^r\left(Y_n^{(k)}\right)^s\right] = \int_0^1\int_0^{y_n} y_m^r\, y_n^s\, f_{Y_m^{(k)},Y_n^{(k)}}(y_m,y_n)\,dy_m dy_n \tag{2.23}$$

Replacing $f_{Y_m^{(k)},Y_n^{(k)}}$ with the joint density in **Error! Reference source not found.**, we obtain

$$\mu_{r,s}^{(k,(m,,n))} = \frac{(kq)^n p^2}{\Gamma(m)\Gamma(n-m)}\int_0^1\int_0^{y_n}\left[-\log\left(1-y_m^p\right)\right]^{m-1}\times \left[-\log\frac{\left(1-y_n^p\right)}{\left(1-y_m^p\right)}\right]^{n-m-1}\frac{y_m^{r+p-1}}{\left(1-y_m^p\right)}\left(1-y_n^p\right)^{kq-1}y_n^{s+p-1}dy_m dy_n. \tag{2.24}$$

In particular, if $n = m+1$, the above reduces to

$$\begin{aligned}\mu_{r,s}^{(k,(m,,m+1))} &= \frac{(kq)^{m+1}p^2}{\Gamma(m)}\int_0^1\int_0^{y_{m+1}}\left[-\log\left(1-y_m^p\right)\right]^{m-1}\times \frac{y_m^{r+p-1}}{\left(1-y_m^p\right)}\left(1-y_{m+1}^p\right)^{kq-1}y_{m+1}^{s+p-1}dy_m dy_{m+1}\\ &= \frac{(kq)^{m+1}p^2}{\Gamma(m)}\int_0^1\left(\frac{\left[-\log\left(1-y_m^p\right)\right]^{m-1}y_m^{r+p-1}}{\left(1-y_m^p\right)}\times \int_{y_m}^1 y_{m+1}^{s+p-1}\left(1-y_{m+1}^p\right)^{kq-1}dy_{m+1}\right)dy_m.\end{aligned} \tag{2.25}$$

Setting $u = 1-y_{m+1}^p$ and applying the Fubini's theorem, we obtain

$$\mu_{r,s}^{(k,(m,,m+1))} = \frac{(kq)^{m+1}p}{\Gamma(m)}\int_0^1\frac{\left[\log\left(1-y_m^p\right)\right]^{m-1}y_m^{r+p-1}}{\left(1-y_m^p\right)}dy_m\times\left[\int_0^{1-y_m^p}(1-u)^{\frac{s}{p}}u^{kq-1}\,du\right]$$

$$= \frac{(kq)^{m+1}p}{\Gamma(m)}\int_0^1\frac{\left[\log\left(1-y_m^p\right)\right]^{m-1}y_m^{r+p-1}}{\left(1-y_m^p\right)}B\left(\frac{s}{p}+1,kq;1-y_m^p\right)dy_m. \tag{2.26}$$

The correlation coefficient between $Y_m^{(k)}$ and $Y_n^{(k)}$ is

$$\rho_{Y_m^{(k)},Y_n^{(k)}} = \frac{\mu_{1,1}^{(k,(m,,n))} - \mu_1^{(k,m))}\mu_1^{(k,n)}}{\sqrt{V\left(Y_m^{(k)}\right)V\left(Y_n^{(k)}\right)}}, \tag{2.27}$$

which can be calculated from (2.16), (2.22), and (2.26).

Based on the conditional density given in Error! Reference source not found., we present the r^{th} conditional mean by the integral

$$E\left(\left(Y_n^{(k)}\right)^r \middle| Y_m\right) = \frac{(kq)^{n-m}\,p}{\Gamma(n-m)} \int_{y_m}^{1} \left[-\log \frac{1-y_n^p}{1-y_m^p}\right]^{n-m-1} \left[\frac{1-y_n^p}{1-y_m^p}\right]^{kq-1} \frac{y_n^{r+p-1}}{1-y_m^p}\, dy_n.$$
(2.28)

For $n = m + 1$, **Error! Reference source not found.** reduces to

$$E\left(\left(Y_{m+1}^{(k)}\right)^r \middle| Y_m\right) = kq\ p \int_{y_m}^{1} \left[\frac{1-y_{m+1}^p}{1-y_m^p}\right]^{kq-1} \frac{y_{m+1}^{r+p-1}}{1-y_m^p}\, dy_{m+1}. \tag{2.29}$$

Set $u = 1 - y_{m+1}^p$, to get

$$\begin{aligned} E\left(\left(Y_{m+1}^{(k)}\right)^r \middle| Y_m\right) &= kq\left(1 - y_m^p\right)^{-kq+1} \int_0^{1-y_m^p} (1-u)^{r/p} u^{kq-1}\, du \\ &= kq\left(1 - y_m^p\right)^{-kq+1} B\left(\frac{r}{p} + 1, kq; 1 - y_m^p\right) \end{aligned} \tag{2.30}$$

The first, second, and first mixed moments, as well as the correlation coefficient between the upper m^{th} and $(m+1)^{th}$ upper records for given values of the parameters p and q are displayed in Table 2.1, Table 2.2, and Table 2.3, respectively. We notice that all of these measures increase with m as expected.

