Effect of Modulating Parameters in Chaos and Lyapunov Exponent

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Abstract
In the last few decades, the field of dynamics has experienced an intensive development in chaos theory. Also, it has found a celebrated place in science and engineering for solving problems of population of species, transportation models, weather forecasting, cryptography, secure communication, etc. This article deals with a simple one-dimensional map which is tested for the dynamical properties in superior fixed-point procedure. The novel system depends on the three control parameters β, r and b and exhibits a reverse representation which is shown in bifurcation and Lyapunov exponent plots. The superior recursive orbit is generated and the analytical results are derived for the fixed states and the stability of fixed states. Further, an experimental analysis is also carried out to examine the periodicity of higher orders and chaotic behavior for different ranges of parameters β, r and b = 2. Moreover, a comparative analysis between bifurcation plot and Lyapunov spectrum is also demonstrated which shows shrinkage in periodicity and chaos as the value of parameter β decreases from 1 to 0.

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1. INTRODUCTION

The concept of dynamics of difference and differential equations plays a vital role in mathematics, science and engineering. The oldest and simplest difference equation is the logistic equation $rx(1-x)$ which was first used in ecology to express the population dynamics. The map was popularized in 1976 in a paper published by the famous biologist Robert May [21], in part as a discrete time demographic model analogous to the logistic equation first created by Pierre Francois Verhulst. A variety of problems in Physics, Chemistry, Biology and Economics can be formulated in terms of nonlinear difference equations. Despite their simplicity the dynamical equations enter into our daily experiences in numerous interesting and useful ways. Hypothetically, it is studied that the discrete maps exhibit unpredictability, aperiodicity and sensitive dependence on initial conditions, such behavior can be termed as chaotic behavior. Many researchers have made efforts to solve different maps and find their behavior so that they can find key to solve more complicated systems. There are several books which gives basic knowledge of chaos theory such as Alligood et al. [1], Ausloos et al. [8], Holmgren [17], Diamond [13] and Devaney [12], and Cao et al. [9].

Dynamical systems are always determined and may be very simple yet they produce completely unpredictable and divergent behavior. The physical models constitute a side of dynamical systems which may be used as a quantitative tool to analyze the environment around us. In 1987, Harikrishanan and Nandakumaran [16] studied the bifurcation structure of the logistic map with a time dependent control parameter. By introducing a specific nonlinear variation for the parameter, they studied that the bifurcation structure is modified qualitatively as well as quantitatively from the first bifurcation onwards. In 2003, Luo et al. [19] proposed a new hybrid control strategy in which state feedback and parameter perturbation are used to control the period-doubling bifurcations and to stabilize unstable periodic orbits embedded in the chaotic attractor of a discrete chaotic dynamical system. In 2008, Salarieh and Alasty [23] studied the problem of chaos control in nonlinear maps using minimization of entropy function. In 2008, Elhadj and Sprott [14, 15] discussed that chaotic systems which are generally characterized by one or more parameters that control the behavior of the system. In 2012, A. G. Radwan [32] introduced the different generalized Logistic maps with arbitrary powers which can be reduced to conventional Logistic map. In 2014, O. Alpar [2] gives a one-dimensional map and also it was discussed that the chaos arises when the parameter raises up to a value. But in this map chaos arises when related parameter decreases and approaches to a constant value. Also, this map is not limited with the constant parameters or intervals. In 2018, Ashish et al. [4] investigated the chaotic behavior such as period doubling, period-3 window and life Lyapunov exponent of the standard logistic map. Also, they described an improved chaos based discrete traffic control model. Newly added parameter α in Mann orbit worked as control variable that increased the stability performance of the traffic model. In 2019, Cao et al. [5, 6] developed a superior technique to control chaos in a class of one-dimensional discrete systems and the unstable fixed and periodic states responsible for chaotic behavior was stabilized. For more applications one may also refer to [3, 10, 18, 24, 25, 26, 27].

