The Lattice of Convex Sublattices of $S^3(B_n)$

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Abstract

In this paper, we prove that $CS[S^3(B_n)]$ is an Eulerian lattice under the set inclusion relation and it is neither simplicial nor dual simplicial, if $n > 1$.

Keywords: Convex sublattice; Simplicial Eulerian lattice; Dual simplicial.

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1 **Introduction**

The lattice of sublattices of a lattice with convex sublattices has been studied in some detail by K. M. Koh [3] in the year 1972. He had investigated the internal structure of a lattice L, in relation to $CS(L)$, like so many other authors for various algebraic structures such as groups, Boolean algebras, directed graphs and so on. In 1992, V. K. Santhi [12] constructed a new Eulerian lattice $S(B_n)$ from a Boolean algebra B_n of rank n. In 2012, R. Subbarayan and A. Vethamanickam [15] have proved in their paper that the lattice of convex sublattices of a Boolean algebra B_n, of rank n, $CS(B_n)$ with respect to the set inclusion relation is a dual simplicial Eulerian lattice. Neither simplicity nor dual simplicity are characteristics associated with the set inclusion relation.

In this paper, we are going to look at the structure of $CS[S^3(B_n)]$ and prove it to be Eulerian under $' \subseteq '$ relation. $S(B_2)$ is shown in figure 1. We note that $S(B_2)$ contains three copies of B_2, we call them left copy, right copy and middle copy of $S(B_2)$.

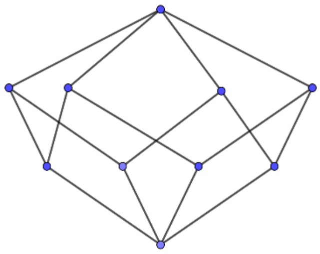

Figure 1

Lemma 1.1. [8] A finite graded poset P is Eulerian if and only if all intervals$[x, y]$ of length $l \geq 1$ in P contain an equal number of elements of odd and even rank.

Lemma 1.2. [13] If L_1 and L_2 are two Eulerian lattices then $L_1 \times L_2$ is also Eulerian.

There is no way to contain a three element chain as an interval. In the case that an undefined term needs to be referred to, we use [2], [11] and [12].

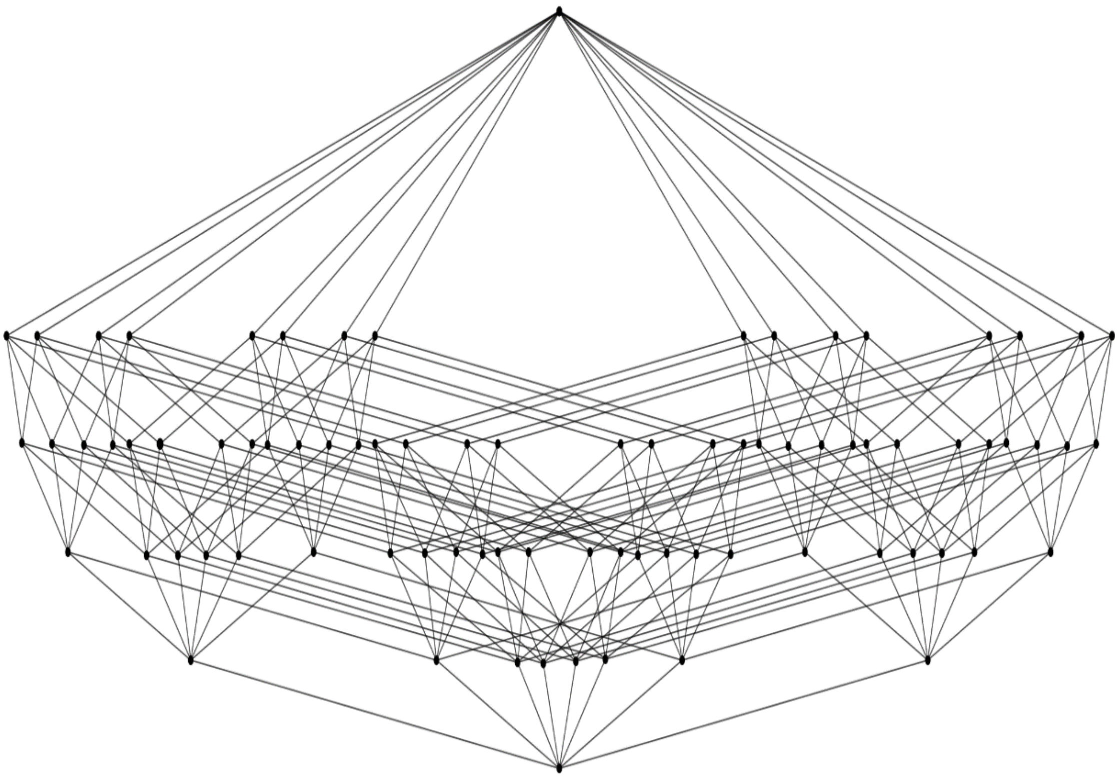

Figure 2-$\boldsymbol{S^3(B_2)}$

2 The Eulerian property of the lattice $CS[S^3(B_n)]$

Lemma 2.1. For $n \geq 1$, we have $1+2+\binom{n}{1}+2+2+2\left[2+\binom{n}{1}+2\right]+2\left[\binom{n}{1}+2\right]+2\binom{n}{1}+\binom{n}{2}+22\left[\binom{n}{1}+2\right]+2\binom{n}{1}+\binom{n}{2}+2\left[2\binom{n}{1}+\binom{n}{2}\right]+2\binom{n}{2}+\binom{n}{3}+22\left[2\binom{n}{1}+\binom{n}{2}\right]+2\binom{n}{2}+\binom{n}{3}+2\left[2\binom{n}{2}+\binom{n}{3}\right]+2\binom{n}{3}+\binom{n}{4}+\cdots+22\left[2\binom{n}{n-3}+\binom{n}{n-2}\right]+2\binom{n}{n-2}+\binom{n}{n-1}+2\left[2\binom{n}{n-2}+\binom{n}{n-1}\right]+2\binom{n}{n-1}+22\left[2\binom{n}{n-2}+\binom{n}{n-1}\right]+2\binom{n}{n-1}+2\left[2\binom{n}{n-1}\right]+22\left[2\binom{n}{n-1}\right]+1=3^3.2^n-26.$

Theorem 2.2 $CS[S^3(B_n)]$, the lattice of convex sublattices of $S^3(B_n)$with respect to the set inclusion relation is an Eulerian lattice.