Table 2.1. The first, second, and first mixed moments of upper records and the correlation coefficient between the upper m^{th} and $(m+1)^{th}$ records for $p = 0.5$ and $q = 1.0, 3.0$.

p	q	m	$\mu_1^{(1,n)}$	$\mu_2^{(1,n)}$	$\mu_{1,1}^{(1,(m,m+1))}$	$\rho_{m,m+1}$	p	q	$\mu_1^{(1,n)}$	$\mu_2^{(1,n)}$	$\mu_{1,1}^{(1,(m,m+1))}$	$\rho_{m,m+1}$
0.5	1	1	0.333	0.200	0.261	0.668	0.5	3	0.100	0.029	0.042	0.698
		2	0.611	0.457	0.530	0.766			0.235	0.094	0.122	0.809
		3	0.787	0.668	0.725	0.807			0.372	0.187	0.227	0.858
		4	0.887	0.810	0.848	0.829			0.497	0.296	0.340	0.885
		5	0.942	0.896	0.918	0.841			0.603	0.407	0.451	0.902

		6	0.970	0.945	0.957	0.849			0.691	0.512	0.553	0.914
		7	0.985	0.971	0.978	0.854			0.761	0.605	0.642	0.923
		8	0.992	0.985	0.989	0.857			0.817	0.686	0.717	0.929
		9	0.996	0.992	0.994	0.860			0.860	0.753	0.779	0.935

Table 2.2. The first, second, and first mixed moments of upper records and the correlation coefficient between the upper m^{th} and $(m+1)^{th}$ records for $p = 1$ and $q = 1.0, 2.0.$

p	q	m	$\mu_1^{(1,n)}$	$\mu_2^{(1,n)}$	$\mu_{1,1}^{(1,(m,m+1))}$	$\rho_{m,m+1}$	p	q	$\mu_1^{(1,n)}$	$\mu_2^{(1,n)}$	$\mu_{1,1}^{(1,(m,m+1))}$	$\rho_{m,m+1}$
1.0	1	1	0.500	0.333	0.417	0.655	1.0	3	0.250	0.100	0.138	0.696
		2	0.750	0.611	0.681	0.753			0.438	0.235	0.286	0.803
		3	0.875	0.787	0.831	0.796			0.578	0.372	0.424	0.852
		4	0.938	0.887	0.912	0.820			0.684	0.497	0.543	0.852
		5	0.969	0.942	0.955	0.834			0.763	0.603	0.643	0.879
		6	0.984	0.970	0.977	0.843			0.822	0.691	0.724	0.910
		7	0.992	0.985	0.989	0.850			0.867	0.761	0.787	0.919
		8	0.996	0.992	0.994	0.854			0.900	0.817	0.837	0.926
		9	0.998	0.996	0.997	0.857			0.925	0.860	0.876	0.932

Table 2.3. The first, second, and first mixed moments of upper records and the correlation coefficient between the upper m^{th} and $(m+1)^{th}$ records for $p = 2$ and $q = 1.0, 3.0$

p	q	m	$\mu_1^{(1,n)}$	$\mu_2^{(1,n)}$	$\mu_{1,1}^{(1,(m,m+1))}$	$\rho_{m,m+1}$	p	q	$\mu_1^{(1,n)}$	$\mu_2^{(1,n)}$	$\mu_{1,1}^{(1,(m,m+1))}$	$\rho_{m,m+1}$
2.0	1	1	0.667	0.500	0.591	0.633	1.0	3	0.457	0.250	0.315	0.676
		2	0.854	0.750	0.804	0.739			0.639	0438	0.497	0.791
		3	0.931	0.875	0.904	0.787			0.748	0.578	0.626	0.843
		4	0.967	0.938	0.952	0.813			0.819	0.684	0.720	0.873
		5	0.984	0.969	0.976	0.829			0.869	0763	0.791	0.893
		6	0.992	0.984	0.988	0.840			0.904	0.822	0.843	0.907
		7	0.996	0.992	0.994	0.847			0.929	0.867	0.883	0.917
		8	0.998	0.996	0.997	0.852			0.948	0.900	0.912	0.924

9	0.999	0.998	0.999	0.856	0.961	0.925	0.943	0.930

3. Characterization

In this section, we will prove a characterization of the Kumaraswamy and other continuous distributions based on upper record values.

