Comparative Study to Solve Differential–Difference Equation using the Laplace Transform and Hybrid Laplace Transformation

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Article Info
Page Number: 1040 – 1045
Publication Issue:
Vol. 71 No. 3s2 (2022)

Article History
Article Received: *28 April 2022*
Revised: *15 May 2022*
Accepted: *20 June 2022*
Publication: *21 July 2022*

Abstract
In this research paper, we present a comparative study between the Laplace transform method and the hybrid Laplace transformation method to solve DDE and also we attained the population mean and its variance directly from the DDE.

Keywords: Differential-Difference Equations(DDE), Laplace Transform (L.T), Z-Transform and Hybrid Laplace Transform (HLT).

Introduction:

Modeling, image processing, computing, tomography& design of many systems involve dealing with differential difference equations. Important of solving DDE in Stochastic Process, Queuing Theory and applied mathematics have played a vital role in addressing DDE in stochastic processes. The Z-transform (also known as the Laplace transformation) is a useful tool for resolving discrete time physical problems. The L.T. and HLT were invented in [4]-[5], and were used to solve DDE in both continuous and discrete variables. The problem also was examined utilizing the methodology outlined in a previous study [1]. Scientists have already found DDE to be a challenging endeavour. In [1], The M/M/1 queue is considered, as well as the probability of disasters. When a catastrophe occurs in the system, the arrival rate and the catastrophe rate follow a poisson process, whereas the service rate follows an exponential distribution. When a catastrophe occurs in the system, all of the customers are terminated at the same time, the server becomes idle and is ready for service and is used typical numerical methods such as Bessal’s functions and Characteristics approaches require a significant amount of computer labour. However, the Laplace approach and HLT [4-6] have been used to solve a wide range of stochastic and deterministic problems in a variety of fields.

The primary goal of this contribution is to present a comparison assessment of the approaches' almost equal and to demonstrate that both methods may be used successfully and efficiently to solve this problem. The probability generating function is used to obtain the general solution and its the mean population and its variance were also collected directly from the DDEs.

Problem Formulation:

The differential-difference equations for single server queue are:

$P'_\ell(\xi) = -(a+b+c)\, P_\ell(\xi) + aP_{\ell-1}(\xi) + b\, P_{\ell+1}(\xi)$, for $\ell=1,2,3,\ldots$ (1)

$P'_0(\xi) = -a\, P_0(\xi) + b\, P_1(\xi) + c(1 - P_0(\xi))$ (2)

We consider the system's initial conditions as follows:

$P_1(0) = 1$ and $P_\ell(0) = 0$ for $\ell \neq 1$.

Method I:

Let $P(z, \xi) = \sum_{\ell=0}^{\infty} P_\ell(\xi)\, z^\ell$. Then $P(z,0) = z$.

Let $f_{i\ell}(s) = L[P_\ell(\xi)]$ and $F(z,s) = L[P(z, \xi)]$.

Multiplying (1) by z^ℓ and summing over $\ell = 1, 2, 3, \ldots$,the resulting equation with (2)

We have

$$\sum_{\ell=1}^{\infty} P'_\ell(\xi) z^\ell + P'_0(t) = -(a+b+c) \sum_{\ell=1}^{\infty} P_\ell(\xi) z^\ell + a \sum_{\ell=1}^{\infty} P_{\ell-1}(\xi) z^\ell + b \sum_{\ell=1}^{\infty} P_{\ell+1}(\xi) z^\ell$$
$$-aP_0(\xi) + bP_1(\xi) + c(1 - P_0(\xi))$$

$$\sum_{\ell=0}^{\infty} P'_\ell(\xi) z^\ell = -(a+b+c) \sum_{\ell=0}^{\infty} P_\ell(\xi) z^\ell + (a+b+c)\, P_0(\xi) + a \sum_{\ell=0}^{\infty} P_\ell(\xi) z^{\ell+1} + b \sum_{\ell=2}^{\infty} P_\ell(\xi) z^{\ell-1}$$
$$- a\, P_0(\xi) + bP_1(\xi) + c(1 - P_0(\xi))$$

$$\frac{\partial}{\partial t} P(z, \xi) = (a z + b z^{-1} - (a+b+c))\, P(z, \xi) + b\,(1 - z^{-1})\, P_0(\xi) + c$$

$$z \frac{\partial}{\partial t} P(z, \xi) = (az^2 + b - (a+b+c) z)\, P(z, \xi) + b\,(z - 1)\, P_0(\xi) + c\, z \tag{3}$$

Applying the Laplace Transform to both sides of (3), we obtain

$$s\, z\, F(z, s) - z^2 = (az^2 + b - (a+b+c)z)\, F(z, s) + b\,(z-1) L[P_0(\xi)] + czs^{-1}$$

$$F(z,s) = \frac{z^2 + b(z-1)L[P_0(\xi)] + czs^{-1}}{[sz + (a+b+c)z - az^2 - b]}.$$

Method II:

By denoting $P_\ell(\xi) = x(\xi, \ell)$ and $L_2[x(\xi, \ell)] = X(s, z)$.

