Asymptotic Behaviour and Boundedness of Fourth Order Difference Equations

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Abstract: We study the asymptotic behavior and boundedness of fourth order difference equation of the form,

$$\Delta(r_n\Delta(s_n\Delta(q_n\Delta x_n))) + p_n g(x_{n-t+1}) - p_{n+1} g(x_{n-t+1}) = \tau_n$$

where $r_n > 0$, $s_n > 0$, $q_n > 0$, $p_n > 0$, $p_{n+1} > 0$, $\tau_n > 0$. The necessary and sufficient conditions for asymptotic behavior and boundedness of above nonlinear difference equation are obtained using summation averaging technique and comparison method. Examples are also provided.

Keywords: Difference equations, asymptotic behavior, boundedness.

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1. Introduction:

This paper deals with the study of asymptotic behavior and boundedness of fourth order nonlinear difference equation of the form,

$$\Delta(r_n\Delta(s_n\Delta(q_n\Delta x_n))) + p_n g(x_{n-t+1}) - p_{n+1} g(x_{n-t+1}) = \tau_n \qquad (1)$$

where $\{r_n\}, \{s_n\}, \{t_n\}, \{p_{n+1}\}, \{p_n\}, \{\tau_n\}$ are sequences of positive real numbers with $r_n > 0$, $s_n > 0$, $q_n > 0$, $p_n > 0$, $\tau_n > 0$, g is real valued function and τ is positive integer. The nontrivial solution of (1) is oscillatory if the terms of sequence $\{x_n\}$ are neither eventually positive nor eventually negative and nonoscillatory otherwise. Also, (1) becomes almost oscillatory if every solution $\{x_n\}$ is either oscillatory or satisfies $\lim_{n\to\infty} \Delta^i x_n = 0$ for $i = 0,1,2$. The purpose of this paper is to establish necessary and sufficient conditions for asymptotic behavior and boundedness of fourth order nonlinear difference equations. The difference equations are applied in the field of statistics, economics, biology, etc see for example [1-8]. The results achieved from this paper motivate the studies on higher order difference equations. The paper is organised as follows: In section 2, the methodology is stated. In section 3, the conditions of almost oscillatory solutions of (1) are obtained. This is followed by determining the conditions for boundedness and asymptotic behavior of (1). In section 4, the conclusion is provided. We illustrate the results with few examples.

2. Methodology

The asymptotic behavior and boundedness of fourth order nonlinear difference equations are studied by means of contraction mapping principle, summation averaging technique and comparison method.

3. Main Results

Theorem 1: Let $r_n = s_n = q_n = 1$ and assume there exists a positive sequence $\{\phi_n\}$ and an oscillatory sequence $\{\psi_n\}$ such that $\Delta(r_n\Delta(s_n\Delta(q_n\Delta x_n)) = \tau_n$ with $\lim \Delta^i \psi_n = 0$ for $i = 0,1,2,3$. If

$$\sum_{n=n_0}^{\infty} n^3 (p_n - p_{n+1}) = \infty \qquad (2)$$

then (1) is almost oscillatory for every bounded solution $\{x_n\}$.

Proof: If $x_n > 0$ and $x_{n-t+1} > 0$ for all n then from (1) we have,

$$\Delta(r_n\Delta(s_n\Delta(q_n\Delta x_n))) = \tau_n + p_{n+1}g(x_{n-t+1}) - p_n g(x_{n-t+1})$$

We consider $\{\Delta^3 x_n\}$ as,

$$\Delta^3 x_n = (p_n - p_{n+1})\phi_{n+3}g(x_{n-t+1}) - 3\Delta\phi_{n+2}\Delta(s_n\Delta(q_n\Delta x_n) + 3\Delta^2\phi_{n+1}\Delta(q_n\Delta x_n) \\ - \Delta^3\phi_n(q_n\Delta x_n)$$

Let $\{x_n\}$ be a nonoscillatory solution such that $\{x_n\}$ is positive. Define a function y_n as,

$y_n = x_n - \psi_n$

To prove the theorem we define the following 2 cases:

Case 1: If $\{y_n\}$ is positive then $\{\Delta y_n\}, \{\Delta^2 y_n\}$ and $\{\Delta^3 y_n\}$ are eventually of one sign and monotonic, hence $\{\Delta^3 y_n\}$ is an increasing sequence. To prove this we assume $\{\Delta^3 y_n\}$ to be eventually negative. Since $\{\Delta y_n\}$ is decreasing, then eventually it must become negative. This is a contradiction and thus $\{\Delta^3 y_n\} \geq 0$.