Recently, Ashish et al. [7] established a modulated logistic system and reported a superior chaos through period-doubling bifurcation, period-three orbits and Lyapunov exponent properties. Moreover, it is speculated that the superiority in chaos may be used in chaos- based cryptography, in the future. The paper is divided into five sections. Section 1 contains an introduction to the dynamical systems. In Section 2, the analytical results followed by theorems and examples are introduced. The numerical simulation to verify the theoretical results is illustrated in Section 3. In Section 4, the largest Lyapunov exponent is determined followed by Lyapunov spectrum plot. Finally, the paper is concluded in Section 5.

2. ANALYTICAL ANALYSIS

This section deals with a few analytical results on the dynamics of a simple discrete one-dimensional equation using superior fixed-point iterative system. The equation was first introduced by O. Alpar [2] in 2014 which admits all the dynamical properties such as stable regular state, novel periodicity and irregular chaotic state. Throughout, let us consider the following one-dimensional chaotic equation:

$$f_r(x,b) = \sqrt{\frac{rx^2}{(x-b)}}, \qquad (1)$$

where $r > 0$, $x > b$ and $x \in [0, 1]$. The dynamics of this equation (1) depends on the control parameters b and r. Now, using the superior fixed-point iterative system [20] and the initiator $x_0 \in [0, 1]$, we obtain the following relation:

$$x_1 = (1-\beta)x_0 + \beta f_r(x_0, b). \qquad (2)$$

In general, for the nth iterative orbit, we can write

$$x_n = (1-\beta)x_{n-1} + \beta f_r(x_{n-1}, b) = N_{\beta,r}(x) \qquad (3)$$

where $\beta \in (0, 1)$ and n is a natural number. Here, the range of the control parameter r and the dynamics of the system depends on the parameter β and b. Also, the orbit evolution of the system (3) is determined as follows:

$$N_{\beta,r}(x_0) = x_1 = (1-\beta)x_0 + \beta f_r(x_0, b)$$
$$N^2{}_{\beta,r}(x_0) = x_2 = (1-\beta)x_1 + \beta f_r(x_1, b)$$
$$\ldots\ldots\ldots\ldots$$
$$N^n{}_{\beta,r}(x_0) = x_n = (1-\beta)x_{n-1} + \beta f_r(x_{n-1}, b)$$

where n is the natural number and $N^n{}_{\beta,r}(x_0)$ is called the nth iterate of the equation $f_r(x, b)$ for all $x > b$. Then, the iterative sequence $(x_0, x_1, x_2, \ldots, x_n)$ for an initiator $x_0 \in [0, 1]$ is known as the orbit of the given equation (1) in the superior fixed-point iterative system. Further, it is trivial that the dynamical system is completely depends on the iterative orbits of the one-dimensional discrete maps. But the orbit of the system attains different kind of properties such as fixed-point orbit, periodic point orbit and the irregular orbits. Therefore, now we demonstrate the following results depending on the nature of the orbit:

Theorem 2.1. Let $f_r(x, b)$ be the original one-dimensional equation (1) and $N_{\beta,r}(x)$ be the modulated superior iterative system, where $r > 0$ and $x > b$. Then, show that there exists two fixed point 0 and $r + b$ for the system $N_{\beta,r}(x)$, for each $\beta \in (0, 1)$ and $r > 0$.

Proof: Since, $f_r(x, b)$ be the original one-dimensional equation and $N_{\beta,r}(x)$ be the modulated superior iterative system. Then, from (1) and (2), we get

$$N_{\beta,r}(x) = (1-\beta)x + \beta\sqrt{\frac{rx^2}{(x-b)}}. \qquad (4)$$

To examine the fixed point of chaotic maps, let us consider Devaney's Definition for fixed points, then we have

$$(1-\beta)x + \beta\sqrt{\frac{rx^2}{(x-b)}} = x,$$

$$(1-\beta)x + \beta\sqrt{\frac{rx^2}{(x-b)}} - x = 0. \qquad (5)$$

Then solving (5), we get the required fixed-point $x^* = 0$ and $x^* = r + b$ which depends on the control parameter r and b. This completes the proof.