Theorem 3.2: Assume $\{X_n, n \geq 1\}$ is a sequence of independent and continuous random variables with a common distribution function $F(x)$ on (α, β) with α and β are possibly finite or infinite. Let $\{Y_n, n \geq 1\}$ denote the corresponding sequence of upper record values. Further, assume $\psi(x)$ is a continuous, monotone decreasing, and differentiable function with $\lim_{x\to-\infty} \psi(x) = 1$ and $\lim_{x\to\infty} \psi(x) = 0$. Then the condition

$$E[\psi^a(Y_s|Y_r = x)] = \frac{\psi^a(x)}{2^{s-r}} \tag{3.1}$$

Determines $F(x)$ uniquely.

Proof: By Error! Reference source not found., the conditional expectation of $\psi^a(Y_s|Y_r = x)$, when $k = 1$, is

$$E[\psi^a(Y_s|Y_r = x)] \\ = \frac{1}{\Gamma(s-r)} \int_x^\beta \psi^a(y) \left[-\log \overline{F}(x) \ + \log \overline{F}(y)\right]^{s-r-1} \frac{f(y)}{\overline{F}(x)} dy \tag{3.2}$$

Replacing $F(x)$ with $F(x) = 1 - \psi^a(x)$ in (3.2), the necessary condition is directly proved.

To prove the sufficient condition, equations (3.1) and (3.2) give

$$\frac{1}{\Gamma(s-r)} \int_x^\beta \psi^a(y) \left[-\log \overline{F}(x) \ + \log \overline{F}(y)\right]^{s-r-1} f(y)dy = \overline{F}(x) \frac{\psi^a(x)}{2^{s-r}}. \tag{3.3}$$

Let

$$E_{r,s}^a(x) = \frac{1}{\Gamma(s-r)} \int_x^\beta \psi^a(y) \left[-\log \overline{F}(x) \ + \log \overline{F}(y)\right]^{s-r-1} f(y)dy$$

so that

$$E_{r,s}^a(x) = \overline{F}(x) \frac{\psi^a(x)}{2^{s-r}}. \tag{3.4}$$

Differentiate both sides of Error! Reference source not found. with respect to x to obtain

$$\frac{-1}{\Gamma(s-r-1)}\int_x^\beta \psi^a(y)[-\log\overline{F}(x) \;\; + \log\overline{F}(y)]^{s-r-2}\frac{f(x)}{\overline{F}(x)}f(y)dy$$
$$= a\overline{F}(x)\frac{\psi^{a-1}(x)\psi'(x)}{2^{s-r}} - f(x)\frac{\psi^a(x)}{2^{s-r}} \tag{3.5}$$

From equations (3.3) and (3.4), the left-hand side of equation (3.5) is$-\frac{f(x)}{\overline{F}\;(x)}E^a_{r+1,s}(x)$, where $E^a_{r+1,s}(x) = \overline{F}(x)\frac{\psi^a(x)}{2^{s-r-1}}$.

Therefore, equation (3.35) can be rewritten as

$$-\frac{f(x)}{\overline{F}(x)}E^\alpha_{r+1,s}(x) = -\frac{\psi^a(x)}{2^{s-r-1}}f(x) = \frac{\psi^{a-1}(x)}{2^{s-r}}\left[a\psi'(x)\overline{F}(x) - \psi(x)f(x)\right], \tag{3.6}$$

which is simplified to

$$-\frac{f(x)}{\overline{F}(x)} = \frac{a\psi'(x)}{\psi(x)},$$

and the solution of which is

$$F(x) = 1 - \psi^a(x). \tag{3.7}$$

Hence the theorem is proved. □

In particular, If $s = r + 1$, then $E[\psi^a(Y_{r+1}|Y_r = x)] = \frac{\psi^a(x)}{2}$ determines $F(x)$ uniquely.

Depending on $\psi(u)$, many continuous distributions, including the Kumaraswamy distribution, can be characterized by Error! Reference source not found.. Some of these are given in the following corollary.