Proof

We first note that, the number of elements of ranks $0,1,2,\ldots,n+1$ in$S(B_n)$are, $1,2+\binom{n}{1},2\binom{n}{1}+\binom{n}{2},2\binom{n}{2}+\binom{n}{3},\ldots,2\binom{n}{n-2}+\binom{n}{n-1},2\binom{n}{n-1},1$ respectively.

The number of elements of ranks $0,1,2,\ldots,n+2$ in $S[S(B_n)]$ are, $1,2+\binom{n}{1}+2,2\left[\binom{n}{1}+2\right]+2\binom{n}{1}+\binom{n}{2},2\left[2\binom{n}{1}+\binom{n}{2}\right]+2\binom{n}{2}+\binom{n}{3},2\left[2\binom{n}{2}+\binom{n}{3}\right]+2\binom{n}{3}+\binom{n}{4},\ldots,2\left[2\binom{n}{n-2}+\binom{n}{n-1}\right]+2\binom{n}{n-1},2\left[2\binom{n}{n-1}\right]$, 1respectively.

The number of elements of ranks $0,1,2,\ldots,n+3$ in $S^3(B_n)$ are, $1,2+\binom{n}{1}+2+2,2\left[2+\binom{n}{1}+2\right]+2\left[\binom{n}{1}+2\right]+2\binom{n}{1}+\binom{n}{2},22\left[\binom{n}{1}+2\right]+2\binom{n}{1}+\binom{n}{2}+2\left[2\binom{n}{1}+\binom{n}{2}\right]+2\binom{n}{2}+\binom{n}{3},22\left[2\binom{n}{1}+\binom{n}{2}\right]+2\binom{n}{2}+\binom{n}{3}+2\left[2\binom{n}{2}+\binom{n}{3}\right]+2\binom{n}{3}+\binom{n}{4},\ldots,22\left[2\binom{n}{n-3}+\binom{n}{n-2}\right]+2\binom{n}{n-2}+\binom{n}{n-1}+2\left[2\binom{n}{n-2}+\binom{n}{n-1}\right]+2\binom{n}{n-1},22\left[2\binom{n}{n-2}+\binom{n}{n-1}\right]+2\binom{n}{n-1}+2\left[2\binom{n}{n-1}\right],22\left[2\binom{n}{n-1}\right]$, 1respectively.

It is clear that the rank of $CS[S^3(B_n)]$, is $n + 4$.

We are going to prove that $CS[S^3(B_n)]$, is Eulerian.

That is, to prove that this interval $[\varphi, S^3(B_n)]$ has the same number of elements of odd and even rank.

Let A_i be the number of elements of rank i in $CS[S^3(B_n)]$, $i=1,2,\ldots,n+3$.

A_1= The number of singleton subsets of $S^3(B_n)$

$=1+2+\binom{n}{1}+2+2+2\left[2+\binom{n}{1}+2\right]+2\left[\binom{n}{1}+2\right]+2\binom{n}{1}+\binom{n}{2}+22\left[\binom{n}{1}+2\right]+2\binom{n}{1}+\binom{n}{2}+2\left[2\binom{n}{1}+\binom{n}{2}\right]+2\binom{n}{2}+\binom{n}{3}+22\left[2\binom{n}{1}+\binom{n}{2}\right]+2\binom{n}{2}+\binom{n}{3}+2\left[2\binom{n}{2}+\binom{n}{3}\right]+2\binom{n}{3}+\binom{n}{4}+\cdots+22\left[2\binom{n}{n-3}+\binom{n}{n-2}\right]+2\binom{n}{n-2}+\binom{n}{n-1}+$

$2\left[2\binom{n}{n-2}+\binom{n}{n-1}\right]+2\binom{n}{n-1}+22\left[2\binom{n}{n-2}+\binom{n}{n-1}\right]+2\binom{n}{n-1}+2\left[2\binom{n}{n-1}\right]+$
$22\left[2\binom{n}{n-1}\right]+1$……………….(2.1.1)

A_2 = The number of rank 2 convex sublattices in $S^3(B_n)$

= The number of edges in $S^3(B_n)$

= The number of edges containing 0 + number of edges with an atom at the bottom + The number of edges from the rank 2 elements$+\cdots+$The number of edges with a coatom of $S^3(B_n)$at the bottom.

Number of edges containing 0 is, $2+\binom{n}{1}+2+2$ ………………(2.2)

The number of edges with an extreme atom at the bottom of the edge$=2+\binom{n}{1}+2$. There are 2 extreme atoms, this means that the total number of these edges will be equal to $2\left[2+\binom{n}{1}+2\right]$

Let x be an atom in the middle copy, then $[x,1]$
$\cong \{\{S^2(B_n)\ \ if\ x\ \ be\ in\ an\ extreme\ copies\ of\ \ S^3(B_n), S^3(B_{n-1})\ \ if x\ be\ in\ the\ middle\ copy\ \ of\ S^3(B_n)\}$

If $[x,1]\ \cong\ S^2(B_n)$, there are $2+\binom{n}{1}+2$ edges.

There are 2 extreme atoms, this means that the total number of these edges will be equal to $2[2+\binom{n}{1}+2]$. If $[x,1]\cong\ S^3(B_{n-1})$, there are $2+2+\binom{n-1}{1}+2$ edges. There are $2+\binom{n}{1}$ such atoms, since, the middle copy of $S^3(B_n)$ is of the form $S^2(B_n)$, whose middle copy is of the form $S\,(B_n)$, this means that the total number of these edges will be equal to $(2+\binom{n}{1})[2+2+\binom{n-1}{1}+2]$. Hence, the number of edges that have an atom at the bottom of the edge is a total of $2[2+\binom{n}{1}+2]+2[2+\binom{n}{1}+2]+(2+\binom{n}{1})[2+2+\binom{n-1}{1}+2]$. ……....(2.3)

Now to find, the number of edges with an element of rank 2 at the bottom.

Let x be a rank 2 element in the left copy. Then,
$$[x,1]\cong\{\{S(B_n)\ \ if\ x\ \in extreme\ copies\ of\ \ left\ copy\ of\ S^3(B_n), S^2(B_{n-1})\ \ if x \in middle\ copy\ \ of\ left\ copy\ S^3(B_n)\}\}$$

If $[x,1]\cong S(B_n)$, there are $\binom{n}{1}+2$ edges in both extreme copies. Totally, $2(\binom{n}{1}+2)$ edges are there. If $[x,1]\cong\ S^2(B_{n-1})$, the number of edges from x is $2+\binom{n-1}{1}+2$. There are $2+\binom{n}{1}$ such elements, since, the middle copy of $S^3(B_n)$ is of the form $S^2(B_n)$ whose middle copy is of the form $S(B_n)$, therefore, totally $2+\binom{n}{1}[2+\binom{n-1}{1}+2]$ edges in the middle of the left copy of $S^3(B_n)$.The number of edges in the left copy that have an element of rank 2 at the bottom is $= 2[\binom{n}{1}+2]+(2+\binom{n}{1})[2+\binom{n-1}{1}+2]$. Similarly, the number of edges in the right copy that have an element of rank 2 at the bottom is therefore $= 2[\binom{n}{1}+2]+(2+\binom{n}{1})[2+\binom{n-1}{1}+2]$.

