Geodetic Dominating Sets and Geodetic Dominating Polynomials of Cycles

P. Arul Paul Sudhahar[1], A.Merin Sherly[2] and T.Premala[3]

[1]Research Supervisor, Department of Mathematics,Rani Anna Government College for Women,Tirunelveli - 627 008, Tamilnadu, India.

[2]Research Scholar(Register No. 20211172092004),Department of Mathematics, Rani Anna Government College for Women, Tirunelveli - 627 008,Affiliated to Manonmaniam Sundaranar University,Abishekapatti,Tirunelveli 627 012, Tamilnadu, India.

[3]Professor of Mathematics, Narayanaguru College of Engineering, Manjalumoodu-629151.

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Abstract

Let Dsg(C_n, i) be the family of split geodetic dominating sets of the cycle graph C_n. with cardinality i and let dsg$(C_n, i) = |Dsg(C_n, i)|$. Then the split geodetic polynomial Dsg(C_n, x) of C_n is defined as Dsg(C_n, x) $=\sum_{i=\gamma_{sg}(C_n)}^{n} dsg(C_n, i)x^i$, where $\gamma_{sg}(C_n)$ is the split geodetic domination number of C_n.In this paper we have determined the family of split geodetic dominating sets of the cycle graph C_n with cardinality i.Also , we have obtained the recursive formula to derive the split geodetic domination polynomials of cycles and also obtain some properties of this polynomial.

Keywords : Cycle, Split geodetic dominating sets , Split geodetic domination Polynomial.

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1 Introduction

Let G $= (V, E)$ be a simple graph of order | V | = n. A dominating set for a graph G $= (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G[1][2]. We call a set of vertices S in a graph G a geodetic dominating set if S is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set of G is its geodetic domination number, and is denoted by $\gamma_g(G)$[3] [4]. Split geodetic number of a graph was studied by in [5]. A geodetic set S of a graph G $= (V, E)$ is the split geodetic set if the induced subgraph $\langle V - S$ is disconnected.The split geodetic number $g_s(G)$ of G is the minimum cardinality of a split geodetic set. A set S $\subseteq V(G)$ is said to be a split geodetic dominating set of G if S is both a split geodetic set and a dominating set of G.The minimum cardinality of the split geodetic dominating set of G is called the split geodetic domination number of G and is denoted by $\gamma_{gs}(G)$.The concept of split geodetic domination number was introduced by P.Arul

Paul Sudhahar and J.Jeba Lisa in [6].A domination polynomial can be studied in [7][8][9] and the geodetic domination polynomial was studied in [10][11].

A simple graph of $'n'$ vertices$(n \geq 3)$ and n edges forming a cycle of length $'n'$ is called as a cycle graph. In a cycle graph, all the vertices are of degree 2. Let Dsg(C_n, i) be the family of split geodetic dominating sets of the cycle graph C_n. with cardinality i and let dsg$(C_n, i) = |Dsg(C_n, i)|$. Then the split geodetic polynomial Dsg(C_n, x) of C_n is defined as Dsg(C_n, x) $=\sum_{i=\gamma_{sg}(C_n)}^{n} dsg(C_n, i)x^i$, where $\gamma_{sg}(C_n)$ is the split geodetic domination number of C_n.

2 Split Geodetic Dominating Set of the Cycle C_n

Lemma 2.1. $\gamma_{sg}(C_n) = \left\lceil \frac{n}{3} \right\rceil$

Lemma 2.2. Dsg$(C_n,\ i) = \phi$ if and only if i $>$ n or i $<$ $\left\lceil \frac{n}{3} \right\rceil$ and Dsg$(C_n, i) > 0$ if $\left\lceil \frac{n}{3} \right\rceil$ $\leq$ i $\leq$ n .

Lemma 2.3. If Y $\in D_{sg}(C_{n-4}, i-1)$ or Y $\in D_{sg}(C_{n-5}, i-1)$ such that Y $\cup\, x \in C_n$, i for some x $\in n$, then Y $\in D_{sg}(C_{n-3}, i-1)$.