Theorem 2.2. Let $f_r(x, b)$ be the original one-dimensional equation (1) and $N_{\beta,r}(x)$ be the modulated superior iterative system, where $r > 0$, $x > b$ and $b = 2$. Then, show that the fixed-point $x^* = r + 2$ is stable for $N_{\beta,r}(x)$.

Proof: As studied in Theorem 2.1, $x^* = r + 2$ is the regular fixed-point for the system $N_{\beta,r}(x)$, where $r > 0$ and $\beta \in (0, 1)$. To prove $r + 2$ is stable fixed-point, let us consider the result, "Let x^* be a regular fixed-point for a one-dimensional map f_r. Then, x^* is said to stable if $|f_r'(x^*)| < 1$ and is said to be unstable if $|f_r'(x^*)| > 1$. Therefore, let us consider

$$N_{\beta,r}(x) = (1-\beta)x + \beta\sqrt{\frac{rx^2}{(x-2)}}. \qquad (6)$$

Differentiating (6) with respect to x, then we get

$$N'_{\beta,r}(x) = (1-\beta) + \beta\sqrt{r}\frac{d}{dx}\left(\frac{x}{\sqrt{x-2}}\right),$$

$$N'_{\beta,r}(x) = (1-\beta) + \beta\sqrt{r}\left(\frac{x-4}{2(x-2)^{3/2}}\right), \qquad (7)$$

Now, putting the fixed-point $x = r+2 = x^*$ in (7), we obtain

$$N'_{\beta,r}(r+2) = (1-\beta) + \beta\sqrt{r}\left(\frac{r-2}{2r^{3/2}}\right), \qquad (8)$$

Then, using the definition of stable fixed-point in the (8), we have

$$\left|N'_{\beta,r}(r+2)\right| > 1 \text{ if } r < -2 \qquad (9)$$

and $$\left|N'_{\beta,r}(r+2)\right| < 1 \text{ if } r > -2 \qquad (10)$$

Since, we are taking $r > 0$, therefore, using (10), the fixed-point $x^* = r + 2$ must be stable for $r > 0$. This completes the proof.

3. EXPERIMENTAL ANALYSIS

This section deals with the numerical simulation of original one-dimensional map $f_r(x, b)$ in superior fixed-point iterative method $N_{\beta,r}(x)$,. As discussed in earlier section, the dynamical properties of the system $N_{\beta,r}(x)$, depends on the three control parameters b, r and β. But the nature of the system $N_{\beta,r}(x)$, is studied for different values of $b > 0$ and $\beta \in (0, 1)$ for which the parameter $r \in (0, 1)$. Therefore, let us consider the three cases for $\beta = 0.9$, 0.6 and 0.3 and examine the dynamical nature as follows:

When $\beta = 0.9$, the map has stable behavior up to value $b = 0.3$, the very first bifurcation is seen for $b = 0.4$ and the full bifurcation behavior is obtained first time for the value $b = 0.5$. Now, fixing the value $b = 2$, we find that this map exhibits sensitive dependence for an initiator $x_0 > 2$ for the full range of the control parameter r and illustrate complete period-doubling bifurcation nature. Therefore, setting step size $r = 0.001$, initiator $x_0 = 2.3$ and controller $\beta = 0.9$ the period-doubling bifurcation is plotted in Figure 1, and the characteristics from routes to chaos are demonstrated. From the bifurcation plot it is examined that the fixed-point $r + 2$ approaches to stable fixed-point state for $r \geq 0.5814$ and it bifurcates into periodicity of order two, that is, in the stable branches D_1 and D_2 for the parameter range $0.4691 \leq r < 0.5814$. Further, the second period-doubling bifurcation occurs when r decreases from 0.4691. Figure 2 shows that the periodicity of order two continues up to 0.4691 and before that up to $r = 0.4529$ we get periodicity of order four, that is, D_1 and D_2 bifurcates into D_{11}, D_{12}, D_{21} and D_{22}. Continuing, in this way, as the parameter β is further decreased, the sequence $\{r_n\}$ of period-doubling bifurcation of order 2^n approaches up to $r_\infty \approx 0.4496$.