Corollary. *If* $s = r + 1$**,** *then equation* (3.34) *yields the following*

a) *If* $\psi(x) = e^{-x}$*, then* $F(x) = 1 - e^{-ax}, x > 0$*; X is distributed as exponential.*
b) *If* $\psi(x) = x$*, then* $F(x) = 1 - x^a, 0 < x < 1$*; X has the power function distribution.*
c) *If* $\psi(x) = e^{-x^\lambda}$*, then* $F(x) = 1 - e^{-ax^\lambda}, 0 < x, \lambda > 0$*; X is distributed as Weibull.*
d) *If* $\psi(x) = 1 - x^p$*, then* $F(x) = 1 - (1 - x^p)^a, 0 < x < 1, p > 0$*; X is distributed as Kumaraswamy.*

4. Prediction of Upper Record Values

Future record predictions are of great interest to researchers in many practical fields, including medicine, industrial production, hydrology, and meteorology. In this section, we look at the problem of predicting future records based on observed records from the two-parameter Kumaraswamy distribution. Three point predictors; the maximum likelihood predictor (MLP), the modified maximum likelihood (MMLP), and the conditional median predictor (CMP) will be derived.

Let $y_1 < y_2 < \cdots < y_m$ represent the first m upper records, and y_s represent the s^{th} future record where $s > m$. Prediction of y_s based on the first m observed records, $\underline{y} = \{y_1, y_2, \ldots, y_m\}$, is primarily determined by the conditional predictive density function of y_s given the observed records $\underline{y} = \{y_1, y_2, \ldots, y_m\}$. Using the Markovian property of records, the conditional distribution of y_s given $\underline{y}$ is simply the conditional distribution of y_s given y_m, as demonstrated by Arnold et al. (1998), as follows:

$$f(y_s|y_m; p, q) = \frac{[H(y_s) - H(y_m)]^{s-m-1}}{\Gamma(s-m)} \frac{f(y_s|p,q)}{1 - F(y_m|p,q)} \tag{4.1}$$

For the $K(p, q)$ distribution, Eq. (4.1) reduces to

$$f(y_s|y_m; p, q) = \frac{pq^{s-m}}{\Gamma(s-m)} [\ln(1 - y_m^p) - \ln(1 - y_s^p)]^{s-m-1} y_s^{p-1} \frac{(1 - y_s^p)^{q-1}}{(1 - y_m^p)^q}, \tag{4.2}$$

Where $0 < y_m < y_s < 1$

In this part, we examine the classical point prediction

4.1 Maximum Likelihood Predictor

In this subsection, we give a point prediction for $y_s, s > m$ using the maximum likelihood prediction (MLP) method. If $\underline{y} = \{y_1, y_2, \ldots, y_m\}$ is a set of observed records from a population with PDF $f(y_s; \theta)$ and CDF $F(y_s; \theta)$, where $\theta = (p, q)$, then the predictive likelihood function (PLF) of y_s, p and q, is given by (see Basak and Balakrishnan, 2003):

$$L\left(y_s;\ \theta, \underline{y}\right) = \prod_{i=1}^{m} h(y_i; \theta) \left(\frac{[H(y_s;\ \theta) - H(y_m;\ \theta)]^{s-m-1}}{\Gamma(s-m)} f(y_s; \theta) \right) \tag{4.3}$$

Generally, if $\hat{y}_{MLP} = u(\underline{y})$, $\hat{p} = v(\underline{y})$, and $\hat{q} = w(\underline{y})$ are statistics for which

$$L\left(u\left(\underline{y}\right), v\left(\underline{y}\right), w\left(\underline{y}\right)\middle|\underline{y}\right) = \sup_{y_s,p,q} L\left(y_s, p, q\middle|\underline{y}\right), \tag{4.4}$$

Then $u(\underline{y})$ is said to be the MLP of y_s, $1 < m < s, 1 < m < s$, and $v(\underline{y})$ and $w(\underline{y})$ are the predictive maximum likelihood estimates (PMLEs) of p and q, respectively. The log-likelihood predictive function (PLF) for the $K(p,q)$ distribution is

$$L(y_s; p, q; \boldsymbol{y}) = q^s p^{m+1} \frac{[\log(1-y_m^p) - \log(1-y_s^p)]^{s-m-1}}{\Gamma(s-m)} \frac{y_s^{p-1}}{(1-y_s^p)^{1-q}} \prod_{j=1}^{m} \frac{y_i^{p-1}}{1-y_i^p} \tag{4.5}$$