To find the split geodetic dominating set of C_n with cardinality i, we can only consider C_{n-1}, i $-$ 1, C_{n-2}, i $-$ 1, C_{n-3}, i $-$ 1.The families of these split geodetic dominating sets will be empty or otherwise.Thus there are eight such combinations among which three of these combinations are not possible that is, if $D_{sg}(C_{n-1}, i-1) = D_{sg}(C_{n-3}, i-1) = \phi$ then $D_{sg}(C_{n-2}, i-1) = \phi$ and if $D_{sg}(C_{n-1}, i-1) \neq \phi$, $D_{sg}(C_{n-3}, i-1) \neq \phi$ then $D_{sg}(C_{n-2}, i-1) \neq$, also if $D_{sg}(C_{n-1}, i-1) = D_{sg}(C_{n-2}, i-1) = D_{sg}(C_{n-3}, i-1) = \phi$ then $D_{sg}(C_n, i) = \phi$. Hence we can consider only the remaining five combinations.

Lemma 2.4. (i) If $D_{sg}(C_{n-1}, i-1) = D_{sg}(C_{n-3}, i-1) = \phi$ then $D_{sg}(C_{n-2}, i-1) = \phi$ (ii) If $D_{sg}(C_{n-1}, i-1) \neq \phi$, $D_{sg}(C_{n-3}, i-1) \neq \phi$ then $D_{sg}(C_{n-2}, i-1) \neq \phi$

(iii) $If\ D_{sg}(C_{n-1}, i-1) = D_{sg}(C_{n-2}, i-1) = D_{sg}(C_{n-3}, i-1) = \Phi$ then $D_{sg}(C_n, i) = \phi$.

Proof.

(i) If $D_{sg}(C_{n-1}, i-1) = D_{sg}(C_{n-3}, i-1) = \phi$ then $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-`1}{3} \right\rceil$ and $i-1 > n-3$ or $i-1 < \left\lceil \frac{n-3}{3} \right\rceil \Rightarrow i-1 < \left\lceil \frac{n-2}{3} \right\rceil$ or$i-1 > n-2$ holds .Therefore $D_{sg}(C_{n-2}, i-1) = \phi$.

(ii) If $D_{sg}(C_{n-1}, i-1) \neq \phi$, $D_{sg}(C_{n-3}, i-1) \neq \phi$ then $\left\lceil \frac{n-1}{3} \right\rceil \leq i-1 \leq n-1\ and \left\lceil \frac{n-3}{3} \right\rceil \leq i-1 \leq n-3 \Rightarrow \left\lceil \frac{n-1}{3} \right\rceil \leq i-1 \leq n-3\ and \left\lceil \frac{n-2}{3} \right\rceil \leq \left\lceil \frac{n-1}{3} \right\rceil \leq i-1 \leq n-3 < n-2 \Rightarrow \left\lceil \frac{n-2}{3} \right\rceil \leq i-1 \leq n-2\}$.Therefore $D_{sg}(C_{n-2}, i-1) \neq \phi$.

(iii) $If\ D_{sg}(C_{n-1}, i-1) = D_{sg}(C_{n-2}, i-1) = D_{sg}(C_{n-3}, i-1) = \Phi$ then $i-1 < \left\lceil\frac{n-1}{3}\right\rceil$ $or\ i-1 > n-1; i-1 < \left\lceil\frac{n-2}{3}\right\rceil\ or\ i-1 > n-2$ and $i-1 < \left\lceil\frac{n-3}{3}\right\rceil ori-1 > n-1 \Rightarrow i < \left\lceil\frac{n-3}{3}\right\rceil + 1\ ori > n \Rightarrow i < \left\lceil\frac{n}{3}\right\rceil ori > n$. Therefore $D_{sg}(C_n, i) = \phi$.

Lemma 2.5. If $D_{sg}(C_n, i) \neq \Phi$ then we have

(i) $D_{sg}(C_{n-1}, i-1) = D_{sg}(C_{n-2}, i-1) = \phi$ and $D_{sg}(C_{n-3}, i-1) \neq \phi$ if and only if $n = 3k$, $i = k$, for some positive integer k.

(ii) $D_{sg}(C_{n-2}, i-1) = D_{sg}(C_{n-3}, i-1) = \phi$ and $D_{sg}(C_{n-1}, i-1) \neq \Phi$ if and only if $i = n$.