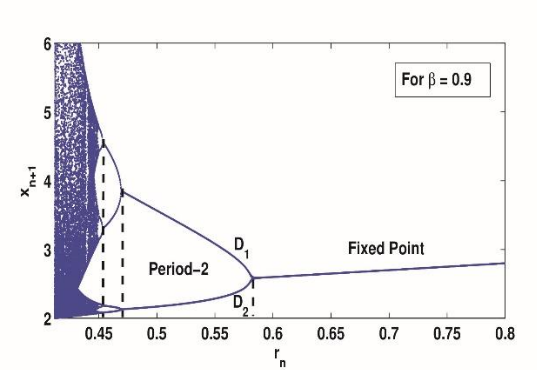

Figure 1: Bifurcation plot for the map $N_{\beta, r}(x)$ for β = 0.9 and $r \in [0.4121, 0.8]$

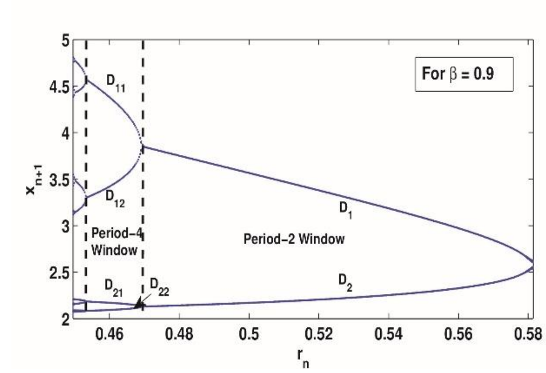

Figure 2: Bifurcation plot for the map $N_{\beta, r}(x)$ for β = 0.9 and $r \in [0.4496, 0.5814]$

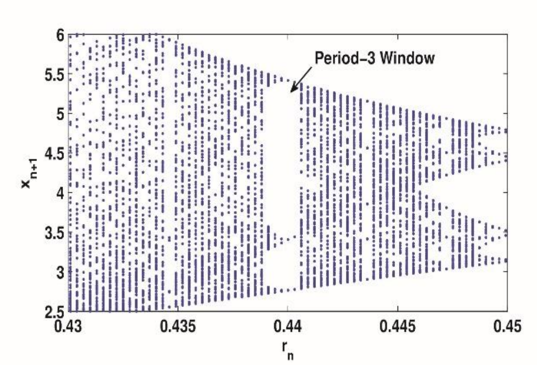

Figure 3: Period-3 window for the map $N_{\beta, r}(x)$ for β = 0.9, $x \geq 2.5$ and $r \in [0.43, 0.45]$

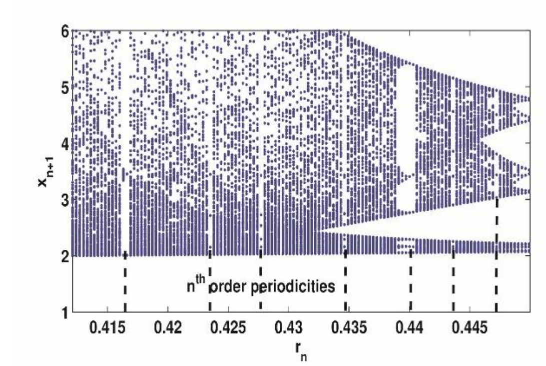

Figure 4: Magnified chaotic region for the map $N_{\beta, r}(x)$ for β = 0.9 and $r \in [0.41, 0.45]$

But for $r < 0.4496$ the bifurcation plot approaches to chaotic regime. The magnified Figure 2 shows the complete periodicity regime, Figure 3 shows the magnified chaotic regime with period-3 window and Figure 4 gives the other periodic window of higher orders. When the dynamical nature of the map $f_r(x, b)$ is tested in Mann orbit for the control parameter $\beta = 0.6$ the first bifurcation is seen at $b = 0.6$. Also, it is illustrated that the full bifurcation with periodicity comes into existence at $b = 0.8$. Therefore, by fixing $b = 2$ and $x_0 = 2.3$ the complete bifurcation diagram is shown in Figure 5. We can see the clear image of period-3 window in Figure 6, for the control parameter range r from 0.2395 to 0.2402. Here, the stability in fixed point state is present for all $r \geq 0.3533$. The first bifurcation starts from r ≈ 0.3533 and two branches can be seen which are named as D_1 and D_2. Because of the behavior of period-doubling bifurcation again the two branches D_1 and D_2 splits into four, four into eight, eight into sixteen and so on.