After some simplifications, the Log-likelihood function, up to a constant, is

$$\begin{aligned} l(y_s, p, q|\boldsymbol{y}) \equiv l(y_s, p, q) &= (p-1)\sum_{i=1}^{m} \log y_i - \sum_{i=1}^{m} \log(1-y_i^p) + (s-m-1)\log[\log(1-y_m^p) - \log(1-y_s^p)] \\ &+ (m+1)\log p + s \log q + (p-1)\log y_s + (q-1)\log(1-y_s^p) \end{aligned} \tag{4.6}$$

Therefore, the likelihood equations are:

$$\begin{aligned} \frac{\partial l(y_s,p,q)}{\partial p} &= \sum_{i=1}^{m} \log y_i + \sum_{i=1}^{m} \frac{y_i^p \log y_i}{(1-y_i^p)} + (s-m-1)\frac{\frac{y_s^p \log y_s}{1-y_s^p} - \frac{y_m^p \log y_m}{1-y_m^p}}{\log(1-y_m^p) - \log(1-y_s^p)} \\ &+ \frac{m+1}{p} + \log y_s - (q-1)\frac{y_s^p \log y_s}{1-y_s^p} = 0 \end{aligned} \tag{4.7}$$

$$\frac{\partial l(y_s,p,q)}{\partial q} = \frac{s}{q} + \log(1-y_s^p) = 0 \tag{4.8}$$

$$\frac{\partial l(y_s,p,q)}{\partial y_s}=(s-m-1)\frac{\frac{py_s^{p-1}}{1-y_s^p}}{[\log(1-y_m^p)-\log(1-y_s^p)]}+\frac{p-1}{y_s}-(q-1)\frac{py_s^{p-1}}{1-y_s^p}=0 \tag{4.9}$$

The PMLE of q, say $\hat{q}$, is obtained from equation (4.8) and is given by

$$\hat{q}=\frac{-s}{\log(1-y_s^{\hat{p}})} \tag{4.10}$$

The PMLE of p, say $\hat{p}$, and MLP of y_s say $\hat{y}_{MLP}$, can be obtained by substituting Eq. (4.10) in Eq (4.7) and (4.9), respectively, to obtain

$$\sum_{i=1}^{m}\log y_i+\sum_{i=1}^{m}\frac{y_i^p\log y_i}{(1-y_i^p)}+(s-m-1)\frac{\frac{y_s^p\log y_s}{1-y_s^p}-\frac{y_m^p\log y_m}{1-y_m^p}}{\log(1-y_m^p)-\log(1-y_s^p)}+$$
$$\frac{m+1}{p}+\log y_s+\left(\frac{s}{\log(1-y_s^p)}+1\right)\frac{y_s^p\log y_s}{1-y_s^p}=0, \tag{4.11}$$

and

$$(s-m-1)\frac{\frac{py_s^{p-1}}{1-y_s^p}}{[\log(1-y_m^p)-\log(1-y_s^p)]}+\frac{p-1}{y_s}+\left(\frac{s}{\log(1-y_s^p)}+1\right)\frac{py_s^{p-1}}{1-y_s^p}=0 \tag{4.12}$$

In particular, when $s=m+1$, (4.11) and (4.12) reduce to

$$\sum_{i=1}^{m}\log y_i+\sum_{i=1}^{m}\frac{y_i^p\log y_i}{(1-y_i^p)}+\frac{m+1}{p}+\log y_s+\left(\frac{m+1}{\log(1-y_s^p)}+1\right)\frac{y_s^p\log y_s}{1-y_s^p}=0, \tag{4.13}$$

and

$$\frac{p-1}{y_s}+\left(\frac{m+1}{\log(1-y_s^p)}+1\right)\frac{py_s^{p-1}}{1-y_s^p}=0 \tag{4.14}$$

Analytical solutions for Eq. (4.13) and (4.14) may not be possible, so they must be solved numerically.

4.2 Modified Maximum Likelihood Predictor

A modified maximum likelihood predictor (MMLP) is obtained as a solution to equation (4.9) after implementing the MLEs of p and q that were derived based on the first m upper records. Thus, the MMLP of y_s is the solution of the following equation

$$(s-m-1)\frac{\frac{\hat{p}y_s^{\hat{p}-1}}{1-y_s^{\hat{p}}}}{[\ln(1-y_m^{\hat{p}})-\ln(1-y_s^{\hat{p}})]}+\frac{\hat{p}-1}{y_s}-(\hat{q}-1)\frac{\hat{p}y_s^{\hat{p}-1}}{1-y_s^{\hat{p}}}=0 \qquad (4.15)$$

Where $y_s > y_m$.

