An Upper Bound on the Partition Dimension of Comb Product Graph Wheel and Path

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Abstract

Research with the title: An Upper Bound on The Partition Dimension of Comb Product Graph Wheel and Path. This research is the development of the results of calculations, research methods with a qualitative model approach to the partition dimensions of the comb product between the wheel and path graph investigated in this study. The main objective in this research is to determine the partition dimension of the upper bound of this graph. To achieve optimal results, the researcher expands the wheel graph completion partition to find the comb product graph completion partition. The results of this study provide an upper bound on the partition dimensions of the comb product graph between the wheel graph and the path as the main result, throughout this section, the graphs W_n for $n \geq 3$ is the wheel graph with n+1 vertices. The path graphs P_m for m≥1 is a graph with $V(P_m) = \{v_1, v_2, \dots, v\_m\}$ and $E(P_m) = \{v_1, v_2, v_2, v_3, \dots, v_{m-1}, v_m\}$. The lower limit of the partition dimensions should be examined further in future research.

Keywords: Partition dimensions, comb product graph, wheel and path.

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1. Introduction

Assume G is a connected graph. Let $v, w \in V(G)$, the distance $d(v, w)$ is the length of a shortest path between them. It is well known that the concept of distance between two vertices in a connected graph can be used to describe the vertices in two ways. First, every vertex in $V(G)$ can be represented by the metric with respect to a subset of $V(G)$. The second representation is the metric with respect to a partition of $V(G)$. For an ordered subset $W = \{w_1, w_2, \dots, w_k\}$ of $V(G)$ and a vertex w of G the metric representation of v with respect to W is $r(w|W) = (d_1, d_2, \dots, d_k)$, where $d_i = d(w, w_i)$. Slater (Slater, 1975) and Harrary and Melter (F. Harary, 1976) independently introduced this representation. Finding the metric dimension and a resolving set of connected graphs is the key difficulty in this area. If every vertex in G have distinct representations with regard to W, W is called a resolving set for G. A basis for G is a resolving set with the minimum cardinality. The metric dimension of G is the number of elements in a basis for G, which is denoted by $\beta(G)$. The $\beta(G)$ is studied in (G. Chartrand L. E., 2000), (G. Chartrand C. P., 2000), (J. Caceres C. H., 2005), (J. Caceres C. H., 2007), (Tomescu, 2008). In chemistry, the idea of metric dimension is used to represent chemical compounds (S. Khuller, 1996) and has certain applications in robot navigation (S. Khuller, 1996). (G. Chartrand L. E., 2000). This principle can also be seen in coin weighing problems (A. Sebo, 2004) and Mastermind game techniques (Goddard, Static Mastermind, 2003). (Goddard, Mastermind revisited, 2004).

A variation representation of the previous one is the metric vector of a vertex with respect to a partition of $V(G)$. We define the distance $d(v,S)$ between v and S by $d(v,S) = \min\{d(v,w)|w \in S\}$ for every vertex $v \in V(G)$ and every subset S of $V(G)$. Given an ordered partition $\Omega = \{S_1, S_2, \dots, S_k\}$ of $V(G)$, the partition representation of a vertex v with respect to the partition Ω is $r(v|\Omega) = \big(d(v,S_1), d(v,S_2), \dots, d(v,S_k)\big)$. If any pair of different vertices $u, v \in V(G)$ has a distinct vector representation with regard to the partition Ω, that is $r(u|\Omega) \neq r(v|\Omega)$, we call Ω a resolving partition of G. For a graph G, the partition dimension $pd(G)$, is the least number of sets in any resolving partition. Chartrand et.al introduced partition dimension and resolving partition of a connected graph (G. Chartrand E. S., 2000). (M. Fehr, 2006), and (Tomescu, 2008) also investigates $pd(G)$. It is natural to think that $\beta(G)$ and $pd(G)$ are linked. It is proved in (G. Chartrand E. S., 2000) that for all connected graph G that isn't trivial, we have $pd(G) \leq \beta(G) + 1$.