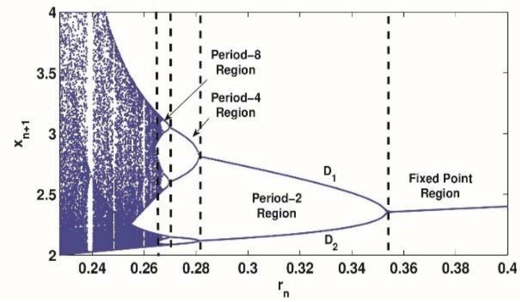

Figure 5: Bifurcation plot for the map $N_{\beta, r}(x)$ for β = 0.6 and $r\in$[0.22, 0.40]

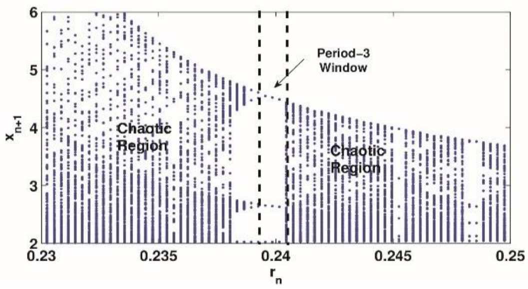

Figure 6: Period-3 window for the map $N_{\beta, r}(x)$ for β = 0.6 and $r\in$[0.23, 0.25]

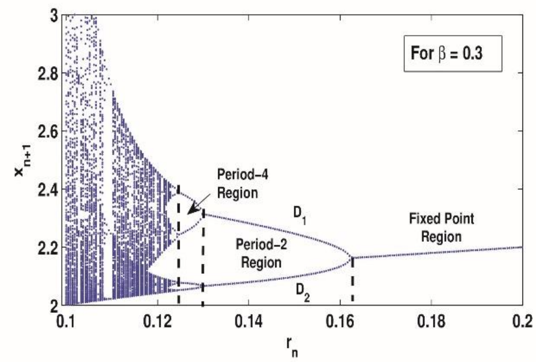

Figure 7: Bifurcation plot for the map $N_{\beta, r}(x)$ for β = 0.3 and $r\in$[0.1, 0.2]

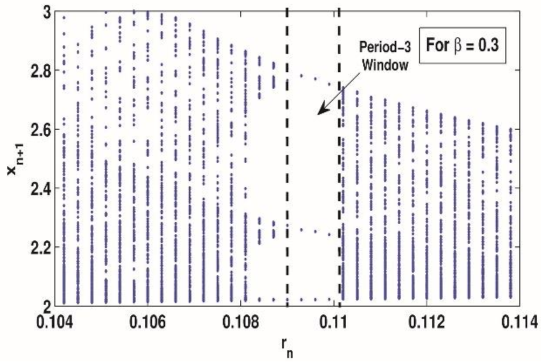

Figure 8: Period-3 window for the map $N_{\beta, r}(x)$ for β = 0.3 and $r\in$[0.104, 0.114]

In this case, the complete chaotic regime is seen for the parameter r ranges from 0.2275 to 0.3533, as for all values of r less than 0.2275 there are no states. Further, for β = 0.3 the given one-dimensional map $f_r(x, b)$, where r > 0 and x > b, is again tested in Mann orbit and as discussed above the first bifurcation occurs at b = 1.3 andon the other side for values of b up to 1.2 the map shows stable nature. Now, continuing withβ = 0.3, b = 2 and x_0 = 2.3, the system gives a fixed-point solution for all $r \geq 0.1624$. As the value of r decreased from 0.1624 the first bifurcation starts and after a periodic window of order 2 ranging for r from 0.1297 to 0.1624, the second period-doubling bifurcation occurs,that is, periodicity of order 4 when r decreases up to 0.1243. Again, in the next period-doubling bifurcation the four branches splits into eight. Further, for $r \leq 0.1003$ the chaotic regime ends as seen in Figure 7. The period-3 window have range from 0.1090 to 0.110. The zoomed view of chaotic regime and period-3 window is seen in Figure 8.