Case 2: If $\{\Delta y_n\}$ is negative then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = h$ where h is a nonnegative number. In order to prove this we take $\{\Delta y_n\}$ to be positive then $\{\Delta^3 y_n\} \geq 0$. This implies $\{y_n\}$ to be unbounded and gives a contradiction. Thus $\{\Delta y_n\}$ must be eventually negative. Now to show $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = h$ it is sufficient to prove $h = 0$. Consider Δx_n with the positive sequence $\{\phi_n\}$ and taking summation from n to j by the fact $\{\Delta^3 y_n\} \geq 0$ then,

$$-\Delta^3 y_n + \sum_{v=n}^{j} (p_v - p_{v+1}) g(x_{v-t+1}) \leq 0$$

Now summing from j to k we have

$$-\Delta^2 y_n + \sum_{n=j}^{k} \left(\sum_{v=n}^{j} (p_v - p_{v+1}) g(x_{v-t+1}) \right) \leq 0$$

Repeating the above procedure we get,

$$\Delta y_n + \sum_{n=j}^{k} \left(\sum_{v=n}^{j} (v-n+1)(p_v - p_{v+1}) g(x_{v-t+1}) \right) \leq 0$$

Taking summation from to ∞,

$$\Delta y_n + \sum_{k=n}^{\infty} \left(\sum_{j=n}^{k} \left(\sum_{v=n}^{j} \frac{(v-n+1)(v-n+2)}{2} \right) \right) (p_v - p_{v+1}) g(x_{v-t+1}) \leq 0$$

Or a final summation from N to ∞ yield,

$$h - y_n + \sum_{v=N}^{\infty} \left(\frac{(v-N+1)(v-N+2)(2v-2N+3)}{6} \right) (p_v - p_{v+1}) g(x_{v-t+1}) \leq 0$$

where h is a positive constant. From (2) and above equation we get,

$$\lim_{n\to\infty} \inf x_n \, g(x_{n-t+1}) = \lim_{n\to\infty} y_n = 0$$

This contradicts our assumption and the theorem is proved.

Example-1: We consider the following difference equation,

$$\Delta^4 x_n - \frac{2^{n-4}}{900} (30 - 21(-1)^n) x_n^2 = \frac{21(-1)^n}{2^{n+4}} \qquad (3)$$

It satisfies all the conditions of Theorem 1 with $\{\psi_n\} = \left\{\frac{(-1)^n}{2^n}\right\}$. Hence the bounded solutions of (3) are almost oscillatory and $\{x_n\} = \left\{\frac{30}{2^n}\right\}$ is one such solution.

ASYMPTOTIC BEHAVIOR OF DIFFERENCE EQUATION

The necessary and sufficient conditions of asymptotic behavior of (1) are obtained. We do not require $p_n > 0$ and $p_{n+1} > 0$. Let R_n, S_n, Q_n, D_n be defined by

$R_n = \sum_{t=n_0}^{n-1} \frac{1}{r_t}$, $S_n = \sum_{t=n_0}^{n-1} \frac{1}{s_t}$, $Q_n = \sum_{t=n_0}^{n-1} \frac{1}{q_t}$, $D_n = \sum_{t=n_0}^{n-1} \frac{R_t S_t}{q_t}$

Theorem 2: Let $f(\alpha)$ be non-decreasing and $h > 0$ be a constant such that $r_n \geq h$ for all $n \geq n_0$. If

$$\sum_{n=n_0}^{\infty} [\, D_{n+1} + R_{n+1}S_{n+1}Q_{n+1}]\, |\tau_n| < \infty$$

and

$$\sum_{n=n_0}^{\infty} [\, D_{n+1} + R_{n+1}S_{n+1}Q_{n+1}]\, |p_n - p_{n+1}| < \infty$$

then (1) has bounded nonoscillatory solution that approaches a nonzero limit.