(iii) $D_{sg}(C_{n-1}, i-1) \neq \phi; D_{sg}(C_{n-2}, i-1) \neq \phi$ and $D_{sg}(C_{n-3}, i-1) = \phi$ if and only if $i = n-1$.

(iv) $D_{sg}(C_{n-1}, i-1) = \phi$; $D_{sg}(C_{n-2}, i-1) \neq \phi$ and $D_{sg}(C_{n-3}, i-1) \neq \phi$ if and only if $n = 3k+2$ and $i = \left\lceil\frac{3k+2}{3}\right\rceil$ for some $k \in N$.

(iv) $D_{sg}(C_{n-1}, i-1) \neq \phi; D_{sg}(C_{n-2}, i-1) \neq \phi$ and $D_{sg}(C_{n-3}, i-1) \neq \phi$ if and only if $\left\lceil\frac{n-1}{3}\right\rceil + 1 \leq i \leq n-2$.

Proof.

(i) Since $D_{sg}(C_{n-1}, i-1) = D_{sg}(C_{n-2}, i-1) = \phi \Rightarrow i-1 > n-1\ ori-1 < \left\lceil\frac{n-1}{3}\right\rceil$ and $i-1 > n-2\ or\ i-1 < \left\lceil\frac{n-2}{3}\right\rceil \Rightarrow i-1 < \left\lceil\frac{n-2}{3}\right\rceil$ or $i-1 > n-1$. If $i-1 > n-1$ then $i > n$ and hence $D_{sg}(C_n, i) = \phi$ which is a contradiction.Therefore $i-1 < \left\lceil\frac{n-2}{3}\right\rceil \Rightarrow i < \left\lceil\frac{n-2}{3}\right\rceil + 1$. Also since $D_{sg}(C_{n-3}, i-1) \neq \phi, then \left\lceil\frac{n-3}{3}\right\rceil \leq i-1 \leq n-3$. Hence $\left\lceil\frac{n-3}{3}\right\rceil + 1 \leq i < \left\lceil\frac{n-2}{3}\right\rceil + 1 \rightarrow \left\lceil\frac{n}{3}\right\rceil \leq i < \left\lceil\frac{n-2}{3}\right\rceil + 1$. This is true only when $n = 3k+2$ and $i = k$ for some $k \in N$.Conversely assume $n = 3k+2$ and $i = k$ for some $k \in N$ then by lemma 2.2 $D_{sg}(C_{n-1}, i-1) = D_{sg}(C_{n-2}, i-1) = \phi and D_{sg}(C_{n-3}, i-1) \neq \phi$.

(ii) Since $D_{sg}(C_{n-2}, i-1) = \phi\ and D_{sg}(C_{n-3}, i-1) = \phi, then\ i-1 < \left\lceil\frac{n-3}{3}\right\rceil ori-1 > n-2$ .If $i-1 < \left\lceil\frac{n-2}{3}\right\rceil ori-1 > n-2$ and $i-1 < \left\lceil\frac{n-3}{3}\right\rceil$ or $i-1 > n-3 \Rightarrow i-1 < \left\lceil\frac{n-3}{3}\right\rceil$ or $i-1 > n-2$.If $i-1 < \left\lceil\frac{n-3}{3}\right\rceil$ then $D_{sg}(C_{n-1}, i-1) = \phi$,which is a contradiction , so we have $i-1 > n-2 \Rightarrow i > n-1 \Rightarrow i \geq n$.Also since $D_{sg}(C_{n-1}, i-1) \neq \phi$ then $\left\lceil\frac{n-1}{3}\right\rceil \leq i-1 \leq n-1 \Rightarrow i \leq n$.Hence $i = n$.Conversely if $i = n$, then $D_{sg}(C_{n-2}, i-1) = D_{sg}(C_{n-2}, n-1) = \phi$, $D_{sg}(C_{n-3}, i-1) = D_{sg}(C_{n-3}, n-1) = \phi$ and $D_{sg}(C_{n-1}, i-1) = D_{sg}(C_{n-1}, n-1) \neq \phi\left[Since D_{sg}(C_{n-1}, n-1) = 1\right]$.