Let x be a rank 2 element in the middle copy of $S^3(B_n)$.

Then, $[x,1] \cong \{\{S^2(B_{n-1})\ if\ x \in extreme\ copies\ of\ middle\ copy\ of\ S^3(B_n),\ S^3(B_{n-2})\ if x \in middle\ copy\ of\ middle\ copy\ S^3(B_n)\}\}$

If $[x,1] \cong S^2(B_{n-1})$, the number of edges from x is $2+\binom{n-1}{1}+2$. There are $2+\binom{n}{1}$ such elements in both extreme copies. Totally, $(2+\binom{n}{1})(2+\binom{n-1}{1}+2)$ edges. If $[x,1] \cong S^3(B_{n-2})$,,the number of edges from x is $2+2+\binom{n-2}{1}+2$. There are $2\binom{n}{1}+\binom{n}{2}$ such elements, therefore, totally $(2\binom{n}{1}+\binom{n}{2})[2+2+\binom{n-2}{1}+2]$ edges in the middle of the middle copy of $S^3(B_n)$. The number of edges in the middle copy that have an element of rank 2 at the bottom is therefore $2[(2+\binom{n}{1})(2+\binom{n-1}{1}+2)]+(2\binom{n}{1}+\binom{n}{2})[2+2+\binom{n-2}{1}+2]$ edges. Hence, the total number of edges from a rank 2 element can be expressed as follows: $2[2[\binom{n}{1}+2]+(2+\binom{n}{1})[2+\binom{n-1}{1}+2]]+2[(2+\binom{n}{1})(2+\binom{n-1}{1}+2)]+(2\binom{n}{1}+\binom{n}{2})[2+2+\binom{n-2}{1}+2]$ ....………..(2.4)

Now to find, the number of edges with an element of rank 3 at the bottom. Let x be a rank 3 element in the extreme copies in the left copy of $S^3(B_n)$.

Then, $[x,1] \cong S(B_{n-1}), if x \in an\ extreme\ copies\ of\ leftcopy\ of\ S^3(B_n)$

$\cong S^2(B_{n-2}), if x \in middle\ copy\ of\ left\ copy\ of\ S^3(B_n)$

If $[x,1] \cong S(B_{n-1})$, the number of edges from x is $2+\binom{n-1}{1}$. There are $2+\binom{n}{1}$such x's in both extreme copies. Totally, $(2+\binom{n}{1})(2+\binom{n-1}{1}$ edges from such x's in the extreme copies of left copy.

If $[x,1] \cong S^2(B_{n-2})$, then the number of edges from x is $2+\binom{n-2}{1}+2$. There are $2\binom{n}{1}+\binom{n}{2}$ such elements in both extreme copies. Totally, $(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)$ edges. If $[x,1] \cong S^3(B_{n-2})$,,the number of edges from x is $2+2+\binom{n-2}{1}+2$. There are $2\binom{n}{1}+\binom{n}{2}$ such elements, therefore, totally $(2\binom{n}{1}+\binom{n}{2})[2+2+\binom{n-2}{1}+2]$ edges in the middle of the left copy of $S^3(B_n)$. The number of edges in the left copy that have an element of rank 3 at the bottom is therefore $2[(2+\binom{n}{1})(2+\binom{n-1}{1})]+(2\binom{n}{1}+\binom{n}{2})[2+\binom{n-2}{1}+2]$ edges.

Similarly, the number of edges in the right copy that have an element of rank 3 at the bottom is therefore, $2[(2+\binom{n}{1})(2+\binom{n-1}{1})]+(2\binom{n}{1}+\binom{n}{2})[2+\binom{n-2}{1}+2]$.

Let x be a rank 3 element in the middle copy of $S^3(B_n)$.

Then, $[x,1]\cong\{\{S^2(B_{n-2})\ if\ x\in extreme\ copies\ of\ middle\ copy\ of\ S^3(B_n),\ S^3(B_{n-3})\ if x\in middle\ copy\ of\ middle\ copy\ S^3(B_n)\}\}$

If $[x,1]\cong S^2(B_{n-2})$, the number of edges from x is $2+\binom{n-2}{1}+2$. There are $2\binom{n}{1}+\binom{n}{2}$ such elements in both extreme copies. Totally, $(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)$ edges.

If $[x,1]\cong S^3(B_{n-3})$,,the number of edges from x is $2+2+\binom{n-3}{1}+2$. There are $2\binom{n}{2}+\binom{n}{3}$ such elements, therefore, totally $(2\binom{n}{2}+\binom{n}{3})[2+2+\binom{n-3}{1}+2]$ edges in the middle of the middle copy of $S^3(B_n)$. The number of edges in the middle copy that have an element of rank 3 at the bottom is therefore $2[(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)]+(2\binom{n}{2}+\binom{n}{3})[2+2+\binom{n-3}{1}+2]$ edges. Hence, the total number of edges from a rank 3 element can be expressed as follows: $2\{2[(2+\binom{n}{1})(2+\binom{n-1}{1})]+(2\binom{n}{1}+\binom{n}{2})[2+\binom{n-2}{1}+2]\}+2[(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)]+2[(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)]+(2\binom{n}{2}+\binom{n}{3})[2+2+\binom{n-3}{1}+2]$ ....…………(2.5)

We can proceed in the same way to find the number of edges from the bottom of a coatom of $S^3(B_n)=$ the number of coatoms in $S^3(B_n)$

$=2\{2[2\binom{n}{n-1}\}$………… (2.6)

Hence, from (2.2), (2.3), (2.4), (2.5) and (2.6) we get, the total number of edges in $S^3(B_n)$ is,

$A_2=2+\binom{n}{1}+2+2+2[2+\binom{n}{1}+2]+2[2+\binom{n}{1}+2]+(2+\binom{n}{1})[2+2+\binom{n-1}{1}+2]+2[2[\binom{n}{1}+2]+(2+\binom{n}{1})[2+\binom{n-1}{1}+2]]+2[(2+\binom{n}{1})(2+\binom{n-1}{1}+2)]+(2\binom{n}{1}+\binom{n}{2})[2+2+\binom{n-2}{1}+2]+2\{2[(2+\binom{n}{1})(2+\binom{n-1}{1})]+(2\binom{n}{1}+\binom{n}{2})[2+\binom{n-2}{1}+2]\}+2[(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)]+2[(2\binom{n}{1}+\binom{n}{2})(2+\binom{n-2}{1}+2)]+(2\binom{n}{2}+\binom{n}{3})[2+2+\binom{n-3}{1}+2]+\ldots+2\{2[2\binom{n}{n-1}\}$………… …… (2.1.2)

$A_3=$ The number of 4 element convex sublattices in $S^3(B_n)$

= The number of B_2's in $S^3(B_n)$

=The number of $B_2's$ containing $0+$ the number of $B_2's$ containing an atom at the bottom $+$ ….+ the number of $B_2's$ containing a rank $n+1$ element at the bottom in $S^3(B_n)$.