Proof: Take $x_{n-t+1} \geq h/2$ for $h > 0$ then we have,

$$\sum_{n=n_0}^{\infty} [\, D_{n+1} + R_{n+1}S_{n+1}Q_{n+1}]\, |\tau_n| < \frac{h}{4}$$

and

$$\sum_{n=n_0}^{\infty} [\, D_{n+1} + R_{n+1}S_{n+1}Q_{n+1}]\, |p_n - p_{n+1}| < \frac{h}{4f(2h)}$$

Let $\mathbb{B}_N$ be the Banach space and $\ell \subseteq \mathbb{B}_N$ and define $W\colon \ell \to \mathbb{B}_N$,

$$(WX)_n = \frac{3h}{2} + \frac{1}{2}\sum_{v=n}^{\infty}\Big((v-n+1)(v-n+2)\Big(\big(p_v g(x_{v-t+1}) - p_{v+1} g(x_{v-t+1})\big) - \tau_v\Big)\Big)$$

$$(WX)_n = \frac{3h}{2} + \sum_{v=n}^{\infty} K(v,n)((p_v - p_{v+1})g(y_{v-t+1}) - \tau_v)$$

where $K(v,n) = D_{v+1} - D_n - R_n S_{v+1} T_{v+1} - R_{v+1} S_{v+1} T_n - R_{v+1} S_n T_{v+1} + R_{v+1} S_{v+1} T_{v+1}$. From Theorem 1 we observe that $X = \{x_n\}$. Thus (1) has nonoscillatory solution that approaches a nonzero limit.

Example-2: Consider the following difference equation,

$$\Delta^2(n^2\Delta^2 x_n) - 2^{-n}\left(1 + \frac{(-1)^n}{2^n} x_{n-m}^{2\gamma-1}\right)\left(\frac{43n^2 - 76n - 60}{8}\right)$$
$$= \frac{(-1)^{n+1}}{2^{2n+4}}\left(\frac{1}{2^{n-m}}\right)^{\gamma}(43n^2 - 76n - 60) \qquad (4)$$

where γ is a ratio of the odd positive integers. Every condition of Theorem 2 are satisfied so (4) has a bounded nonoscillatory solution that approaches a nonzero limit.

The succeeding theorem is a special case of Theorem 2. We provide examples for proving the results.

Theorem 3: Let $r_n = s_n = q_n \equiv 1$ and g be non-decreasing. If

$$\sum_{n=n_0}^{\infty} n^2 |p_n - p_{n+1}| < \infty$$

and

$$\sum_{n=n_0}^{\infty} n^2 |\tau_n| < \infty$$

then (1) has nonoscillatory solution that approaches a non-zero real number as $n \to \infty$.

Proof: Let $h > 0$ be given so that

$$\sum_{n=N}^{\infty} n^2 |p_n - p_{n+1}| < \frac{h}{2} \frac{1}{g(2h)}$$

and

$$\sum_{n=N}^{\infty} n^2 |\tau_n| < \frac{h}{2}$$

Let $\mathbb{B}_N$ be a Banach space for every real sequence $X = \{x_n\}$ with norm,

$$\|X\| = \sup_{n \geq N} |x_n|$$

Define $W: l \to \mathbb{B}_N$ and by $h \leq x_n \leq 2h$ we get,

$$(WX)_N = \frac{3h}{2} + \sum_{v=n}^{\infty} \left(\frac{(v-n+1)(v-n+2)(2v-2n+3)}{6} (p_v g(x_{v-t+1}) - p_{v+1} g(x_{v-t+1}) - \tau_n) \right)$$

Thus ℓ is closed, bounded and convex subset of $\mathbb{B}_N$. Now to show T maps ℓ into itself then,

$$\left|(WX)_N - \frac{3h}{2}\right| \leq \sum_{t=n}^{\infty} v^2 (|p_v - p_{v+1}| g(2h) + |\tau_v|) \leq \frac{h}{2}$$

Let $Y = \{y_n\}$ and $X^i = \{x_n^i\}$ such that $\|X^i - Y\| = 0$ then the continuity of g shows that

$$\lim_{i \to \infty} \|(WX^i)_N - (WY)_N\| = 0$$

Hence W is continuous. Now to show $W\ell$ is relatively compact. Let $X = \{x_n\} \in \ell$ for any $k > n > N$ and $W\ell$ is uniformly Cauchy we have,

$$|Wx_n - Wx_k| \leq \sum_{t=n}^{\infty} v^2(|p_v - p_{v+1}|g(2h) + |\tau_v|)$$

For given ε there exist an integer N_1 such that $k > n \geq N_1$ then $|Wx_n - Wy_k| < \varepsilon$. Thus $W\ell$ is uniformly Cauchy and hence $W\ell$ becomes relatively compact. Thus (1) has nonoscillatory solution for $X = \{x_n\}$ with required properties.