The systems (1) and (2) becomes a DDE which is transformed {by [5,Theorem 2.6]} as above, using the HLT ,we obtain

$$L_2[P'_\ell(\xi)] = (-(a+b+c) + az^{-1} + bz)\, X(s, z) + b(1-z)\, L_2[P_0(\xi)] + cs^{-1}$$

$Z(sL(P_\ell(\xi))- P_\ell(0))= (-(a+b+c)+az^{-1}+bz)\ X(s, z)+b(1-z)\ L_2[P_0(\xi)]\ +cs^{-1}$

$sX(s, z)-Z(P_\ell(0)) = (-(a+b+c)+az^{-1}+bz)\ X(s, z) + b(1-z)\ L_2[P_0(\xi)]+cs^{-1}$

$(s z +(a+b+c)z-a-bz^2)X(s, z)= zZ[P_\ell(0)]+bz(1-z)\ L_2[P_0(\xi)]+cs^{-1}\ z$

$X(s, z) = \frac{1+bz(1-z)L_2[\,P_0(\xi)]+czs^{-1}}{(sz+(a+b+c)z-a-bz^2)}.$

Determination of the Mean and Variance:

We can calculate the mean population size by partially differentiating $P(z, \xi)$ with respect to z and setting z =1. It is possible to obtain it directly from (1) and (2) without first obtaining $P(z, \xi)$ as follows:

Let $\in [X(\xi)]=M_1(\xi) = \sum_{\ell=1}^{\infty} \ell P_\ell(\xi)$ and $\in [X^2(\xi)]= M_2(\xi) = \sum_{\ell=1}^{\infty} \ell^2 P_\ell(\xi)$.

Summing for ℓ =1, 2, 3,...., and multiplying both sides of (1) by ℓ ,We now have

$\sum_{\ell=1}^{\infty} \ell P'_\ell(\xi) = -(a+b+c) \sum_{\ell=1}^{\infty} \ell P_\ell(\xi)+a \sum_{\ell=1}^{\infty} \ell P_{\ell-1}(\xi) +b\sum_{\ell=1}^{\infty} \ell P_{\ell+1}(\xi)$ (4)

$\sum_{\ell=1}^{\infty} \ell P_{\ell-1}(\xi) = \sum_{\ell=0}^{\infty}(\ell + 1)P_\ell(\xi)= M_1(\xi) +1$ (5)

$\sum_{\ell=1}^{\infty} \ell P_{\ell+1}(\xi) = \sum_{\ell=2}^{\infty}(\ell - 1)P_\ell(\xi)$

$= M_1(\xi)-1+P_0(\xi)$ (6)

$M'_1(\xi) = \sum_{\ell=1}^{\infty} \ell P'_\ell(\xi).$

Put (5) & (6) in (4), we get

$M'_1(\xi)=-c\, M_1(\xi)+a-b-c$ (7)

The solution of (7) is

$M_1(\xi)=k_1e^{-c\xi} +c^{-1}(a-b-c).$

The initial condition provides

$M_1(0) = \sum_{\ell=1}^{\infty} \ell P_\ell(0) = 1$, whence $k_1=c^{-1}(2c-a+b)$

We have therefore,

$$\text{Mean } [\in(X(\xi))]= M_1(\xi) = c^{-1}(2c + b - a)e^{-c\xi} + c^{-1}(a-b-c)$$

Let $M_2(\xi) = \sum_{\ell=1}^{\infty} \ell^2 P_\ell(\xi)$ and $M_3(\xi) = \sum_{\ell=1}^{\infty} \ell^3 P_\ell(\xi)$.

Then $M'_2(\xi) = \sum_{\ell=1}^{\infty} \ell^2 P'_\ell(\xi)$.

Summing for ℓ =1, 2, 3,...., and multiplying both sides of (1) by ℓ^2 ,We get

$$\sum_{\ell=1}^{\infty}\ell^2 P'_\ell(\xi) = -(a+b+c)\sum_{\ell=1}^{\infty}\ell^2 P_\ell(\xi)+a\sum_{\ell=1}^{\infty}\ell^2 P_{\ell-1}(\xi) +b\sum_{\ell=1}^{\infty}\ell^2 P_{\ell+1}(\xi)$$

$$M'_2(\xi)= -(a+b+c)\sum_{\ell=1}^{\infty}\ell^2 P_\ell(\xi)+a\sum_{\ell=0}^{\infty}(\ell^2+2\ell+1)P_\ell(\xi) +b\sum_{\ell=2}^{\infty}(\ell^2-2\ell+1)P_\ell(\xi)$$

$$M'_2(\xi) =-c\, M_2(\xi)+2(a-b)\, M_1(\xi)+ a\sum_{\ell=0}^{\infty} P_\ell(\xi)+b\sum_{\ell=0}^{\infty} P_\ell(\xi)-b\, P_0(\xi)$$

$$M'_2(\xi)+c\, M_2(\xi)=2(a-b)\, M_1(\xi)+a+b+c \tag{8}$$

$$M'_2(\xi)+c\, M_2(\xi)= 2k_1(a-b)e^{-c\xi}+2c^{-1}(a-b)(a-b-c)+a+b+c \tag{9}$$

The solution of (9) is

$$M_2(\xi) = k_2 e^{-c\xi}+2k_1\xi(a-b)e^{-c\xi}+2c^{-2}(a-b)(a-b-c)+c^{-1}(a+b+c)$$