The number of 4 element convex sublattices in $S^3(B_n)$ containing 0 as the bottom element is,

$$2[2+\binom{n}{1}+2]+\ 2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}\quad \dots\dots\dots (2.7)$$

Next, we find the number of 4 element convex sublattices containing an atom as the bottom element.

Fix an atom $x\in S^3(B_n)$. If x is the bottom element of the left copy of $S^3(B_n)$, then $[x,1]\cong S^2(B_n)$. Therefore, the number of B_2's containing x at the bottom is $2[\binom{n}{1}+2]+\ 2\binom{n}{1}+\binom{n}{2}$.Similarly, the number of $B_2's$ containing the bottom element of the right copy is $2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}$.

If x is in the middle copy of $S^3(B_n)$, then, $[x,1]\cong\{\{S^2(B_n)\ if\ x\ \in extreme\ copies\ of\ \ middle\ copy\ of\ S^3(B_n),\ S^3(B_{n-1})\ \ ifx\ middle\ copy\ \ of\ middle\ copy\ S^3(B_n)\}\}$
If $[x,1]\cong S^2(B_n)$, there are $2[\binom{n}{1}+2]+\ 2\binom{n}{1}+\binom{n}{2}$ B_2's in both extreme copies. Totally, $2\{2[\binom{n}{1}+2]+\ 2\binom{n}{1}+\binom{n}{2}\}$ such B_2's . If $[x,1]\cong S^3(B_{n-1})$, then the number of B_2's containing x is $2[2+\binom{n-1}{1}+2]+2[\binom{n-1}{1}+2]+2\binom{n-1}{1}+\binom{n-1}{2}$. There are $2+\binom{n}{1}$ such elements, therefore, the total number of B_2's containing all the atoms at the bottom in the middle of the middle copy is $2\{2[\binom{n}{1}+2]+\ 2\binom{n}{1}+\binom{n}{2}\}+(2+\binom{n}{1})\{2[2+\binom{n-1}{1}+2]+2[\binom{n-1}{1}+2]+2\binom{n-1}{1}+\binom{n-1}{2}\}$.

Therefore, the number of B_2's containing all the atoms of $S^3(B_n)$ is,$2[2[\binom{n}{1}+2]+2\binom{n}{1}+\binom{n}{2}]+2\{2[\binom{n}{1}+2]+\ 2\binom{n}{1}+\binom{n}{2}\}+(2+\binom{n}{1})\{2[2+\binom{n-1}{1}+2]+2[\binom{n-1}{1}+2]+2\binom{n-1}{1}+\binom{n-1}{2}\}$.

.............(2.8)

Next, fix an element x of rank 2 in $S^3(B_n)$

If x is in the left copy of $S^3(B_n)$.

$$\text{Then, } [x,1]\cong S(B_n), ifx\in an\ extreme\ copies\ of\ leftcopy\ of\ S^3(B_n)$$

$$\cong S^2(B_{n-1}), ifx\in middle\ copy\ of\ left\ copy\ of\ S^3(B_n)$$

If $[x,1]\cong S(B_n)$, the number of B_2's from x is $2\binom{n}{1}+\binom{n}{2}$. There are 2 such extreme copies. Totally, $2(2\binom{n}{1}+\binom{n}{2})$ such B_2's in the extreme copies of left copy.

If $[x,1] \cong S^2(B_{n-1})$, then the number of B_2's from x is $2(\binom{n-1}{1}+2)+2\binom{n-1}{1}+\binom{n-1}{2}$. There are $2+\binom{n}{1}$ such elements x of rank 2in the middle of the left copy. Therefore, the total number of B_2's containing a rank 2 element at the bottom in the left copy is , $2(2\binom{n}{1}+\binom{n}{2})(2+\binom{n}{1})[2(\binom{n-1}{1}+2)+2\binom{n-1}{1}+\binom{n-1}{2}]$. Similarly, we have the same number in the right copy. Therefore, the total number of B_2's containing a rank 2 element at the bottom in the extreme copies $= 2(2\binom{n}{1}+\binom{n}{2})(2+\binom{n}{1})[2(\binom{n-1}{1}+2)+2\binom{n-1}{1}+\binom{n-1}{2}]$.

If x is in the middle copy of $S^3(B_n)$, then

$$[x,1] \cong S^2(B_{n-1}), if x \in an\ extreme\ copies\ of\ middle\ copy\ of\ S^3(B_n)$$

$$\cong S^3(B_{n-2}), if x \in middle\ copy\ of\ middle\ copy\ of\ S^3(B_n)$$

If $[x,1] \cong S^2(B_{n-1})$, there are $2(\binom{n-1}{1}+2)+2\binom{n-1}{1}+\binom{n-1}{2}$ B_2's with x at the bottom. There are $2+\binom{n}{1}$ such x's. Totally, $2+\binom{n}{1}$ $\{2(\binom{n-1}{1}+2)+2\binom{n-1}{1}+\binom{n-1}{2}\}$ B_2's in the extreme copies of the middle copy.

If $[x,1] \cong S^3(B_{n-2})$, then the number of B_2's containing x is $2[2+\binom{n-2}{1}+2]+2[\binom{n-2}{1}+2]+2\binom{n-2}{1}+\binom{n-2}{2}$. There are $2\binom{n}{1}+\binom{n}{2}$ such elements x of rank 2 in the middle of the middle copy. Therefore, the total number of B_2's containing a rank 2 element at the bottom in the middle of the middle copy is $(2\binom{n}{1}+\binom{n}{2})[\ 2[2+\binom{n-2}{1}+2]+2[\binom{n-2}{1}+2]+2\binom{n-2}{1}+\binom{n-2}{2}]$. Therefore, the number of B_2's in the middle copy containing all the elements of rank 2 in the middle copy is, $2\{(2+\binom{n}{1})\ \{2(\binom{n-1}{1}+2)+2\binom{n-1}{1}+\binom{n-1}{2}\}\}+(2\binom{n}{1}+\binom{n}{2})[\ 2[2+\binom{n-2}{1}+2]+2[\binom{n-2}{1}+2]+2\binom{n-2}{1}+\binom{n-2}{2}]$. Therefore, the total number of B_2's containing all the rank 2 elements in $S^3(B_n)$ is, $2\{2\{2\binom{n}{1}+\binom{n}{2}\}+(2+\binom{n}{1})[2(\binom{n-1}{1}+2)+2\binom{n-1}{1}+\binom{n-1}{2}]\}+2\{(2+\binom{n}{1})[2[\binom{n-1}{1}+2]+2\binom{n-1}{1}+\binom{n-1}{2}]\}+(2\binom{n}{1}+\binom{n}{2})[2[2+\binom{n-1}{1}+2]+2[\binom{n-2}{1}+2]+2\binom{n-2}{1}+\binom{n-2}{2}]$