Example-3: We consider the following difference equation,

$$\Delta^4 x_n - 2^n\left(\frac{1}{16} + (-1)^n 2^{(n+1)/2}\right)x_n^2 = (-1)^{n+1} 2^{\left(\frac{1-n}{2}\right)} \tag{5}$$

for $n \geq 3$ that satisfies all the conditions of Theorem 3. Thus (5) has nonoscillatory solution that approaches a nonzero real number. In fact, $\{x_n\} = \left\{\frac{1}{2^n}\right\}$ is one such solution.

BOUNDED SOLUTION OF DIFFERENCE EQUATION

We establish the necessary and sufficient conditions of boundedness of every nonoscillatory solution of (1). Assume the following,

$$\sum_{n=1}^{\infty} \frac{1}{r_n} = \sum_{n=1}^{\infty} \frac{1}{s_n} = \sum_{n=1}^{\infty} \frac{1}{q_n} < \infty$$

then solutions of (1) are either type (i) or type (ii).

Lemma 1: Any nonnegative solution $\{x_n\}$ of (1) belong to the below classes:

If $x_n > 0$ and $\Delta x_n > 0$ then $\Delta(s_n \Delta(q_n \Delta x_n)) > 0$ (i)

If $x_n > 0$ and $\Delta x_n > 0$ then $\Delta(s_n \Delta(q_n \Delta x_n)) < 0$ (ii)

If $x_n > 0$ and $\Delta x_n < 0$ then $\Delta(s_n \Delta(q_n \Delta x_n)) > 0$ (iii)

If $x_n > 0$ and $\Delta x_n < 0$ then $\Delta(s_n \Delta(q_n \Delta x_n)) < 0$ (iv)

for all sufficiently large n.

Proof: Assume $\{x_n\}$ to be an nonnegative solution of (1). From (1) we have $\Delta(q_n \Delta x_n) > 0$ and $\Delta(r_n \Delta(s_n \Delta(q_n \Delta x_n)) > 0$ so $\{\Delta(s_n \Delta(q_n \Delta x_n))\}$, $\{\Delta x_n\}$ and $\{x_n\}$ are eventually of one sign. Hence the lemma is proved.

We consider solutions of types (i) and (ii) only for proving the rest of the theorems.

Theorem 4: If g is an increasing function and $g(x)/x$ is a decreasing function for $x > 0$ then,

$$\sum_{l=1}^{\infty}\frac{1}{q_l}\sum_{k=1}^{\infty}\frac{1}{s_k}\sum_{j=1}^{k-1}\frac{1}{r_j}\sum_{i=1}^{j-1}\left(\tau_i-(p_i-p_{i+1})\right)<\infty \qquad \text{(E)}$$

Here every solution of type (i) is said to be bounded.

Proof: Assume $\{x_n\}$ to be an unbounded solution of (1) and if $x_n > 0$, $\Delta x_n > 0$, $\Delta\left(s_n\Delta(t_n\Delta x_n)\right) > 0$ then,

$$\tau_i-[p_n-p_{n+1}]g(x_{n-m+1})=\frac{\Delta(r_n\Delta\left(s_n\Delta(q_n\Delta x_n)\right))}{g(x_{n-t+1})}$$

$$=\frac{r_{n+1}\Delta\left(s_{n+1}\Delta(q_{n+1}\Delta x_{n+1})\right))}{g(x_{n-t+1})}-\frac{r_n\Delta\left(s_n\Delta(q_n\Delta x_n)\right))}{g(x_{n-t+1})}$$

$$\geq\frac{r_{n+1}\Delta\left(s_{n+1}\Delta(q_{n+1}\Delta x_{n+1})\right)}{g(x_{n-t+1})}-\frac{r_n\Delta\left(s_n\Delta(q_n\Delta x_n)\right)}{g(x_{n-t})}$$

$$=\Delta\left[\frac{r_n\Delta\left(s_n\Delta(q_n\Delta x_n)\right)}{g(x_{n-t})}\right]$$