The $pd(G)$ for some specific classes of graphs have been known up to now. The values of the partition dimension is usually the results achieved but some other graph only get the bound of the partition dimension. Rodríguez et.al (Juan A. Rodríguez-Velázquez, 2014) put an upper bound of $pd(T)$ where T is a tree. For circulant graphs, its partition dimension investigated by Grigorius et.al (Cyriac Grigorious, 2014). Tomescu et.al (I. Tomescu, 2007) provide a bound for $pd(W_n)$ with $n \geq 4$.

Theorem 1. Let $n > 3$ be an integer. If p is the lowest prime number with $p(p-1) \geq n$, then $\lceil (2n)^{1/3} \rceil \leq pd(W_n) \leq p + 1$. In graph theory, there are known operations of one graph such as subgraphs, subdivision. Also, there are operations between two or more graphs, for example the corona product, Cartesian product, union, normal, addition, and comb product. Many authors have looked into the graph's partition dimension from graph operations, including as corona product graphs (E. T. Baskoro, 2012), Cartesian product graphs (I. G. Yero J. A.-V., 2010), strong product graphs (I. G. Yero M. J., 2014), subdivision of complete graphs (Amrullah, 2015). In addition, Haryeni et.al (D. O. Haryeni, 2017) have studied the partition dimensions of disconnected graphs. Especially for the comb product graphs, Suhadi et al. conducted research on the graph's metric dimension (Suhadi W. S., 2017). While the partition dimensions of the comb product graphs in some cases have been studied in (Alfarisi, 2017), (Faisal, 2019).

Comb product graphs were originally introduced by Hora and Obata (Akihito Hora, 2007). These graphs are interesting to study because the structure is similar to chemical molecules, so it can be used to model specific chemical molecules. Consider two connected graphs, G and Γ.

Assume that o is a vertex in Γ. The comb product between G and Γ, denoted by $G \rhd \Gamma$, is the graph with $V(G \rhd \Gamma) = \{(x,y)|x \in V(G), y \in V(\Gamma)\}$ and $(x,y)(w,z) \in E(G \rhd \Gamma)$ whenever $x = w$ and $yz \in E(\Gamma)$, or $xw \in E(G)$ and $y = z = o$. In this work, we give an upper bound for $pd(W_n \rhd_o P_m)$ where $W_n \rhd_o P_m$ is the comb product of the wheel and path graph. For $n \geq 3$, wheel graph W_n is a graph built by linking the single vertex in graph K_1 to all vertices of the cycle C_n with $V(C_n) = \{c_0, c_1, \ldots, c_{n-1}\}$. As a result, W_n has $n + 1$ vertices. The center is the single vertex of K_1 in the W_n, while the rim is the vertices of $V(C_n)$. We consider the partition dimension of $W_n \rhd_o P_m$ where P_m is a path with m vertices and o is a vertex of P_m with degree 1.

2. Method

This type of research is qualitative research. This research aims to check the partition dimensions for a special graph class, namely the graph of the comb product between two connected graphs. In this study, we take a comb product between wheel graphs and a path. We gain an upper bound on the $pd(W_n \rhd_o P_m)$ this scenario. The first step in this investigation is to review the literature on the $pd(W_n)$. Then we construct a resolving partition of $W_n \rhd_o P_m$ using resolving partition of W_n to obtain the upper bound of $pd(W_n \rhd_o P_m)$.

3. Result and Discussion

Throughout this section, the graphs W_n for $n \geq 3$ is the wheel graph with $n + 1$ vertices. The path graphs P_m for $m \geq 1$ is a graph with $V(P_m) = \{v_1, v_2, \ldots, v_m\}$, and $E(P_m) = \{v_1v_2, v_2v_3, \ldots, v_{m-1}v_m\}$. We denote by $\mathbf{1_i}$ the vector whose coordinates are all 1, except one that equals 0 in the i-th coordinate. In general, we denote $\mathbf{k}_i$ vector whose coordinates are all k, except one that equals 0 in the i-th coordinate.

Lemma 1. If $\Omega = \{S_1, S_2, \ldots, S_k\}$ is a resolving partition for W_n with $c \in S_t$ then $r(c|\Omega) = \mathbf{1_t} \in \mathbb{Z}^k$.

Proof. It is clear that $d(c, S_t) = 0$. Since $d(c, v) = 1$ for every $v \in V(W_n)$ with $v \neq c$, hence $d(v, S_i) = 1$ for every $i \neq t$.