(iii) Assume $D_{sg}(C_{n-1}, i-1) \neq \phi, D_{sg}(C_{n-2}, i-1) \neq \phi\ and\ D_{sg}(C_{n-3}, i-1) = \phi. Since D_{sg}(C_{n-3}, i-1) = \phi, i-1 > n-3ori-1 < \left\lceil\frac{n-3}{3}\right\rceil$. Since $D_{sg}(C_{n-2}, i-1) \neq$

$\phi, \left\lceil\frac{n-2}{3}\right\rceil < i-1 \leq n-2$. That is , $i-1 < \left\lceil\frac{n-3}{3}\right\rceil$ is not possible. Therefore , $i-1 > n-3 \Rightarrow i-1 \geq n-2$, But $i-1 \leq n-2 \Rightarrow i-1 = n-2 \Rightarrow i = n-1$. Conversely suppose $i = n-1$, then $D_{sg}(C_{n-1}, i-1) = D_{sg}(C_{n-1}, n-2) \neq \phi, D_{sg}(C_{n-2}, i-1) = D_{sg}(C_{n-2}, n-2) \neq \phi, but D_{sg}(C_{n-3}, i-1) = D_{sg}(C_{n-3}, n-2) = \phi$.

(iv) Assume $D_{sg}(c_{n-1}, i-1) = \phi; D_{sg}(C_{n-2}, i-1) \neq \phi$ and $D_{sg}(C_{n-3}, i-1) \neq \phi$. Since $D_{sg}(C_{n-1}, i-1) = \phi, i-1 > n-1$ *and* $i-1 < \left\lceil\frac{n-1}{3}\right\rceil$. *If* $i-1 > n-2$ then $D_{sg}(C_{n-2}, i-1)$ *and* $D_{sg}(C_{n-3}, i-1)$ are empty , which is a contradiction. Therefore $i-1 < \left\lceil\frac{n-1}{3}\right\rceil \Rightarrow i < \left\lceil\frac{n-1}{3}\right\rceil + 1$. Since $D_{sg}(C_{n-2}, i-1) \neq \phi$ and $D_{sg}(C_{n-3}, i-1) \neq \phi$, we have $\left\lceil\frac{n-2}{3}\right\rceil \leq i-1 \leq n-2$ *and* $\left\lceil\frac{n-3}{3}\right\rceil \leq i-1 \leq n-3$. Therefore $\left\lceil\frac{n-2}{3}\right\rceil \leq i-1 \leq n-3$. Hence $\left\lceil\frac{n-2}{3}\right\rceil + 1 \leq i < \left\lceil\frac{n-1}{3}\right\rceil + 1$. This holds only when $n = 3k+2$ and $i = k+1$ for some $k \in N$. Conversely , assume $n = 3k+2$ and $i = k+1$, then $D_{sg}(C_{n-1}, i-1) = \phi; D_{sg}(C_{n-2}, i-1) \neq \phi and D_{sg}(C_{n-3}, i-1) \neq \phi$.

(v) Assume $D_{sg}(C_{n-1}, i-1) \neq \phi; D_{sg}(C_{n-2}, i-1) \neq \phi; D_{sg}(C_{n-3}, i-1) \neq \phi$. Then $\left\lceil\frac{n-1}{3}\right\rceil \leq i-1 \leq n-1; \left\lceil\frac{n-2}{3}\right\rceil \leq i-1 \leq n-2$ and $\left\lceil\frac{n-3}{3}\right\rceil \leq i-1 \leq n-3 \Rightarrow \left\lceil\frac{n-1}{3}\right\rceil \leq i-1 \leq n-3 \Rightarrow \left\lceil\frac{n-1}{3}\right\rceil + 1 \leq i \leq n-2$. Conversely, suppose $\left\lceil\frac{n-1}{3}\right\rceil + 1 \leq i \leq n-2$. Therefore $\left\lceil\frac{n-1}{3}\right\rceil \leq i-1 \leq n-1$; $\left\lceil\frac{n-2}{3}\right\rceil \leq i-1 \leq n-2$ *and* $\left\lceil\frac{n-3}{3}\right\rceil \leq i-1 \leq n-3$. From these we obtain $D_{sg}(C_{n-1}, i-1) \neq \phi; D_{sg}(C_{n-2}, i-1) \neq \phi and D_{sg}(C_{n-3}, i-1) \neq \phi$.