Summing from $i=N$ to $i=j-1$,

$$\sum_{i=N}^{j-1}\left(\tau_i-[p_i-p_{i+1}]g(x_{i-t+1})\right)+\frac{r_N\Delta\left(s_N\Delta(q_N\Delta x_N)\right)}{g(x_{N-t})}\geq\frac{r_j\Delta\left(s_j\Delta(q_j\Delta x_j)\right)}{g\left(x_{j-t}\right)}$$

Hence,

$$\frac{1}{r_j}\sum_{i=N}^{j-1}\left(\tau_i-[p_i-p_{i+1}]g(x_{i-t+1})\right)+\frac{r_N\Delta\left(s_N\Delta(q_N\Delta x_N)\right)}{r_j g(x_{N-t})}\geq\frac{\Delta\left(s_j\Delta(q_j\Delta x_j)\right)}{g\left(x_{j-t}\right)}$$

$$\geq\frac{s_{j+1}\Delta\left(q_{j+1}\Delta x_{j+1}\right)}{g\left(x_{j-t}\right)}-\frac{s_j\Delta\left(q_j\Delta x_j\right)}{g\left(x_{j-t}\right)}$$

$$=\Delta\left[\frac{s_j\Delta\left(q_j\Delta x_j\right)}{g\left(x_{j-t-1}\right)}\right]$$

Summing again from $j=N$ to $j=k-1$,

$$\sum_{j=N}^{k-1}\frac{1}{r_j}\sum_{i=N}^{j-1}\left(\tau_i-[p_i-p_{i+1}]g(x_{i-t+1})\right)+\sum_{j=N}^{k-1}\frac{r_N\Delta\left(s_N\Delta(q_N\Delta x_N)\right)}{r_j g(x_{N-t})}$$

$$\geq\frac{s_k\Delta(q_k\Delta x_k)}{g\left(x_{k-t-1}\right)}-\frac{s_N\Delta(q_N\Delta x_N)}{g(x_{N-t-1})}$$

Hence

$$\frac{1}{s_k}\sum_{j=N}^{k-1}\frac{1}{r_j}\sum_{i=N}^{j-1}\left(\tau_i-[p_i-p_{i+1}]g(x_{i-t+1})\right)+\frac{1}{s_k}\sum_{j=N}^{k-1}\frac{r_N\Delta\left(s_N\Delta(q_N\Delta x_N)\right)}{r_jg(x_{N-t})}+\frac{s_N\Delta(q_N\Delta x_N)}{s_kg(x_{N-t-1})}$$

$$\geq\frac{\Delta(q_k\Delta x_k)}{f\left(x_{k-t-1}\right)}$$

$$\geq\frac{q_{k+1}\Delta x_{k+1}}{g\left(x_{k-t-1}\right)}-\frac{q_k\Delta x_k}{g\left(x_{k-t-1}\right)}$$

$$\geq\frac{q_{k+1}\Delta x_{k+1}}{g\left(x_{k-t-1}\right)}-\frac{q_k\Delta x_k}{g\left(x_{k-t-2}\right)}$$

$$=\Delta\left[\frac{q_k\Delta x_k}{g\left(x_{k-t-2}\right)}\right]$$

A final summation from $k=N$ to $k=l-1$ yields,

$$\sum_{k=N}^{l-1}\frac{1}{s_k}\sum_{j=N}^{k-1}\frac{1}{r_j}\sum_{i=N}^{j-1}\left(\tau_i-[p_i-p_{i+1}]g(x_{i-t+1})\right)+\sum_{k=N}^{l-1}\frac{1}{s_k}\sum_{j=N}^{k-1}\frac{r_N\Delta\left(s_N\Delta(q_N\Delta x_N)\right)}{r_jg(x_{N-t})}$$
$$+\sum_{k=N}^{l-1}\frac{s_N\Delta(q_N\Delta x_N)}{s_kg(x_{N-t-1})}\geq\frac{q_l\Delta x_l}{g\left(x_{l-t-2}\right)}-\frac{q_N\Delta x_N}{g(x_{N-t-2})}$$

$$\sum_{k=N}^{l-1}\frac{1}{s_k}\sum_{j=N}^{k-1}\frac{1}{r_j}\sum_{i=N}^{j-1}\left(\tau_i-[p_i-p_{i+1}]g(x_{i-m+1})\right)+\sum_{k=N}^{l-1}\frac{1}{s_k}\sum_{j=N}^{k-1}\frac{r_N\Delta\left(s_N\Delta(q_N\Delta x_N)\right)}{r_jg(x_{N-t})}$$
$$+\sum_{k=N}^{l-1}\frac{s_N\Delta(q_N\Delta x_N)}{s_kg(x_{N-t-1})}+\frac{q_N\Delta x_N}{g(x_{N-t-2})}\geq\frac{q_l\Delta x_l}{g\left(x_{l-t-2}\right)}$$