The initial condition gives $M_2(0) = \sum_{\ell=1}^{\infty}\ell^2 P_\ell(0) = 1$, whence $k_2 = 1-2c^{-2}(a-b)(a-b-c)-c^{-1}(a+b+c)$.

We have therefore,

$$M_2(\xi) = c^{-2}(4ab+ac-3bc-2a^2-2b^2)e^{-ct}-2c^{-1}(a-b)(2c-a+b)\xi e^{-c\xi}$$
$$+2c^{-2}(a-b)(a-b-c)+ c^{-1}(a+b+c)$$

$$\text{Variance }[X(\xi)]= -c^{-2}(2c-a+b)^2e^{-2c\xi}+c^{-2}(4c^2+3bc-5ac)\, e^{-c\xi}+2c^{-1}(a-b)(2c-a+b)\,\xi\, e^{-c\xi}$$
$$+2c^{-2}(a-b)(a-b-c)-c^{-2}(a-b-c)^2+c^{-1}(a+b+c).$$

Numerical Illustrations:

We have shown the average and variation of population size as time ' ξ ' changes in this section. For a=4,b=1, Figures 1 and 2 show the predicted value and variance of the population size for various values of 'c.' The average and variance number of units in the population size decreased as rate 'c' increased, as shown in figures 1 and 2.

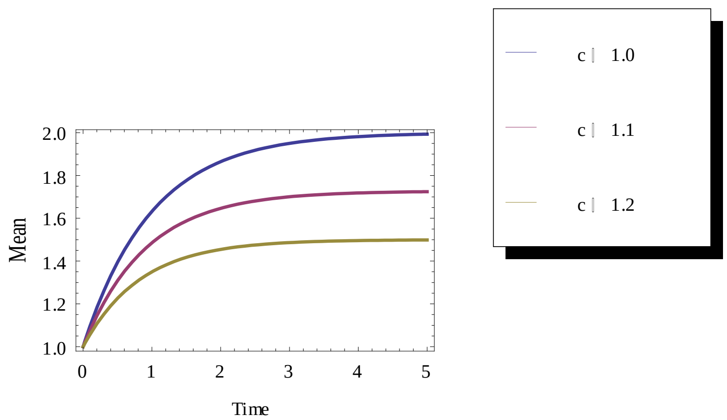

Figure 1 : Mean population size verses time ξ for different values of 'c'.

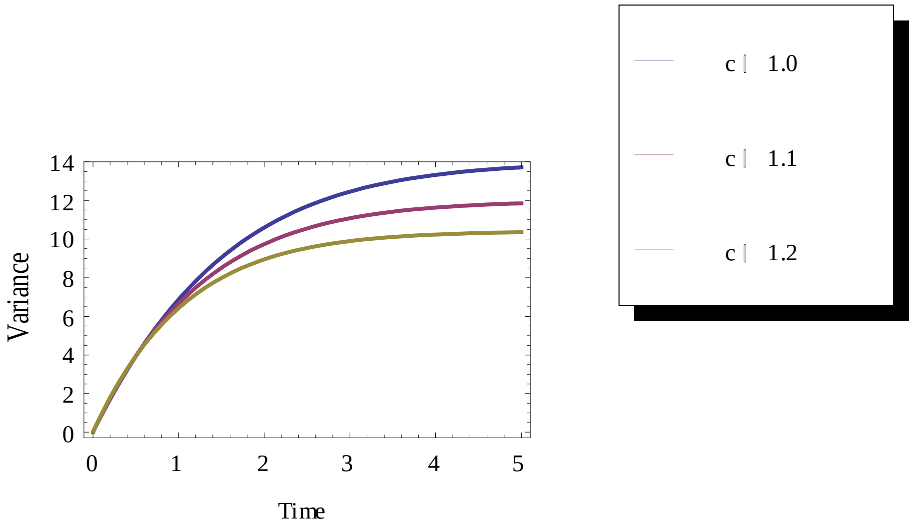

Figure 2 : Variance of population size verses time ξ for different values of 'c'.

Conclusion:

The L.T & HLT may be used to DDE for the Single Server Queue in this work, and we get almost equal solution in both the transforms. We also acquired the population mean and its variance directly from the DDEs. Overall, the findings demonstrate that L.T. and HLT are dependable and efficient for analyzing transient solutions in single server Queueing models.

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