………(2.9)

In the same manner, the total number of B_2's containing all the rank 3 elements in $S^3(B_n)$ is, $2\{2\{(2+\binom{n}{1})[2\binom{n-1}{1}+\binom{n-1}{2}]\}+(2\binom{n}{1}+\binom{n}{2})[2(\binom{n-2}{1}+2)+2\binom{n-2}{1}+\binom{n-2}{2}]\}+2\{(2\binom{n}{1}+\binom{n}{2})[2[2+\binom{n-2}{1}]+2\binom{n-2}{1}+\binom{n-2}{2}]\}+(2\binom{n}{2}+\binom{n}{3})[2[2+\binom{n-3}{1}+2]+2[\binom{n-3}{1}+2]+2\binom{n-3}{1}+\binom{n-3}{2}]$ …………..(2.10)

Proceeding like this, we find the number of B_2's containing all the rank $n+1$ element at the bottom in $S^3(B_n) =$ the number of rank $n+1$ elements in $S^3(B_n) = 2\{2[2\binom{n}{n-2} + \binom{n}{n-1}] + 2\binom{n}{n-1}\} + 2[2\binom{n}{n-1}]$ …….(2.11)

Hence, using (2.7),(2.8),(2.9), (2.10) and (2.11) we get the total number of 4 element convex sublattices in $S^3(B_n)$ is

$$A_3 = 2[2+\binom{n}{1}+2] + 2[\binom{n}{1}+2] + 2\binom{n}{1} + \binom{n}{2} + 2[2[\binom{n}{1}+2] + 2\binom{n}{1} + \binom{n}{2}] + 2\{2[\binom{n}{1}+2] + 2\binom{n}{1} + \binom{n}{2}\} + (2+\binom{n}{1})\{2[2+\binom{n-1}{1}+2] + 2[\binom{n-1}{1}+2] + 2\binom{n-1}{1} + \binom{n-1}{2}\} + 2\{2\{2\binom{n}{1} + \binom{n}{2}\} + (2+\binom{n}{1})[2(\binom{n-1}{1}+2) + 2\binom{n-1}{1} + \binom{n-1}{2}]\} + 2\{(2+\binom{n}{1})[2[\binom{n-1}{1}+2] + 2\binom{n-1}{1} + \binom{n-1}{2}]\} + (2\binom{n}{1} + \binom{n}{2})[2[2+\binom{n-1}{1}+2] + 2[\binom{n-2}{1}+2] + 2\binom{n-2}{1} + \binom{n-2}{2}] + 2\{2\{(2+\binom{n}{1})[2\binom{n-1}{1} + \binom{n-1}{2}]\} + (2\binom{n}{1} + \binom{n}{2})[2(\binom{n-2}{1}+2) + 2\binom{n-2}{1} + \binom{n-2}{2}]\} + 2\{(2\binom{n}{1} + \binom{n}{2})[2[2+\binom{n-2}{1}] + 2\binom{n-2}{1} + \binom{n-2}{2}]\} + (2\binom{n}{2} + \binom{n}{3})[2[2+\binom{n-3}{1}+2] + 2[\binom{n-3}{1}+2] + 2\binom{n-3}{1} + \binom{n-3}{2}] + \ldots + 2\{2[2\binom{n}{n-2} + \binom{n}{n-1}] + 2\binom{n}{n-1}\} + 2[2\binom{n}{n-1}] \quad \ldots\ldots\ldots..(2.1.3)$$

Proceeding like this, we find that $A_4, A_5, \ldots. A_{n+3}$

$$A_4 = 2[2(\binom{n}{1}+2) + 2\binom{n}{1} + \binom{n}{2}] + 2[2\binom{n}{1} + \binom{n}{2}] + 2\binom{n}{2} + \binom{n}{3} + 2\{2[2\binom{n}{1} + \binom{n}{2}] + 2\binom{n}{2} + \binom{n}{3}\} + 2\{2[2\binom{n}{1} + \binom{n}{2}] + 2\binom{n}{2} + \binom{n}{3}\} + (2+\binom{n}{1})[2[2(\binom{n-1}{1}+2) + 2\binom{n-1}{1} + \binom{n-1}{2}] + 2[2\binom{n-1}{1} + \binom{n-1}{2}] + 2\binom{n-1}{2} + \binom{n-1}{3}] + 2\{2\{2\binom{n}{2} + \binom{n}{3}\} + (2+\binom{n}{1})[2(2\binom{n-1}{1} + \binom{n-1}{2}) + 2\binom{n-1}{2} + \binom{n-1}{3}]\} + 2\{(2+\binom{n}{1})\{2[2\binom{n-1}{1} + \binom{n-1}{2}] + 2\binom{n-1}{2} + \binom{n-1}{3}\}\} + (2\binom{n}{1} + \binom{n}{2})[2[2[\binom{n-2}{2}+2] + 2\binom{n-2}{1} + \binom{n-2}{2}] + 2[2\binom{n-2}{1} + \binom{n-2}{2}] + 2\binom{n-2}{2} + \binom{n-2}{3}] + \ldots + 2\{2[2\binom{n}{n-3} + \binom{n}{n-2}] + 2\binom{n}{n-2} + \binom{n}{n-1}\} + 2[2\binom{n}{n-2} + \binom{n}{n-1}] + 2\binom{n}{n-1} \quad \ldots\ldots\ldots\ldots..\ (2.1.4)$$