4. LARGEST LYAPUNOV EXPONENT

This section deals with the study of the largest Lyapunov exponent which measures the sensitivity of a system with some slight changes in the initial conditions, and the positive value indicates the chaotic behavior. The largest Lyapunov exponent value is determined to identify the chaotic nature and it is observed that the larger exponent is more chaotic and less predictable. For the stable periodicity, it measures the rate of convergence towards stable point, whereas for the chaotic behavior, it measures the rate of divergence between the orbits. We compute largest Lyapunov exponent by calculating the average of local Lyapunov exponents.

In this section, to determine a time-series analysis versus Lyapunov exponent, we take some particular values of the parameter r in its prescribed range at $\beta = 0.9$, 0.6 and 0.3 as discussed in Section 3. For the map $f_r(x, b)$ under the superior fixed-point procedure, Lyapunov exponent is defined in the following way:

$$LE(\lambda) = \lim_{n\to\infty} \frac{1}{n}\sum \; log\left|N'_{\beta,r}(x_n)\right| \qquad (11)$$

where the quantity $log\left|N'_{\beta,r}(x_n)\right|$ is the local Lyapunov exponent that can be easily derived from

$$\lambda = \log|\Delta x_{n+1}/\Delta x_n| \approx log\left|N'_{\beta,r}(x_n)\right|$$

where $$\Delta x_{n+1} = N_{\beta,r}(x_n + \delta x_n) - N_{\beta,r}(x_n) \approx \Delta x_n log\left|N'_{\beta,r}(x_n)\right|$$

The largest Lyapunov exponent number indicates the separation ratio between two nearby points x_{n+1} and x_n. The largest exponent is found as $\lambda = 0.5456$, after 1000 iterations and the approximate convergence of Lyapunov exponent for $\beta = 0.9$, $b = 2$ and $x_0 = 2.3$, is drawn in Figure 9 and 10. For the orbit of periodicity of order-p, let us take

$$\mathrm{LE}(\lambda) = \frac{1}{p}\sum_{i=1}^{p} log\left|N'_{\beta,r}(x_i)\right| \qquad (12)$$

Further, it is observed that for the periodic orbits, we use the full length of an orbit. But for aperiodic orbits it is not possible to use the full length of an orbit, therefore, we estimate the Lyapunov exponent using only finite length of an aperiodic orbit. Moreover, for $\lambda < 0$ the system admits stable fixed or periodic states and for $\lambda > 0$ the system shows chaotic irregular state.

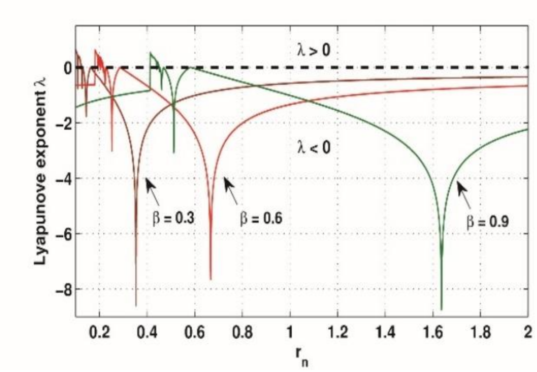

Figure 9: Lyapunov spectrum plot for the map $N_{\beta, r}(x)$ for $\beta = 0.9$, 0.6 and 0.3

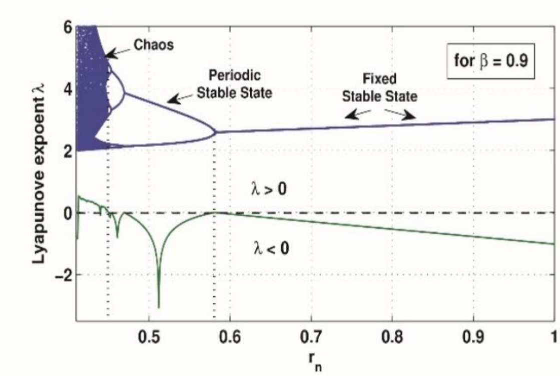

Figure 10: Lyapunov spectrum versus bifurcation plot for $N_{\beta, r}(x)$ when $\beta = 0.9$