To compute the MMLP of y_s, say $\hat{Y}_{MMLP}$, an analytical method has to be applied to solve Eq. (4.15). When $s = m+1$, the one-step MMLP is

$$\hat{y}_{m+1} = \left(1+\frac{\hat{p}(\hat{q}-1)}{\hat{p}-1}\right)^{-1/\hat{p}} \qquad (4.16)$$

4.3 Conditional Median Predictor

Let $\hat{Y}_{CMP}$ be the conditional median predictor of y_s. Assuming that $\hat{Y}_{CMP} = k(y_m;p,q)$ is a function of y_m, then $P(y_s|y_m;p,q) \le k(y_m,p,q) = \frac{1}{2} = P(y_s|y_m;p,q) \ge k(y_m,p,q)$. As a result of Eq. (4.2), we have

$$\int_{y_m}^{k(y_m,p,q)} \frac{pq^{s-m}}{\Gamma(s-m)}[\ln(1-y_m^p) - \ln(1-y_s^p)]^{s-m-1}\frac{[1-y_s^p]^{q-1}}{(1-y_m^p)^q}y_s^{p-1}dy_s = \frac{1}{2} \qquad (4.17)$$

Let $t = [\ln(1-y_m^p) - \ln(1-y_s^p)]$, then $dt = \frac{py_s^{p-1}}{1-y_s^p}dy_s$ and the integral equation (4.17) becomes

$$\int_0^{\ln(1-y_m^p)-\ln[1-k(y_m;p,q)^p]} \frac{q^{s-m}}{\Gamma(s-m)}t^{s-m-1}e^{-tq}dt = \frac{1}{2}.$$

Thus,

$$\log(1-y_m^p) - \log[1-k(y_m;p,q)^p] = F^{-1}\left(\frac{1}{2}\right) = Med(W) \qquad (4.18)$$

Where F is the cumulative distribution function of $W \sim Gamma(s-m, \frac{1}{q})$.

Solving Eq. (4.18) for $k(y_m; p, q)$, the conditional median predictor is

$$\hat{Y}_{CMP} = \left[1 - \left(1 - y_m^p\right)e^{-Med\,(W)}\right]^{\frac{1}{p}} \tag{4.19}$$

Assume that we are interested in predicting the first future record, i.e., $s = m + 1$, then $W \sim Exp(q)$, with$Med\ (W) = \frac{1}{q}\log 2$, therefore

$$\hat{Y}_{CMP} = \left[1 - \left(1 - y_m^p\right)e^{-\frac{1}{q}\log 2}\right]^{\frac{1}{p}} = \left[1 - \left(1 - y_m^p\right)2^{-\frac{1}{q}}\right]^{\frac{1}{p}} \tag{4.20}$$

If p and q are unknown, they are replaced by their MLEs, in which case an approximate CMP is derived from (4.20).

5 Real data application

In this section, we analyze the rainfall for the month of January recorded at the Changi Climate Station in Singapore. Monthly data available from 1982 to 2021 were downloaded from the open source https://data.gov.sg/dataset/rainfall-monthly-total. For the month of January, the rainfalls in millimeter are:

107.1, 246.0, 251.2, 111.1, 308.2, 568.6, 237.5, 189.7, 147.4, 123.9, 83.9, 176.4, 56.9, 349.4, 173.2, 15.4, 268.8, 193.9, 275.2, 425.8, 221.2, 444.2, 600, 9, 163.2, 454.4, 450.1, 262.6, 38.3, 69.5, 513.2, 106.1, 262.0, 75.4, 79.6, 126.6, 197.6, 287.0, 63.6, 88.4, 692.8.

Therefore, the upper rainfall records are 107.1, 246, 251.2, 308.2, 568.6, 600.9, and 692.8.

To test the randomness of the data, we performed the Wald-Wolfowitz Runs test, and a p-value of 0.786 was calculated, suggesting that the data are random. A histogram of the precipitation data and a graph of the smoothed empirical distribution overlaid with the cumulative distribution, evaluated at the MLEs, are shown in Figure 1 and Figure 2, respectively.