Lemma 2. Let $\Omega = \{S_1, S_2, \ldots, S_k\}$ be a minimal resolving partition for W_n and let $c, c_j \in S_\ell$. There exist S_u with $u \neq \ell$ such that $d(c_j, S_u) = 1$ if and only if $c_{j-1} \notin S_\ell$ or $c_{j+1} \notin S_\ell$ for every $j \in \mathbb{Z}_n$.

Proof. Since $d(c_j, c_{j-1}) = d(c_j, c_{j+1}) = 1$, there exists $S_u \in \Omega$ with $u \neq \ell$ such that $d(c_j, S_u) = 1$ if and only if $c_{j-1} \in S_u$ or $c_{j+1} \in S_u$.

Corollary 1. Let $\Omega = \{S_1, S_2, \ldots, S_k\}$ be a minimal resolving partition for W_n such that $c_i \in S_t$ and either $c_{i-1} \notin S_t$ or $c_{i+1} \notin S_t$ for every $i \in \mathbb{Z}_n$. Then there is no $v \in V(W_n)$ such that $r(v|\Omega)2(\mathbf{1_j}) \in \mathbb{Z}^k$ for every. Type equation here.

Proof. If $v = c$ then $(v|\Omega) \neq 2(\mathbf{1_j})$. Let $v = c_i \in S_t$ for $i \in \mathbb{Z}_n$ and suppose, without sacrificing generality, that $c \in S_1$. If v is not in S_1, then $d(v, S_1) = d(v, c) = 1$. Hence, $r(v|\Omega) \neq 2(\mathbf{1_j})$. If $v \in S_1$ then $d(v, S_1) = 0$. By Lemma 2, there is a number u different with t such that $d(v, S_u) = 1$. Therefore, $r(v|\Omega) \neq 2(\mathbf{1_j})$.

Lemma 3. Let $\Omega = \{S_1, S_2, \dots, S_k\}$ be a resolving partition for W_n and define $S_i' = \{(v, v_j) | v \in S_i, 1 \leq j \leq m\}$ for every i. If $\Omega' = \{S_1', S_2', \dots, S_k'\}$ then $r((v, v_1)|\Omega') = r(v|\Omega)$ for every $v \in V(W_n)$.

Proof. Pick a fix vertex $v \in S_t$, we know that $0 \leq d(v, S_i) \leq 2$ for every i. Consider any vertex w in W_n with $w \notin S_t$. It follows that $d\left((v, v_1), (w, v_j)\right) \geq 2$ for $j \geq 2$. Consequently, $d((v, v_1), S_t') = 0$ and for $i \neq t$ $d((v_1, v), S_i') = d((v_1, v), (w, v_1))$ for some $w \in S_i$.Therefore, $d((v_1, v), S_i') = d((v, v_1), (v_1, w)) = d(v, w) = d(v, S_i)$.

Lemma 4. Let Ω be a minimal resolving partition that satisfy the necessary condition in Corollary 1. If $W_n \rhd_o P_m$ is a comb product graph where o is a vertex of P_m with degree 1, then $pd(W_n \rhd_o P_m) \leq pd(W_n)$.

Proof. Let $\Omega = \{S_1, S_2, \dots, S_k\}$ is a minimal resolving partition for W_n. Let c be the central vertex and $c_0, c_1, \dots, c_{n-1}$ be the rim vertices of W_n. Let $v_1, v_2, \dots, v_m$ be the vertices of P_m and $E(P_m) = \{v_1, v_2, \dots, v_m\}$. We assume that $V(W_n \rhd_o P_m) = \{(c_k, v_l)|k \in [1, n]\}$, and $l \in [1, m]\} \cup \{(c, v_k)|, k \in \{1, \dots, m]\}$. The resolving partition Ω will induces a resolving partition of $W_n \rhd_o P_m$. We define a subset of $V(W_n \rhd_o P_m)$ as follows: $S_i' = \{(v, v_j) | v \in S_i, j \in [1, m]\}$ for all i. Therefore, $\Omega' = \{S_1', S_2', \dots, S_k'\}$ is a partition of $V(W_n \rhd_o P_m)$. Consider Ω' as an ordered partition, we claim that Ω' is a resolving partition. Assume, without losing generality, that $c \in S_1$. We have that $r\left((c, v_j)|\Omega'\right) = j(\mathbf{1_1})$ for $1 \leq j \leq m$. Let $\mathbf{x}, \mathbf{y} \in S_t'$ for some t. Let $\mathbf{x}$ and $\mathbf{y}$ be two vertices of $W_n \rhd_o P_m$ with $\mathbf{x}, \mathbf{y} \in S_t'$. Case 1. $\mathbf{x} = (c, v_j), \mathbf{y} = (c_i, v_\ell)$ with $c_i \in S_1$