Lemma 2.6 If $D_{sg}(C_n, i) \neq \phi$, then

(i)$D_{sg}(C_{n-1}, i-1) = D_{sg}(C_{n-2}, i-1) = \phi and D_{sg}(C_{n-3}, i-1) \neq \phi, then D_{sg}(C_n, i) = \{1,4, \ldots, n-2\}, \{2,5, \ldots, n-1\}, \{3,6, \ldots, n\}$.

(ii) $D_{sg}(C_{n-2}, i-1) = D_{sg}(C_{n-3}, i-1) = \phi and D_{sg}(C_{n-1}, i-1) \neq \phi$ $then D_{sg}(C_n, i) = \{1,2, \ldots, n\}$.

(iii) $D_{sg}(C_{n-1}, i-1) \neq \phi; D_{sg}(C_{n-2}, i-1) \neq \phi$ and $DD_{sg}(C_{n-3}, i-1) = \phi$ then $D_{sg}(C_n, i) = \{[n] - x / x \in n\}$.

(iv) $D_{sg}(C_{n-1}, i-1) = \phi; D_{sg}(C_{n-2}, i-1) \neq \phi and D_{sg}(C_{n-3}, i-1) \neq \phi$, then $D_{sg}(C_n, i) = \{\{1,4, .., n-4, n-1\}, \{2,5, \ldots., n-3, n\}, \{3,6, \ldots., n-2, n\}\} \cup$

$$\{\ X \cup \begin{cases} n-2 & if \quad 1 \in X \\ n-1 & if \ \ 1 \notin X, 2 \in X \ / X \in C_{n-3}, i-1 \\ n & otherwise \end{cases}$$

(v)$D_{sg}(C_{n-1}, i-1) \neq \phi; D_{sg}(C_{n-2}, i-1) \neq \phi and D_{sg}(C_{n-3}, i-1) \neq \phi$ then $D_{sg}(C_n, i) = \{\{n\} \cup X / X \in C_{n-1}, i-1\} \cup$

$$\{\ X_1 \cup \begin{cases} \{n\}\ if\ n-2\ or\ n-3\ \in\ X_1\ for\ X_1 \in C_{n-2}, i-1 \backslash C_{n-1}, i-1 \\ \{n-1\}\ if\ n-2 \notin\ X_1, n-3 \notin\ X_1\ or\ X_1 \in C_{n-1}, i-1\ \cap C_{n-1}, i-1 \end{cases} \cup$$

$$\{\ X_1 \cup \begin{cases} \{n-2\}\ if\ 1\ \in\ X_2\ for\ X_2 \in C_{n-3}, i-1 or\ X_2 \in C_{n-3}, i-1 \cap C_{n-2}, i-1 \\ \{n-1\}\ if\ n-3 \in\ X_2\ or\ n-4\ \in\ X_2\ \ for\ X_2 \in C_{n-3}, i-1/C_{n-2}, i-1 \end{cases}$$

Proof.

(i)Since $D_{sg}(C_{n-1}, i-1) = D_{sg}(C_{n-2}, i-1) = \phi and D_{sg}(C_{n-3}, i-1) \neq \phi$,then by Lemma2.5$(i) n = 3k and i = k\ for some\ k \in N$. Hence $D_{sg}(P_n, i) = \{1,4,7, \ldots, n-2\}, \{2,5,8, \ldots, n-1\}, \{3,6,9, \ldots, n\}$.

(ii)Since $D_{sg}(C_{n-2}, i-1) = D_{sg}(C_{n-3}, i-1) = \phi$ and $D_{sg}(C_{n-1}, i-1) \neq \phi$, then by lemma 2.5 (ii) $i = n$ .Therefore $D_{sg}(C_n, i) = \{1,2, \ldots, n\}$.