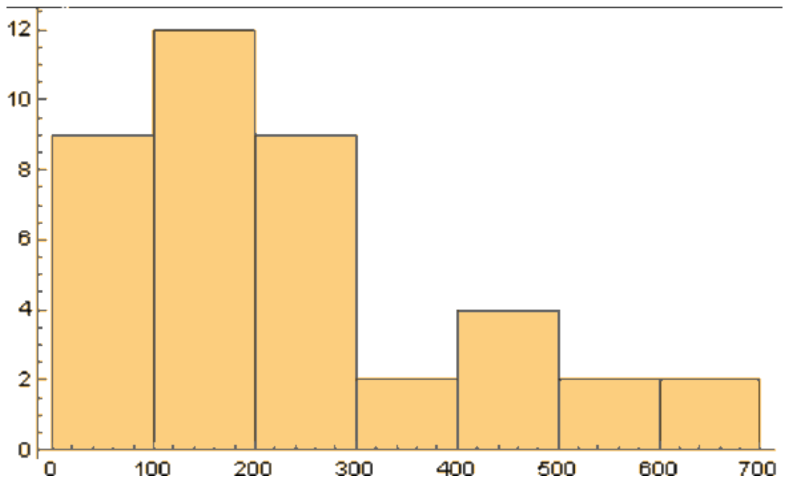

Figure1: Histogram for the January rainfall data recorded at the Changi Climate Station in Singapore for theperiod 1982-2021.

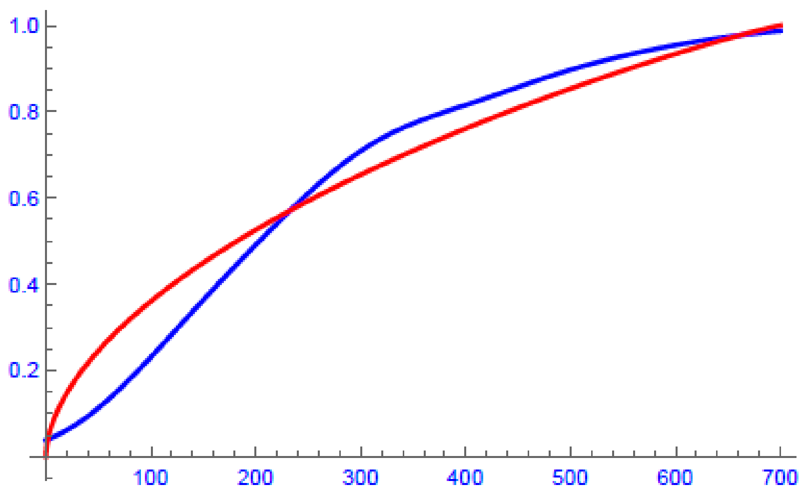

Figure2: The smoothed empirical distribution (blue) overlaid with the Kumaraswamy cumulative distribution (red), evaluated at the MLEs, for the January rainfall data recorded at the Changi ClimateStation in Singapore for the period 1982-20

We will apply the following parametric bootstrap procedure on the Anderson-Darling (AD) test to test the conformity of the rainfall data with the Kumaraswamy distribution:

a) Assuming the data is from $K(p,q)$, obtain the MLEs of p and q, say $\hat{p}$ and $\hat{q}$.
b) Evaluate the AD test, say T_{AD}, at the MLEs obtained in (1)
c) Simulate 10000 bootstrap samples from $K(\hat{p},\hat{q})$, and for each sample, find the MLEs of p and q and the corresponding value of the AD test, say $T^*{}_{AD}$.
d) Calculate the p-value of the test as the proportion of times the value of $T^*{}_{AD}$ exceeds T_{AD}.

Thus, the first step is to find the MLEs of p and q. The likelihood function for an observed sample $x_1, \dots, x_n$ from $K(p,q)$ is

$$L(p,q|x_1,\dots,x_n) \equiv L(p,q) = p^n q^n \prod_{j=1}^{n} x_j{}^{p-1}(1-x_j{}^p)^{q-1}$$

hence, the log-likelihood function is

$$l(p,q) = n\log p + n\log q + (p-1)\sum_{j=1}^{n}\log\, x_j + (q-1)\sum_{j=1}^{n}\log\,(1-x_j{}^p).$$

Therefore, the likelihood equations are:

$$\frac{\partial l(p,q)}{\partial p} = \frac{n}{p} + \sum_{j=1}^{n}\log\, x_j - (q-1)\sum_{j=1}^{n}\frac{x_j{}^p\log x_j}{(1-x_j{}^p)} = 0 \tag{5.1}$$

$$\frac{\partial l(p,q)}{\partial q} = \frac{n}{q} + \sum_{j=1}^{n}\log\,(1-x_j{}^p) = 0 \tag{5.2}$$

Solving Eq. (5.2) for q, we have

$$q = -\frac{n}{\sum_{j=1}^{n}\log\,(1-x_j{}^p)} \tag{5.3}$$

Replacing Eq. (5.3) in Eq. (5.1), we get

$$\frac{n}{p} + \sum_{j=1}^{n}\log\, x_j + \left(\frac{n}{\sum_{j=1}^{n}\log\,(1-x_j{}^p)} + 1\right)\sum_{j=1}^{n}\frac{x_j{}^p\log x_j}{(1-x_j{}^p)} = 0 \tag{5.4}$$

The MLEs for p and q are obtained by first solving $Eq.$ (5.4) numerically for p and then solving eq. (5.3) for q.