We have that $r(\mathbf{y}|\Omega') = r(c_i|\Omega) + (\ell - 1)(\mathbf{1_1})$ for $1 \leq \ell \leq m$. By Lemma 2, there exists some coordinates in $r(c_i|\Omega)$ whose value are equal to 1. Since Ω is a resolving partition, there exist some coordinates of $r(c_i|\Omega)$ whose value are equal to 2. Hence, $r(c_i|\Omega) \neq s(\mathbf{1_1})$ for all integer s. We conclude, $r(\mathbf{x}|\Omega')$ is not the same as $r(\mathbf{y}|\Omega')$.

Case 2. $\mathbf{x} = (c_i, v_j), \mathbf{y} = (c_i, v_\ell)$ with $j \neq \ell$.

Since $r(\mathbf{x}|\Omega') = r(c_i|\Omega) + (j - 1)(\mathbf{1_t})$ and $r(\mathbf{x}|\Omega') = r(c_i|\Omega) + (\ell - 1)(\mathbf{1_t})$, we can deduce that $r(\mathbf{x}|\Omega') \neq r(\mathbf{y}|\Omega')$.

Case 3. $\mathbf{x} = (c_i, v_j), \mathbf{y} = (c_q, v_\ell)$ with $i \neq q$.

We have that $r(\mathbf{x}|\Omega') = r(c_i|\Omega) + (j - 1)(\mathbf{1_t})$ and $r(\mathbf{y}|\Omega') = r(c_q|\Omega) + (\ell - 1)(\mathbf{1_t})$. Assume that $r(\mathbf{x}|\Omega') = r(\mathbf{y}|\Omega')$. We know that, every nonzero coordinate of $r(c_i|\Omega)$ is either 1 or 2. If there exist a natural number u such that the u-th coordinate of $r(c_i|\Omega)$ is equal to the u-th coordinate of $r(c_q|\Omega)$, then $j = \ell$. It follows that, $r(c_i|\Omega) = r(c_q|\Omega)$, a contradiction. Next, by Lemma 2, we may assume that there exist natural numbers u and v such that $d(c_i, S_u) = 1, d(c_i, S_v) = 2$ and $d(c_q, S_u) = 2, d(c_q, S_v) = 1$. From the u-th coordinate, $\ell + 1 = j$ and from the v-th coordinate we have that,

$\ell = j + 1$

Therefore, j is equal to $j + 2$, a contradiction.

Theorem 2. Let Π be a minimal resolving of W_n. If $W_n \rhd_o P_m$ is a comb product graph where o is a vertex of P_m with degree 1, then $pd(W_n \rhd_o P_m) \leq pd(W_n) + 1$.

Proof. By Lemma 4, the comb product graph $W_n \rhd_o P_m$ satisfy the inequality $pd(W_n \rhd_o P_m) \leq pd(W_n) < pd(W_n) + 1$. Next, we assume that there exist $v \in V(W_n)$ such that $r(v|\Omega) = 2(\mathbf{1}_{\ell})$ for some ℓ. It is clear that $v \neq c$ where c is the center of the graph W_n. Let $\Omega = \{S_1, S_2, \dots, S_k\}$. Assume, without loss of generality, that $c \in S_1$. We define the subsets of $V(W_n \rhd_o P_m)$ listed below: $S_1' = \{(c, v_1\} \cup \{(v, v_j)|v \in S_1, v \neq c, j \in [1, m]\}$, $S_i' = \{(v, v_j)|v \in S_i, j \in [1, m]\}$ for every $i \geq 2$, and $S_{k+1}' = \{(c, v_j)|\ 2 \leq j \leq m\}$. It follows that $\Omega' = \{S_1', S_2', \dots, S_k', S_{k+1}'\}$ is a partition of $V(W_n \rhd_o P_m)$. The next step is to show Ω' is a resolving partition.