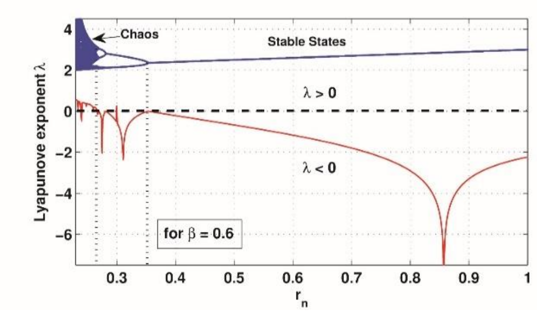

Figure 11: Lyapunov spectrum versus bifurcation plot for $N_{\beta, r}(x)$ when $\beta = 0.6$

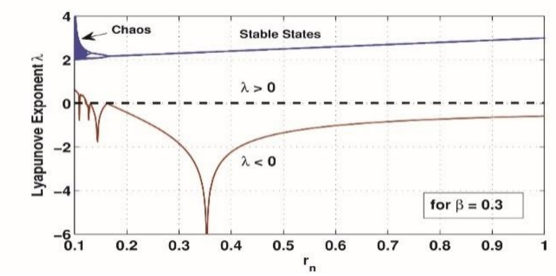

Figure 12: Lyapunov spectrum versus bifurcation plot for $N_{\beta, r}(x)$ when $\beta = 0.3$

For example, For r = 0.5 and x > 2, the orbit of the map $N_{\beta,r}(x)$ exhibits periodic behavior of order two. Therefore, the periodic points are measured as x^* = 2.1702 and x^* = 3.5655. Then, we get $N'_{0.9,\ 0.5}(2.1702)$ = −8.19203 and $N'_{0.9,\ 0.5}(3.5655)$ = 0.029417. Now, substituting these values in (12), we obtain

$$\lambda = \frac{1}{2}[\log|-8.19203| + \log|0.029417|] = -0.3090 \quad (13)$$

Thus, the required largest Lyapunov exponent at r = 0.5 is −0.3090, which is a negative value and hence by the definition of Lyapunov exponent the periodic states are stable. Similarly, we can calculate the Lyapunov exponent value for higher order periodic orbits and irregular chaotic orbits also. Moreover, the Lyapunov exponent versus parameter r for β = 0.9, 0.6 and 0.3 is also illustrated in Figure 9. While in Figures 10-12, a comparative analysis versus bifurcation plot and Lyapunov spectrum is also shown which determines a symmetric nature between fixed, periodic and chaotic states.

Remark 4.1. It is examined that as the value of parameter β decreases from 1 to 0 the periodic and chaotic states shrinks continuously as shown in Figures 9-12.

5. CONCLUSION

In this article, we deal with a novel simple one-dimensional difference map which shows the dynamical properties for different ranges of the parameter r depending on the parameter β. Therefore, the dynamics of the original map is examined analytically as well as numerically using superior fixed-point procedure. It is observed that the modified system depends on the three control parameters β, r and b. The divorce effect is observed in the bifurcation plot, that is, the chaos starts from the smallest control parameter value r and approaches to stable states as r increases. Since the dynamics depends on the parameter β, r and b, therefore, we fix b = 2 and the dynamics is examined for the variations in the control parameters β and r. In Section 2, the analytical results are derived using superior fixed-point procedure. The fixed-point states are determined for r > 0 and x > b followed by the stability result. In Section 3, an experimental analysis is carried out for the parameter β = 0.9, 0.6 and 0.3 followed by the bifurcation plots in Figures 1-8. Similarly, the largest Lyapunov exponent λ is determined using Lyapunov spectrum in Figures 9-12. It is also observed that the map exhibits its dynamical properties for a large range of parameter r, as compared to the existing methods.

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