In the same manner, A_{n+1} =The number of convex sublattices of rank n in $S^3(B_n)$

$$=2\{2(2\binom{n}{n-3} + \binom{n}{n-2}) + 2\binom{n}{n-2} + \binom{n}{n-1}\} + 2[2\binom{n}{n-2} + \binom{n}{n-1}] + 2\binom{n}{n-1} + 2\{2(2\binom{n}{n-2} + \binom{n}{n-1}) + 2\binom{n}{n-1}\} + 2\{2[2\binom{n}{n-2} + \binom{n}{n-1}] + 2\binom{n}{n-1}\} + (2+\binom{n}{1})\{2[2(2\binom{n-1}{n-2}) + 2\binom{n-1}{n-2} + 2\binom{n-1}{n-2}] + 2[2\binom{n-1}{n-2}]\} + 2\{2\{2\binom{n}{n-1}\} + (2+\binom{n}{1})\{2(2\binom{n-1}{n-2})\}\} + 2\{(2+\binom{n}{1})\{2[2\binom{n-1}{n-2}]\} + (2\binom{n}{1} + \binom{n}{2})\{2[2[2\binom{n-2}{n-3}]]\}\} + 2\{2[\binom{n}{1}+2] + 2\binom{n}{1} + \binom{n}{2}\} + 2[2\binom{n}{1} + \binom{n}{2}] + 2\binom{n}{2} + \binom{n}{3}.$$

……..(2.1.5)

$A_{n+2} = 2\{2(2\binom{n}{n-2} + \binom{n}{n-1}) + 2\binom{n}{n-1}\} + 2[2\binom{n}{n-1}] + 2\{2[2\binom{n}{n-1}]\} + 2\{2[2\binom{n}{n-1}]\} + (2+\binom{n}{1})[2\{2[2\binom{n-1}{n-2}]\}] + 2[2+\binom{n}{1}] + 2] + 2[\binom{n}{1} + 2\binom{n}{1} + \binom{n}{2}$ ....……(2.1.6)

$A_{n+3} = 2\{2[2\binom{n}{n-1}]\} + 2 + \binom{n}{1} + 2 + 2.$ …………(2.1.6)

Case(i): Suppose that n is odd. Therefore, $n+4$ is odd.

$A_1 - A_2 + A_3 - \ldots - A_{n+1} + A_{n+2} - A_{n+3} = 1 + 2 + \binom{n}{1} + 2 + 2 + 2\left[2 + \binom{n}{1} + 2\right] + 2\left[\binom{n}{1} + 2\right] + 2\binom{n}{1} + \binom{n}{2} + 22\left[\binom{n}{1} + 2\right] + 2\binom{n}{1} + \binom{n}{2} + 2\left[2\binom{n}{1} + \binom{n}{2}\right] + 2\binom{n}{2} + \binom{n}{3} + 22\left[2\binom{n}{1} + \binom{n}{2}\right] + 2\binom{n}{2} + \binom{n}{3} + 2\left[2\binom{n}{2} + \binom{n}{3}\right] + 2\binom{n}{3} + \binom{n}{4} + \cdots + 22\left[2\binom{n}{n-3} + \binom{n}{n-2}\right] + 2\binom{n}{n-2} + \binom{n}{n-1} + 2\left[2\binom{n}{n-2} + \binom{n}{n-1}\right] + 2\binom{n}{n-1} + 22\left[2\binom{n}{n-2} + \binom{n}{n-1}\right] + 2\binom{n}{n-1} + 2\left[2\binom{n}{n-1}\right] + 22\left[2\binom{n}{n-1}\right] + 1 - 2 + \binom{n}{1} + 2 + 2 + 2[2 + \binom{n}{1} + 2] + 2[2 + \binom{n}{1} + 2] + (2 + \binom{n}{1})[2 + 2 + \binom{n-1}{1} + 2] + 2[2[\binom{n}{1} + 2] + (2 + \binom{n}{1})[2 + \binom{n-1}{1} + 2]] + 2[(2 + \binom{n}{1})(2 + \binom{n-1}{1} + 2)] + (2\binom{n}{1} + \binom{n}{2})[2 + 2 + \binom{n-2}{1} + 2] + 2\{2[(2 + \binom{n}{1})(2 + \binom{n-1}{1})] + (2\binom{n}{1} + \binom{n}{2})[2 + \binom{n-2}{1} + 2]\} + 2[(2\binom{n}{1} + \binom{n}{2})(2 + \binom{n-2}{1} + 2)] + 2[(2\binom{n}{1} + \binom{n}{2})(2 + \binom{n-2}{1} + 2)] + (2\binom{n}{2} + \binom{n}{3})[2 + 2 + \binom{n-3}{1} + 2] + \ldots + 2\{2[2\binom{n}{n-1}]\} + 2[2 + \binom{n}{1} + 2] + 2[\binom{n}{1} + 2] + 2\binom{n}{1} + \binom{n}{2} + 2[2[\binom{n}{1} + 2] + 2\binom{n}{1} + \binom{n}{2}] + 2\{2[\binom{n}{1} + 2] + 2\binom{n}{1} + \binom{n}{2}\} + (2 + \binom{n}{1})\{2[2 + \binom{n-1}{1} + 2] + 2[\binom{n-1}{1} + 2] + 2\binom{n-1}{1} + \binom{n-1}{2}\} + 2\{2\{2\binom{n}{1} + \binom{n}{2}\} + (2 + \binom{n}{1})[2(\binom{n-1}{1} + 2) + 2\binom{n-1}{1} + \binom{n-1}{2}]\} + 2\{(2 + \binom{n}{1})[2[\binom{n-1}{1} + 2] + 2\binom{n-1}{1} + \binom{n-1}{2}]\} + (2\binom{n}{1} + \binom{n}{2})[2[2 + \binom{n-1}{1} + 2] + 2[\binom{n-2}{1} + 2] + 2\binom{n-2}{1} + \binom{n-2}{2}] + 2\{2\{(2 + \binom{n}{1})[2\binom{n-1}{1} + \binom{n-1}{2}]\} + (2\binom{n}{1} + \binom{n}{2})[2(\binom{n-2}{1} + 2) + 2\binom{n-2}{1} + \binom{n-2}{2}]\} + 2\{(2\binom{n}{1} + \binom{n}{2})[2[2 + \binom{n-2}{1}] + 2\binom{n-2}{1} + \binom{n-2}{2}]\} + (2\binom{n}{2} + \binom{n}{3})[2[2 + \binom{n-3}{1} + 2] + 2[\binom{n-3}{1} + 2] + 2\binom{n-3}{1} + \binom{n-3}{2}] + \ldots + 2\{2[2\binom{n}{n-2} + \binom{n}{n-1}] + 2\binom{n}{n-1}\} + 2[2\binom{n}{n-1}]$ $-2[2(\binom{n}{1} + 2) + 2\binom{n}{1} + \binom{n}{2}] + 2[2\binom{n}{1} + \binom{n}{2}] + 2\binom{n}{2} + \binom{n}{3} + 2\{2[2\binom{n}{1} + \binom{n}{2}] + 2\binom{n}{2} + \binom{n}{3}\} + 2\{2[2\binom{n}{1} + \binom{n}{2}] + 2\binom{n}{2} + \binom{n}{3}\} + (2 + \binom{n}{1})[2[2(\binom{n-1}{1} + 2) + 2\binom{n-1}{1} + \binom{n-1}{2}] + 2[2\binom{n-1}{1} + \binom{n-1}{2}] + 2\binom{n-1}{2} + \binom{n-1}{3}] + 2\{2\{2\binom{n}{2} + \binom{n}{3}\} + (2 + \binom{n}{1})[2(2\binom{n-1}{1} + \binom{n-1}{2}) + 2\binom{n-1}{2} + \binom{n-1}{3}]\} + 2\{(2 + \binom{n}{1})\{2[2\binom{n-1}{1} + \binom{n-1}{2}] + 2\binom{n-1}{2} + \binom{n-1}{3}\}\} + (2\binom{n}{1} + \binom{n}{2})[2[2[\binom{n-2}{2} + 2] + 2\binom{n-2}{1} + \binom{n-2}{2}] + 2[2\binom{n-2}{1} + \binom{n-2}{2}] + 2\binom{n-2}{2} + \binom{n-2}{3}] + \ldots + 2\{2[2\binom{n}{n-3} + \binom{n}{n-2}] + 2\binom{n}{n-2} + \binom{n}{n-1}\} + 2[2\binom{n}{n-2} + \binom{n}{n-1}] + 2\binom{n}{n-1}$ $+ \ldots -$ $2\{2(2\binom{n}{n-3} + \binom{n}{n-2}) + 2\binom{n}{n-2} + \binom{n}{n-1}\} + 2[2\binom{n}{n-2} + \binom{n}{n-1}] + 2\binom{n}{n-1} + 2\{2(2\binom{n}{n-2} + \binom{n}{n-1}) + 2\binom{n}{n-1}\} + 2\{2[2\binom{n}{n-2} +$