Hence,

$$\frac{\Delta x_l}{x_l}\leq\frac{1}{q_l}\sum_{k=N}^{l-1}\frac{1}{s_k}\sum_{j=N}^{k-1}\frac{1}{r_j}\sum_{i=N}^{j-1}\left(\tau_i-[p_i-p_{i+1}]g(x_{i-t+1})\right)+\frac{1}{q_l}\sum_{k=N}^{l-1}\frac{1}{s_k}\sum_{j=N}^{k-1}\frac{r_N\Delta\left(s_N\Delta(q_N\Delta x_N)\right)}{r_jg(x_{N-t})}$$
$$+\frac{1}{q_l}\sum_{k=N}^{l-1}\frac{s_N\Delta(q_N\Delta x_N)}{s_kg(x_{N-t-1})}+\frac{q_N\Delta x_N}{q_lg(x_{N-t-2})}$$

Here $g(x)/x$ is non-increasing for $x>0$, so we have,

$$\frac{\Delta x_l}{x_l}\leq\frac{g(x_N)}{x_N}\frac{\Delta x_l}{x_l}\leq\frac{g(x_N)}{x_N}\frac{1}{q_l}\sum_{k=N}^{l-1}\frac{1}{s_k}\sum_{j=N}^{k-1}\frac{1}{r_j}\sum_{i=N}^{j-1}\left(\tau_i-[p_i-p_{i+1}]g(x_{n-t+1})\right)$$

$$+\frac{g(x_N)}{x_N}\frac{1}{q_l}\sum_{k=N}^{l-1}\frac{1}{s_k}\sum_{j=N}^{k-1}\frac{r_N\Delta\left(s_N\Delta(q_N\Delta x_N)\right)}{r_j g(x_{N-t})}+\frac{g(x_N)}{x_N}\left(\frac{1}{q_l}\right)\sum_{k=N}^{l-1}\frac{s_N\Delta(q_N\Delta x_N)}{s_k g(x_{N-t-1})}$$
$$+\frac{g(x_N)}{x_N}\left(\frac{q_N\Delta x_N}{q_l f(x_{N-t-2})}\right) \qquad (E1)$$

Final summation from $l=N$ to $k=n-1$

$$\sum_{l=N}^{n-1}\frac{\Delta x_l}{x_l}=\frac{\Delta x_n}{x_n}-\frac{\Delta x_N}{x_N}=\log(\Delta x_n)-\log(x_N)$$

Now summing both sides of (E1) we get,

$$\sum_{l=N}^{n-1}\frac{\Delta x_l}{x_l}\le\frac{g(x_N)}{x_N}\sum_{l=N}^{n-1}\frac{1}{q_l}\sum_{k=N}^{l-1}\frac{1}{s_k}\sum_{j=N}^{k-1}\frac{1}{r_j}\sum_{i=N}^{j-1}\left(\tau_i-[p_i-p_{i+1}]g(x_{i-t+1})\right)$$
$$+\frac{g(x_N)}{x_N}\sum_{l=N}^{n-1}\frac{1}{q_l}\sum_{k=N}^{l-1}\frac{1}{s_k}\sum_{j=N}^{k-1}\frac{r_N\Delta\left(s_N\Delta(q_N\Delta x_N)\right)}{r_j g(x_{N-t})}$$
$$+\frac{g(x_N)}{x_N}\sum_{l=N}^{n-1}\frac{1}{q_l}\sum_{k=N}^{l-1}\frac{s_N\Delta(q_N\Delta x_N)}{s_k f(x_{N-t-1})}+\frac{g(x_N)}{x_N}\sum_{l=N}^{n-1}\frac{1}{q_l}\left(\frac{q_N\Delta x_N}{g(x_{N-t-2})}\right)$$