A scale parameter should be incorporated in the Kumaraswamy model for the rainfall data under consideration. The maximum observed value (692.8) could be considered an approximation of this parameter. Although the maximum value may not necessarily be the MLE of the scale parameter, it is a good approximation, especially for large observations. To avoid logarithms of zero or zero denominator, we rescaled the data by dividing by 700 instead of 692.8. Based on the rescaled data, the MLEs of p and q are calculated as $\hat{p} = 0.56$ and $\hat{q} = 1.09$.

Applying the parametric bootstrap procedure described previously, we calculate the p-value at 0.064, indicating a significant fit of the data to the two-parameter Kumaraswamy distribution at the nominal level of 0.05.

To obtain the MMLP and the CMP for the next rainfall record, we will first get the MLEs of p and q based on the set of upper records. In general, given the observed records $y = y_1, \dots, y_n$, the log-likelihood function is

$$l(p,q|y) = n\log p + n\log q + q\log\left(1 - y_n^p\right) + p\sum_{i=1}^{n}\log y_i - \sum_{i=1}^{n}\log\left(1 - y_i^p\right). \quad (5.5)$$

Therefore, the likelihood equations are:

$$\frac{\partial l(p,q)}{\partial p} = \frac{n}{p} - \frac{q y_n^p \log y_n}{1-y_n^p} + \sum_{i=1}^{n}\log y_i + \sum_{i=1}^{n}\frac{y_i^p \log y_i}{1-y_i^p} = 0$$
(5.6)

$$\frac{\partial l(p,q)}{\partial q} = \frac{n}{q} + \log\left(1 - y_n^p\right) = 0$$
(5.7)

From Eq. (4.27), we have

$$q = -\frac{n}{\log(1-y_n^p)} \quad (5.8)$$

Substitution of Eq. (4.28) in (4.26) gives,

$$\frac{n}{p} + \frac{n}{\log(1-y_n^p)}\frac{y_n^p \log y_n}{1-y_n^p} + \sum_{i=1}^{n}\log y_i + \sum_{i=1}^{n}\frac{y_i^p \log y_i}{1-y_i^p} = 0$$
(5.9)

The MLEs, $\hat{p}_y$ and $\hat{q}_y$, are obtained by first solving Eq. (5.9) numerically for p and then obtain $\hat{q}_y$ from Eq. (5.8).

Based on the set of upper precipitation records $\{107.1, \ 246, 251.2, 308.2, 568, 600.9, 692.8\}$, the MLE values for p and q are, respectively, $\hat{p}_y = 0.38$ and $\hat{q}_y = 1.04$. as expected, these estimates are different from the corresponding MLEs based on the complete sample.

Solving Eqs. (4.10), (4.13), and (4.14) for p, q and $y_{m+1,}$ one has $\hat{p}_{MLE} = 0.32, \hat{q}_{MLP} = 0.88$, and $\hat{y}_{MLP} = 699.8$.

The MMLP one-step predictor, obtained as the solution of Eq. (4.16) when p and q are, respectively, replaced by their MLEs, $\hat{p}_y = 0.38$ and $\hat{q}_y = 1.04$, is 747.3 ml.

Finally, the CMP obtained from Eq. (4.20), with $\hat{p}_y = 0.38$ and $\hat{q}_y = 1.04$, is 700 ml.

5. Conclusions

We have studied some properties of the Kumaraswamy distribution based on upper record values. Exact forms and representations of the distributions, joint distributions, and conditional distributions were derived. We also derived exact representations for moments, mixed moments, conditional moments, and correlation coefficients. We further proved characterizations for severalknown continuous distributions based on a conditional functional expectation. As for the inferenceanalysis, we derived the likelihood equations based on the upper records sample as well as for thefull sample. We also derived the MLP, MMLP, and CMP predictors for future records. In addition,we fitted the rainfall data for the month of January recorded at the Changi Climate Station in Singapore for the period from 1982-2021 with the Kumaraswamy distribution. Based on the set ofavailable upper rainfall records, we have predicted the next record for the month of January.

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