It is obvious to verify that $r[(c, v_1)|\Omega'] = (r(c|\Omega), 1) = (\mathbf{1_1}, 1) = (0,1,1, \dots, 1) \in \mathbb{Z}^{k+1}$ and $r[(v, v_1)|\Omega'] = (\mathbf{2}_{\ell}, 2) = (2, ..2,0,2, \dots, 2) \in \mathbb{Z}^{k+1}$. If $\mathbf{x} \in S_{k+1}'$, then $r[(c, v_j)|\Omega'] = (j - 1, j, j, \dots, j, 0) \in \mathbb{Z}^{k+1}$. We conclude that $r[(c, v_1)|\Omega'] \neq r[(v, v_1)|\Omega']$. Since $d((c_i, v_j), S_{k+1}') = j + 1 \geq 2$, we also have that $r[(c, v_1)|\Omega'] \neq r[(c_i, v_j)|\Omega']$. Note that $r[(v, v_1)|\Omega'] = (r(v|\Omega), 2)$ and $r[(c_i, v_1)|\Omega'] = (r(c_i|\Omega), 2)$. Since Ω is a resolving partition, $r[(v, v_1)|\Omega'] \neq r[(c_i, v_1)|\Omega']$ for all i. For $j \geq 2$, we have that $d[(c_i, v_j), S_{k+1}'] = j + 1 \geq 3$. Therefore, $r[(v, v_1)|\Omega'] \neq r[(c_i, v_j)|\Omega']$ for all $j \geq 2$. Let $\mathbf{x}$ and $\mathbf{y}$ be two vertices of $W_n \rhd_o P_m$ with $\mathbf{x}, \mathbf{y} \in S_t'$. It is obvious that $r(\mathbf{x}|\Omega') \neq r(\mathbf{y}|\Omega')$ if $\mathbf{x}, \mathbf{y} \in S_{k+1}'$. For the last case, assume that $\mathbf{x} = (c_i, v_j), \mathbf{y} = (c_q, v_{\ell}) \in S_t'$ with $q \neq i$ and $k + 1 \neq t$. If $j \neq \ell$, then $d((c_i, v_j), S_{k+1}') \neq d((c_i, v_{\ell}), S_{k+1}')$. Hence, $r(\mathbf{x}|\Omega') \neq r(\mathbf{y}|\Omega')$. If $j = \ell$, then $r(\mathbf{x}|\Omega') = (r(c_i|\Omega) + (j - 1)(\mathbf{1_t}), 1 + j)$ and $r(\mathbf{y}|\Omega') = (r(c_q|\Omega) + (j - 1)(\mathbf{1_t}), 1 + j)$. If $r(\mathbf{x}|\Omega') = r(\mathbf{y}|\Omega')$, then $r(c_i|\Omega) = r(c_q|\Omega)$. This contradicts Ω as a resolving partition. We have proved that $\Omega' = \{S_1', S_2', \dots, S_k', S_{k+1}'\}$ is a resolving partition of $W_n \rhd_o P_m$. Consequently, $pd(W_n \rhd_o P_m) \leq k + 1 = pd(W_n) + 1$.

We get the following result because of Theorem 1.

Corollary 2. For all integer $n \geq 4$ we have $pd(W_n \rhd_o P_m) \leq p + 2$, where p is the lowest prime number that has the property of $p(p - 1) \geq n$.

4. Conclusion

The $pd(W_n \rhd_o P_m)$ of the comb product is closely related to $pd(W_n)$. The partition dimension of the comb product graph between the wheel and path graph has an upper bound, which we present by constructing its resolving partition from a resolving partition of W_n. We have proved that if $W_n \rhd_o P_m$ is a comb product graph where o is a vertex of P_m with degree 1, then $pd(W_n \rhd_o P_m) \leq pd(W_n) + 1$.

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