This implies,

$$\log(\Delta x_n)-\log(x_N)\le\frac{g(x_N)}{x_N}\sum_{l=N}^{n-1}\frac{1}{q_l}\sum_{k=N}^{l-1}\frac{1}{s_k}\sum_{j=N}^{k-1}\frac{1}{r_j}\sum_{i=N}^{j-1}\left(\tau_i-[p_i-p_{i+1}]g(x_{n-t+1})\right)$$
$$+\frac{g(x_N)}{x_N}\sum_{l=N}^{n-1}\frac{1}{q_l}\sum_{k=N}^{l-1}\frac{1}{s_k}\sum_{j=N}^{k-1}\frac{r_N\Delta\left(s_N\Delta(q_N\Delta x_N)\right)}{r_j g(x_{N-t})}$$
$$+\frac{g(x_N)}{x_N}\sum_{l=N}^{n-1}\frac{1}{q_l}\sum_{k=N}^{l-1}\frac{s_N\Delta(q_N\Delta x_N)}{s_k g(x_{N-t-1})}+\frac{g(x_N)}{x_N}\sum_{l=N}^{n-1}\frac{1}{q_l}\left(\frac{q_N\Delta x_N}{g(x_{N-t-2})}\right)$$

From (1) it follows the convergence of series given as,

$$\sum_{l=1}^{\infty}\frac{1}{q_l}\sum_{k=1}^{\infty}\frac{1}{s_k}\sum_{j=1}^{\infty}\frac{1}{r_j}$$

Here we observe that x_n converges, so $\{x_n\}$ is bounded which is a contradiction. Thus the proof is completed.

Example-4: We consider the following difference equation,

$$\Delta^2\big((n+1)\Delta^2 x_n\big) = (n+3)x_{n+4} + (5n+9)x_{n+2} - 2[(2n+5)x_{n+3} + (n+1)x_{n+1}] + 1$$

(6)

Here every condition of Theorem 4 is satisfied and $\{x_n\} = \left\{\frac{n}{n+1}\right\}$ is one such solution of (6).

Theorem 5: If the condition (E) holds then every solution of type (ii) is bounded.

Proof: Let $\{x_n\}$ be a solution of (1). If $x_n > 0$ and $\Delta x_n > 0$ then $\Delta\big(s_n\Delta(q_n\Delta x_n)\big) < 0$, hence summing from $n = N$ to $n = i - 1$ we have,

$$s_i\Delta(q_i\Delta x_i) < s_N\Delta(q_N\Delta x_N)$$

$$\Delta(q_i\Delta x_i) < \frac{s_N\Delta(q_N\Delta x_N)}{s_i}$$

Summing again from $i = N$ to $i = j - 1$ then

$$\Delta x_j < \frac{s_N\Delta(q_N\Delta x_N)}{q_j}\sum_{i=N}^{j-1}\frac{1}{s_i} + \frac{q_N\Delta x_N}{q_j}$$

Final summation from $j = N$ to $j = n - 1$ then

$$x_n < s_N\Delta(q_N\Delta x_N)\sum_{i=N}^{j-1}\frac{1}{s_i}\sum_{j=N}^{n-1}\frac{1}{q_j} + q_N\Delta x_N\sum_{j=N}^{n-1}\frac{1}{q_j} + x_N$$

As (E) implies below condition,

$$\sum_{n=1}^{\infty}\frac{1}{s_n} = \sum_{n=1}^{\infty}\frac{1}{q_n} < \infty \quad \text{and} \quad s_N\Delta(q_N\Delta x_N) > 0$$

the solution $\{x_n\}$ must be bounded. This completes the proof.

Example-5: We consider the below difference equation,

$$\Delta(n\Delta((n+1)\Delta^2 x_n)) = \frac{(-1)\,n^5}{(n+4)(n+3)(n+2)(n+1)}[38x_n^5 + 26x_n^4 + 153x_n^3 + 122x_n^2 + 38x_n + 4]$$

(7)

Thus every condition of Theorem 5 is satisfied. Here $\{x_n\} = \left\{\frac{1}{n}\right\}$ is one such solution of (7).

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4. Conclusion:

From this paper we conclude that the necessary and sufficient conditions for asymptotic behavior and boundedness of (1) are established using contraction mapping principle, summation averaging technique and comparison method.

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