(iii)Since $D_{sg}(C_{n-1}, i-1) \neq \phi; D_{sg}(C_{n-2}, i-1) \neq \phi\ and D_{sg}(C_{n-3}, i-1) = \phi$, then by lemma 2.5 (iii) $i = n-1$, then $D_{sg}(C_n, i) = \{[n] - x/x \in [n]\}$.

(iv) $D_{sg}(C_{n-1}, i-1) = \phi$; $D_{sg}(C_{n-2}, i-1) \neq \phi\ and\ D_{sg}(C_{n-3}, i-1) \neq \phi$, then by theorem,we have $n = 3k+2, i = k+1$,for some $k\ \in N$.We denote $\{\{1,4, .., n-4, n-1\}, \{2,5, \ldots., n-3, n\}, \{3,6, \ldots., n-2, n\}\}$ and

$$\{\ \mathrm{X} \cup \begin{cases} 3k & if\ \ \ 1\ \in\ X \\ 3k+1 \quad if\ \ 1 \notin\ X, 2 \in X\ /X \in C_{n-3}, i-1 & \\ 3k+2 & otherwise \end{cases}$$

as Y_1 and Y_2.We have to prove that $C_{3k+2}, k+1 = Y_1 \cup Y_2$.Since $C_{3k}, k = \{1,4,7, \ldots, n-2\}, \{2,5,8, \ldots, n-1\}, \{3,6,9, \ldots, n\}$. ,then $Y_1 \subseteq C_{3k+2}, k+1$.Also it is obvious that $Y_2 \subseteq C_{3k+2}, k+1$.Hence $C_{3k+2}, k+1 = Y_1 \cup Y_2$. Now let $Y \in C_{3k+2}, k+1$, then , at least one of the vertices labelled $3k\ +\ 2, 3k\ +\ 1$ or $3k$ is in Y . Suppose that $3k\ +\ 2$ \in Y , then , atleast one of the vertices labeled $1, 2\ or\ 3$ and $3k\ +\ 1, 3k$ or $3k-1$ are in Y. If $3k\ +\ 1$ and at least one of $\{1,2,3\}$, and also $3k$ and at least one of $\{1,\ 2\}$ are in Y , then $Y - \{3k+2\} \in C_{3k+1}, k$, a contradiction. If $\{3k, k\}$ or$\{2,3k-1\}$ is a subset of Y ,then $Y = X \cup \{3k+2\}$ for some $X \in C_{3k}, k$. Hence $Y \in Y_1$. If $\{1,3k-1\}$ is a subset of Y , then $Y - \{3k+2\} \in C_{3k+1}, k$, a contradiction. If $\{3,3k-1\}$ is a subset of Y and $\{3k,\ 3k\ +\ 1\}$ is not a subset of Y , then $Y - 3k+2 \in C_{3k-1}, k$. Hence $Y \in Y_2$. If $3k\ +\ 1$ or $3k$ is in Y , we also have the result by the similar argument as above.

(iv) $D_{sg}(C_{n-1}, i-1) \neq \phi; D_{sg}(C_{n-2}, i-1) \neq \phi and D_{sg}(C_{n-3}, i-1) \neq \phi$.First suppose that $X \in C_{n-1}, i-1$,then $X \cup \{n\} \in C_n, i$.So $Y_1 = \{\{n\} \cup\ X\ /\ X\ \in\ D_{n-1}, i-1\} \subseteq\ D_{sg}(C_n, i)$w suppose that $C_{n-2}, i-1 \neq \phi$ .Let $X_1 \in C_{n-2}, i-1$.We denote

$$\{\ X_1 \cup \begin{cases} \{n\}\ if\ n-2\ or\ n-3\ \in\ X_1\ for\ X_1 \in C_{n-2}, i-1 \backslash C_{n-1}, i-1 \\ \{n-1\}\ if\ n-2 \notin\ X_1, n-3 \notin\ X_1\ or\ X_1 \in C_{n-1}, i-1\ \cap C_{n-1}, i-1 \end{cases}$$

by Y_2.we know that at least one of the vertices labeled $n\ -\ 3, n\ -\ 2$ or 1 is in X_1. If $n-2$ or $n-3$ is in X_1, then $X_1 \cup \{n\} \in C_n, i$, otherwise $X_1 \cup \{n-1\} \in C_n, i$.Hence $Y_2 \subseteq C_n, i$.Consider $C_{n-3}, i-1 \neq \phi$.Let $X_2 \in C_{n-3}, i-1$.We denote