$\binom{n}{n-1}] + 2\binom{n}{n-1}\} + (2 + \binom{n}{1})\{2[2(2\binom{n-1}{n-2}) + 2\binom{n-1}{n-2} + 2\binom{n-1}{n-2}] + 2[2\binom{n-1}{n-2}]]\} +$
$2\{2\{2\binom{n}{n-1}\} + (2 + \binom{n}{1})\{2(2\binom{n-1}{n-2})\}\} + 2\{(2 + \binom{n}{1})\{2[2\binom{n-1}{n-2}]]\} + (2\binom{n}{1} +$
$\binom{n}{2})\{2[2[2\binom{n-2}{n-3}]]]\} + 2\{2[\binom{n}{1} + 2] + 2\binom{n}{1} + \binom{n}{2}\} + 2[2\binom{n}{1} + \binom{n}{2}] + 2\binom{n}{2} + \binom{n}{3} +$
$2\{2(2\binom{n}{n-2} + \binom{n}{n-1}) + 2\binom{n}{n-1}\} + 2[2\binom{n}{n-1}] + 2\{2[2\binom{n}{n-1}]]\} + 2\{2[2\binom{n}{n-1}]]\} + (2 +$
$\binom{n}{1})[2\{2[2\binom{n-1}{n-2}]]\}] + 2[2 + \binom{n}{1}] + 2] + 2[\binom{n}{1} + 2\binom{n}{1} + \binom{n}{2}$ - $2\{2[2\binom{n}{n-1}]]\} + 2 + \binom{n}{1} +$
$2 + 2$

$$= 0.$$

Case(ii): Suppose that n is even. Therefore, $n + 4$ is even.

$A_1 - A_2 + A_3 - \cdots + A_{n+1} - A_{n+2} + A_{n+3} = 1 + 2 + \binom{n}{1} + 2 + 2 + 2\left[2 + \binom{n}{1} + 2\right] +$
$2\left[\binom{n}{1} + 2\right] + 2\binom{n}{1} + \binom{n}{2} + 22\left[\binom{n}{1} + 2\right] + 2\binom{n}{1} + \binom{n}{2} + 2\left[2\binom{n}{1} + \binom{n}{2}\right] + 2\binom{n}{2} + \binom{n}{3} +$
$22\left[2\binom{n}{1} + \binom{n}{2}\right] + 2\binom{n}{2} + \binom{n}{3} + 2\left[2\binom{n}{2} + \binom{n}{3}\right] + 2\binom{n}{3} + \binom{n}{4} + \cdots + 22\left[2\binom{n}{n-3} +\right.$
$\left.\binom{n}{n-2}\right] + 2\binom{n}{n-2} + \binom{n}{n-1} + 2\left[2\binom{n}{n-2} + \binom{n}{n-1}\right] + 2\binom{n}{n-1} + 22\left[2\binom{n}{n-2} + \binom{n}{n-1}\right] +$
$2\binom{n}{n-1} + 2\left[2\binom{n}{n-1}\right] + 22\left[2\binom{n}{n-1}\right] + 1 - 2 + \binom{n}{1} + 2 + 2 + 2[2 + \binom{n}{1} + 2] + 2[2 +$
$\binom{n}{1} + 2] + (2 + \binom{n}{1})[2 + 2 + \binom{n-1}{1} + 2] + 2[2[\binom{n}{1} + 2] + (2 + \binom{n}{1})[2 + \binom{n-1}{1} + 2]] +$
$2[(2 + \binom{n}{1})(2 + \binom{n-1}{1} + 2)] + (2\binom{n}{1} + \binom{n}{2})[2 + 2 + \binom{n-2}{1} + 2] + 2\{2[(2 + \binom{n}{1})(2 +$
$\binom{n-1}{1})] + (2\binom{n}{1} + \binom{n}{2})[2 + \binom{n-2}{1} + 2]\} + 2[(2\binom{n}{1} + \binom{n}{2})(2 + \binom{n-2}{1} + 2)] + 2[(2\binom{n}{1} +$
$\binom{n}{2})(2 + \binom{n-2}{1} + 2)] + (2\binom{n}{2} + \binom{n}{3})[2 + 2 + \binom{n-3}{1} + 2] + \cdots + 2\{2[2\binom{n}{n-1}\} + 2[2 +$
$\binom{n}{1} + 2] + 2[\binom{n}{1} + 2] + \quad 2\binom{n}{1} + \binom{n}{2} + 2[2[\binom{n}{1} + 2] + \quad 2\binom{n}{1} + \binom{n}{2}] + 2\{2[\binom{n}{1} + 2] +$
$2\binom{n}{1} + \binom{n}{2}\} + (2 + \binom{n}{1})\{2[2 + \binom{n-1}{1} + 2] + 2[\binom{n-1}{1} + 2] + 2\binom{n-1}{1} + \binom{n-1}{2}\} +$
$2\{2\{2\binom{n}{1} + \binom{n}{2}\} + (2 + \binom{n}{1})[2(\binom{n-1}{1} + 2) + 2\binom{n-1}{1} + \binom{n-1}{2}]]\} + 2\{(2 +$
$\binom{n}{1})[2[\binom{n-1}{1} + 2] + 2\binom{n-1}{1} + \binom{n-1}{2}]]\} + (2\binom{n}{1} + \binom{n}{2})[2[2 + \binom{n-1}{1} + 2] + 2[\binom{n-2}{1} +$
$2] + 2\binom{n-2}{1} + \binom{n-2}{2}] + 2\{2\{(2 + \binom{n}{1})[2\binom{n-1}{1} + \binom{n-1}{2}]]\} + (2\binom{n}{1} + \binom{n}{2})[2(\binom{n-2}{1} +$
$2) + 2\binom{n-2}{1} + \binom{n-2}{2}]]\} + 2\{(2\binom{n}{1} + \binom{n}{2})[2[2 + \binom{n-2}{1}] + 2\binom{n-2}{1} + \binom{n-2}{2}]]\} + (2\binom{n}{2} +$
$\binom{n}{3})[2[2 + \binom{n-3}{1} + 2] + 2[\binom{n-3}{1} + 2] + 2\binom{n-3}{1} + \binom{n-3}{2}] + \ldots + 2\{2[2\binom{n}{n-2} + \binom{n}{n-1}] +$
$2\binom{n}{n-1}\} + 2[2\binom{n}{n-1}] \qquad -2[2(\binom{n}{1} + 2) + 2\binom{n}{1} + \binom{n}{2}] + 2[2\binom{n}{1} + \binom{n}{2}] + 2\binom{n}{2} + \binom{n}{3} +$
$2\{2[2\binom{n}{1} + \binom{n}{2}] + 2\binom{n}{2} + \binom{n}{3}\} + 2\{2[2\binom{n}{1} + \binom{n}{2}] + 2\binom{n}{2} + \binom{n}{3}\} + (2 +$
$\binom{n}{1})[2[2(\binom{n-1}{1} + 2) + 2\binom{n-1}{1} + \binom{n-1}{2}] + 2[2\binom{n-1}{1} + \binom{n-1}{2}] + 2\binom{n-1}{2} + \binom{n-1}{3}] +$
$2\{2\{2\binom{n}{2} + \binom{n}{3}\} + (2 + \binom{n}{1})[2(2\binom{n-1}{1} + \binom{n-1}{2}) + 2\binom{n-1}{2} + \binom{n-1}{3}]]\} + 2\{(2 +$
$\binom{n}{1})\{2[2\binom{n-1}{1} + \binom{n-1}{2}] + 2\binom{n-1}{2} + \binom{n-1}{3}\}\} + (2\binom{n}{1} + \binom{n}{2})[2[2[\binom{n-2}{2} + 2] + 2\binom{n-2}{1} +$

$\binom{n-2}{2}] + 2[2\binom{n-2}{1} + \binom{n-2}{2}] + 2\binom{n-2}{2} + \binom{n-2}{3}] + \ldots + 2\{2[2\binom{n}{n-3} + \binom{n}{n-2}] + 2\binom{n}{n-2} + \binom{n}{n-1}\} + 2[2\binom{n}{n-2} + \binom{n}{n-1}] + 2\binom{n}{n-1}$ +…+ $2\{2(2\binom{n}{n-3} + \binom{n}{n-2}) + 2\binom{n}{n-2} + \binom{n}{n-1}\} + 2[2\binom{n}{n-2} + \binom{n}{n-1}] + 2\binom{n}{n-1} + 2\{2(2\binom{n}{n-2} + \binom{n}{n-1}) + 2\binom{n}{n-1}\} + 2\{2[2\binom{n}{n-2} + \binom{n}{n-1}] + 2\binom{n}{n-1}\} + (2 + \binom{n}{1})\{2[2(2\binom{n-1}{n-2}) + 2\binom{n-1}{n-2} + 2\binom{n-1}{n-2}] + 2[2\binom{n-1}{n-2}]\} + 2\{2\{2\binom{n}{n-1}\} + (2 + \binom{n}{1}\{2(2\binom{n-1}{n-2})\}\} + 2\{(2 + \binom{n}{1})\{2[2\binom{n-1}{n-2}]\} + (2\binom{n}{1} + \binom{n}{2})\{2[2[2\binom{n-2}{n-3}]]\} + 2\{2[\binom{n}{1} + 2] + 2\binom{n}{1} + \binom{n}{2}\} + 2[2\binom{n}{1} + \binom{n}{2}] + 2\binom{n}{2} + \binom{n}{3} + 2\{2(2\binom{n}{n-2} + \binom{n}{n-1}) + 2\binom{n}{n-1}\} + 2[2\binom{n}{n-1}] + 2\{2[2\binom{n}{n-1}]\} + 2\{2[2\binom{n}{n-1}]\} + (2 + \binom{n}{1})[2\{2[2\binom{n-1}{n-2}]\}] - 2[2 + \binom{n}{1}] + 2] + 2[\binom{n}{1} + 2\binom{n}{1} + \binom{n}{2} + 2\{2[2\binom{n}{n-1}]\} + 2 + \binom{n}{1} + 2 + 2$

$$= 2.$$

Hence the interval $[\emptyset, S^3(B_n)]$ has the same number of elements of odd and even rank.

Though in the above theorem we have proved that $CS[S^3(B_n)]$ is Eulerian, it is neither Simplicial nor dual simplicial.

$CS[S^3(B_n)]$is not dual simplicial since, the upper interval $[\{1\}, S^3(B_n)]$ in $CS[S^3(B_n)]$contains $8\binom{n}{n-1}$ number of atoms which is greater than $n+3$, the rank of $[\{1\}, S^3(B_n)]$,implying that $[\{1\}, S^3(B_n)]$is not Boolean.

$CS[S^3(B_n)]$is not simplicial since, the lower interval $[\emptyset, S^3(B_n)]$ where l_1 is the left extreme atom of $S^3(B_n)$ contains $3^3.2^n - 26$ number of atoms by Lemma 2.1, which cannot be equal to $n+3$, the rank of $[\emptyset, [l_1, 1]]$, implying that $[\emptyset, [l_1, 1]]$ is not Boolean.

Conclusions

In this paper, we have proved that $CS[S^3(B_n)]$ is an Eulerian lattice under the set inclusion

relation which is neither simplicial nor dual simplicial, if $n > 1$. We strongly believe that the result proved in this paper, can be extended to more general Eulerian lattices and any other general lattices.

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