$$\{ \ X_1 \cup \begin{cases} \{n-2\}\ if\ 1 \ \in \ X_2\ for\ X_2 \in C_{n-3}, i-1 or \ \ X_2 \in C_{n-3}, i-1 \cap C_{n-2}, i-1 \\ \{n-1\}\ if\ n-3 \in \ X_2\ or\ n-4 \ \in \ X_2 \ \ for\ X_2 \in C_{n-3}, i-1/C_{n-2}, i-1 \end{cases}$$

by Y_3. If $n-3\ or\ n-4$ is in X, then $X \cup \{n-1\} \in C_n, i, otherwise X_2 \cup \{n-2\} \in C_n, i$.Hence $Y_3 \subseteq Y$ .Therefore we have proved that $Y_1 \cup Y_2 \cup Y_3 \subseteq C_n, i$.Now suppose that $Y \in C_n, i$, so , Y contain at least one of the vertices labeled $n, n-1\ or\ n-2$. If $n \in Y$, so at least one of the vertices labeled $n-1, n-2 or\ n-3$ and $1, 2\ or\ 3$ are in Y . If $n-2 \in Y$ or $n-3 \in Y$,then $Y = X \cup \{n\}$ for some $X \in C_{n-1}, i-1$.Hence $Y \in Y_2$. Otherwise $Y = X \cup \{n-1\}$ for some $X \in C_{n-2}, i-1$.Hence $Y \in Y_2$ If $n-1\ or\ n-2$ is in Y , we also have the result by the similar argument as above.

3 Split Geodetic Domination Polynomial of a cycle.

Let Dsg(C_n, i) be the family of split geodetic dominating sets of the cycle graph C_n. with cardinality i and let dsg$(C_n, i) = |Dsg(C_n, i)|$. Then the split geodetic polynomial Dsg(C_n, x) of C_n is defined as Dsg(C_n, x) =$\sum_{i=\gamma_{sg}(C_n)}^{n} dsg(C_n, i)x^i$, where $\gamma_{sg}(C_n)$ is the split geodetic domination number of C_n.In this paper we have determined the family of split geodetic dominating sets of the cycle graph C_n with cardinality i.

Lemma 3.1. For every $n \geq 9$, $D_{sg}(C_n, x) = x\left[D_{sg}(C_{n-1}, x) + D_{sg}(C_{n-2}, x) + D_{sg}(C_{n-3}, x)\right]$ with initial values

$D_{sg}(C_3, x) = x^3$;

$D_{sg}(C_4, x) = 4x^3 + x^4$;

$D_{sg}(C_5, x) = 5x^3 + 5x^4 + x^5$;

$D_{sg}(C_6, x) = 12x^3 + 10x^4 + 6x^5 + x^6$;

$D_{sg}(C_7, x) = 14x^3 + 21x^4 + 16x^5 + 7x^6 + x^7$;

$D_{sg}(C_8, x) = 8x^3 + 31x^4 + 36x^5 + 23x^{+}8x^7 + x^8$;

$D_{sg}(C_9, x) = 3x^3 + 34x^4 + 62x^5 + 58x^{+}31x^7 + 9x^8 + x^9$

Table1: $D_{sg}(C_n, j)$,the number of split geodetic dominating set of C_n with cardinality j

n/j	3	4	5	6	7	8	9	10	11	12	13	14
3	1											

4	4	1										
5	5	5	1									
6	12	10	6	1								
7	14	21	16	7	1							
8	8	31	36	23	8	1						
9	3	34	62	58	31	9	1					
10	0	25	86	114	88	40	10	1				
11	0	11	90	184	195	127	50	11	1			
12	0	3	70	238	356	314	176	61	12	1		
13	0	0	39	246	536	639	481	236	73	13	1	
14	0	0	14	199	668	1087	1080	707	308	86	14	1

Using lemma 2.6 ,we obtain $d_{sg}(C_n, j) for 3 \leq n \leq 14$ as shown in Table1.

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